International Journal of Pure and Applied Mathematics
Volume 62 No. 2 2010, 221-225

THE DYNAMIC OF THE AK RAMSEY GROWTH MODEL WITH QUADRATIC UTILITY AND LOGISTIC POPULATION CHANGE

Luca Guerrini
Department of Mathematics for Economic and Social Sciences
University of Bologna
5, Viale Filopanti, Bologna, 40126, ITALY
e-mail: luca.guerrini@unibo.it

Abstract: This paper incorporates the hypothesis of quadratic utility and logistic population growth into the Ramsey growth model. It is shown that the economy has a unique non-trivial steady state equilibrium, which happens to be a saddle point with a one or two dimensional stable manifold.

AMS Subject Classification: 91B62
Key Words: Ramsey model, quadratic utility, logistic population

1. Introduction

The Ramsey [4] growth model belongs to the class of the neoclassical models of exogenous growth. It is a simple, general equilibrium model where economic agents (households and firms) behave in the way to maximize their objectives. The key element to the model dynamics is the specifications of households behavior. Given households preferences and an intertemporal budget constraint, they maximize their lifetime utility. This is what makes the Ramsey model different from the well known Solow [5] growth model. While the saving rate is constant and exogenous in the Solow model, the Ramsey model predicts the saving rate to vary. In this paper we argue that the form of preferences matters to the Ramsey model. We investigate the model for the quadratic preferences. We build our model modification on Guerrini [3]. More concretely, within the neoclassical Ramsey growth model we assume a linear (AK) aggregate produc-

Received: June 17, 2010

tion function without diminishing returns to capital, a quadratic utility function, and a logistic-type population growth law. Within this framework, we demonstrate that the model has a unique non-trivial steady state equilibrium, which is a saddle point with a one or two dimensional stable manifold. Now, many models of growth, including Ramsey model, have the property that the transitional dynamics are determined by a one dimensional stable manifold. As a consequence, all the variables converge to their respective steady states at the same constant speed, which is equal to the magnitude of the unique stable eigenvalue. By contrast, in the present model, the stable transitional path may be a two dimensional locus, thereby introducing important flexibility to the convergence and transition characteristics.

2. The Model

Following Barro and Sala-i-Martin [1], we define the Ramsey model with quadratic utility function as follows. There is a closed economy inhabited by many identical agents facing the optimization problem

$$\max \int_0^{\infty} \left(c - \frac{c^2}{2}\right) e^{-\rho t} dt,$$

subject to the constraint

$$\dot{k} = Ak - (a - bL)k - c. \tag{1}$$

In the instantaneous utility function, $\rho > 0$ is the rate of time preference, c is per capita consumption of a single good, and k denotes a stock of productive capital. Output y is produced according to the linear production function $f(k) = Ak$, $A > 0$. For simplicity, let us assume $A \neq \rho$. As well, we assume that there is no capital depreciation. Population L evolves according to the following law

$$\dot{L} = L(a - bL), \quad a > b > 0, \tag{2}$$

where today's population is supposed to be equal to $L_0 = 1$. Equation (2) is called the Verhulst equation [6], and the underlying population model is known as the logistic model. Solving this continuous time dynamic problem involves using calculus of variations. The current-value Hamiltonian of this optimization problem writes as

$$H(k, c, L, \lambda) = c - \frac{c^2}{2} + \lambda \left[Ak - (a - bL)k - c\right],$$

where λ is the costate variable associated to the budget constraint (1). The Pontryagin conditions for optimality are

$$H_c = 0 \quad \Rightarrow \quad 1 - c = \lambda, \tag{3}$$

$$\dot{\lambda} = \rho\lambda - H_k \quad \Rightarrow \quad \dot{\lambda} = -\lambda[A - \rho - (a - bL)], \tag{4}$$

together with equations (1), (2), the boundary condition $k_0 > 0$, and the transversality condition $\lim\limits_{t\to\infty} e^{-\rho t}\lambda k = 0$. Differentiating (3) with respect to time, and using formula (3), we can rid (4) of the $\dot{\lambda}$, and λ expressions. In this way, we derive the dynamical system

$$\begin{aligned} \dot{k} &= Ak - (a - bL)k - c, && (5)\\ \dot{c} &= (1 - c)\left[A - \rho - (a - bL)\right], && (6)\\ \dot{L} &= L(a - bL). && (7) \end{aligned}$$

These equations, together with the initial condition k_0, and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\,(1 - c)\,k = 0, \tag{8}$$

constitute the dynamic system which drives the economy over time.

3. Local Stability Analysis

We are now ready to analyze the steady state equilibria of the model, which is defined as a situation in which the growth rates of consumption, capital and population are zero. An asterisk below a variable will denote its stationary value.

Lemma 1. *The economy has a unique non-trivial steady state equilibrium* (k_*, c_*, L_*), *namely*

$$k_* = \frac{1}{A}, \quad c_* = 1, \quad L_* = \frac{a}{b}. \tag{9}$$

Proof. To solve for the steady-state equilibrium we impose the derivatives in system (5)-(7) to be zero. This leads to $c = Ak$, $(1 - c)\,(A - \rho) = 0$, $L = a/b$. The statement is now straightforward. □

Proposition 1. *The steady state equilibrium* (k_*, c_*, L_*) *is a saddle point with a one or two dimensional stable manifold.*

Proof. Linearizing system (5)-(7) around the steady state yields the following approximated dynamic system

$$\begin{bmatrix} \dot{k} \\ \dot{c} \\ \dot{L} \end{bmatrix} = J^* \begin{bmatrix} k - k_* \\ c - c_* \\ L - L_* \end{bmatrix}, \text{ with } J^* = \begin{bmatrix} A & -1 & \frac{b}{A} \\ 0 & -A+\rho & 0 \\ 0 & 0 & -a \end{bmatrix}.$$

The Jacobian matrix J^* is obtained by differentiating the right-hand sides of equations (5)-(7) with respect to the variables k, c, L, and then evaluating them at the steady state. In order to characterize the local stability of our system, we need to compute the eigenvalues of J^*. It is now immediate that they are given by $\lambda_1 = A > 0$, $\lambda_2 = -A + \rho \gtrless 0$, $\lambda_3 = -a < 0$. In conclusion, we have found that J^* may have one real positive (unstable) and two real negative (stable) roots, or two real positive (unstable) and one real negative (stable) roots. This proves that the steady state is (locally) a saddle point (Blume and Simon, [2]). The dimension of the stable arm is by definition the number of negative eigenvalues. Hence, if $A < \rho$, then there is only one negative eigenvalue, so that the stable arm is a line going through the steady state. If $A > \rho$, then there are two negative eigenvalues, and the stable arm is a plane going through the steady state. □

4. Conclusion

This paper has examined a Ramsey model with quadratic utility and logistic population growth. Abandoning the standard constant-relative-risk-aversion utility function, and introducing a logistic-type population growth law in an otherwise standard Ramsey model with linear aggregate technology, leads to a three dimensional dynamical system with a unique non-trivial steady state equilibrium (a saddle). Contrary to standard Ramsey model, the dimension of the stable manifold is proved to be one or two, thus yielding important flexibility to the convergence and transition characteristics of the model.

References

[1] R.J. Barro, X. Sala-i-Martin, *Economic Growth*, MIT Press, Cambridge, Massachusetts (2004).

[2] L.E. Blume, C.P. Simon, *Mathematics for Economists*, Norton, New York (1994).

[3] L. Guerrini, The Ramsey model with a bonded population growth rate, *Journal of Macroeconomics*, **32** (2010), 872-878.

[4] F.P. Ramsey, A mathematical theory of saving, *Economic Journal*, 38 (1928), 543-559.

[5] R.M. Solow, A contribution to the theory of economic growth, *Quarterly Journal of Economics*, **70** (1956), 65-94.

[6] P.F. Verhulst, Notice sur la loi que la population suit dans son accroissement, *Correspondance Mathématique et Physique*, **10** (1838), 113-121.