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VORONOVSKAYA FORMULAE FOR KANTOROVICH TYPE GENERALIZED SAMPLING SERIES

Carlo Bardaro[1 §], Ilaria Mantellini[2]
[1,2]Department of Mathematics and Informatics
University of Perugia
1, Via Vanvitelli, Perugia, 06123, ITALY
[1]e-mail: bardaro@unipg.it
[2]e-mail: mantell@dmi.unipg.it

Abstract: Here we give a Voronovskaya type formula for Kantorovich generalized sampling series and a corresponding quantitative version in terms of some moduli of smoothness.

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1. Introduction

The theory of univariate generalized sampling series of a function f was introduced in [11] and [12] by P.L. Butzer and his school in Aachen and was developed by many authors (we quote here for example [27], [23], [28], [7], [19], [18], [5], [6]). A generalized sampling operators generated by a kernel function φ is defined by

$$(T_w^\varphi f)(x) = \sum_{k=-\infty}^{+\infty} \varphi(wx-k) f(\frac{k}{w}), \quad w > 0, \ x \in I\!R, \tag{1}$$

and it is very useful in the reconstruction of a signal, in an approximate sense, using the nodes k/w. One of the most interesting applications is the prediction theory, in which one can obtain good approximation of the value $f(x)$ at the point x using only sample values k/w in the past of x (see [12]). However in

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§Correspondence author

some cases one cannot determine the exact value $f(k/w)$ at the node k/w. Thus it seems to be useful to consider in place of $f(k/w)$ a mean value of f in a small interval $[k/w, (k+1)/w]$. This leads to a new version of the operator (1) of type

$$(S_w^\varphi f)(x) = \sum_{k=-\infty}^{+\infty} \varphi(wx-k)\left[w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)du\right], \quad w>0, \ x\in I\!R. \tag{2}$$

This operator can be considered as a “Kantorovich” version of the generalized sampling series (1), taking inspiration in the classical Kantorovich version of the Bernstein polynomials (see [17], [8], [31], [20], [15]). It was introduced in [4] where some modular convergence results are obtained in Orlicz spaces (for a nonlinear version of the results from [4], see [32]).

In [6], using a class of kernels φ with null first order (discrete) moment, we obtained an asymptotic formula of the second order of type

$$\lim_{w\to+\infty} w^2[(T_w^\varphi f)(x) - f(x)] = Af''(x), \tag{3}$$

where A is a constant depending on the second order moment of φ and the function f is twice differentiable at the point x.

In this paper using the same class of kernels, we obtain an asymptotic formula of type

$$\lim_{w\to+\infty} w[(S_w^\varphi f)(x) - f(x)] = \frac{1}{2}f'(x), \tag{4}$$

where f is only differentiable at the point x. It is interesting to remark here that the limit in (4) is fully independent of the kernel φ. Moreover we notice that the order of pointwise approximation in (4) cannot be improved also for more regular functions f.

The second main result is a quantitative version of (4) which shows that when f belongs to $C^1(I\!R)$ the convergence is uniform. As in [6] for the quantitative version of (3), the main tool is an approach considered in [16] which is based on the classical K-functional introduced by J. Peetre (see [25] and [26]) and widely used in various context (see [21], [22], [14], [13], [2] and [3]).

Finally we give some examples of kernels for which the theory developed can be applied. In particular we examine the one dimensional Bochner-Riesz kernel, a Blackman-Harris type kernel, the generalized Jackson kernels and some combinations of central B-splines. As a final remark note that quantitative Voronovskaya formulae have important links with the theory of semi-groups of operators (see [9]). The strict connections between the two theories were described in [1].

2. The Voronovskaya Formula

Let us denote by $C^0 = C^0(\mathbb{R})$ the space of all uniformly continuous and bounded functions $f : \mathbb{R} \to \mathbb{R}$, endowed with the usual supnorm $\|f\|_\infty$ and for $k \geq 1$ by $C^k = C^k(\mathbb{R})$ the subspace of C^0 whose elements f are k-times continuosly differentiable and $\|f^{(k)}\|_\infty < +\infty$.

Let $\varphi \in C^0$ be fixed. For any $\nu \in \mathbb{N}_0$, $u \in \mathbb{R}$ let us define the algebraic moments

$$m_\nu(\varphi, u) := \sum_{k=-\infty}^{+\infty} \varphi(u-k)(k-u)^\nu$$

and the absolute moments

$$M_\nu(\varphi) := \sup_{u \in \mathbb{R}} \sum_{k=-\infty}^{+\infty} |\varphi(u-k)||k-u|^\nu.$$

Remark. Note that for $\mu,\ \nu \in \mathbb{N}_0$ with $\mu < \nu$, $M_\nu(\varphi) < +\infty$ implies $M_\mu(\varphi) < +\infty$. Indeed for $\mu < \nu$ we have

$$\begin{aligned} & \sum_{k=-\infty}^{+\infty} |\varphi(u-k)||k-u|^\mu \\ = \ & \sum_{|u-k|<1} |\varphi(u-k)||k-u|^\mu + \sum_{|u-k|\geq 1} |\varphi(u-k)||k-u|^\mu \\ \leq \ & 2\|\varphi\|_\infty + \sum_{|u-k|\geq 1} |\varphi(u-k)| \frac{|k-u|^\nu}{|k-u|^{\nu-\mu}} \leq 2\|\varphi\|_\infty + M_\nu(\varphi). \end{aligned}$$

When φ has compact support, we immediately have that $M_\nu(\varphi) < +\infty$ for every $\nu \in \mathbb{N}_0$.

We suppose that the following assumptions hold:

i) for every $u \in \mathbb{R}$, we have

$$\sum_{k=-\infty}^{+\infty} \varphi(u-k) = 1,$$

ii) $M_2(\varphi) < +\infty$ and there holds

$$\lim_{r \to +\infty} \sum_{|u-k|>r} |\varphi(u-k)|(k-u)^2 = 0$$

uniformly with respect to $u \in I\!R$,

iii) for every $u \in I\!R$ we have

$$m_1(\varphi, u) \equiv m_1(\varphi) = \sum_{k=-\infty}^{+\infty} \varphi(u-k)(k-u) = 0.$$

Remark. Note that assumption ii) implies that for $j = 0, 1$, there holds

$$\lim_{r \to +\infty} \sum_{|u-k|>r} |\varphi(u-k)||k-u|^j = 0$$

uniformly with respect to $u \in I\!R$. Indeed, for example

$$\sum_{|u-k|>r} |\varphi(u-k)| < \frac{1}{r^2} \sum_{|u-k|>r} |\varphi(u-k)|(k-u)^2.$$

For $w > 0$ and for a kernel φ we define a family of operators $(S_w^\varphi)_{w>0}$ by (see [4])

$$(S_w^\varphi f)(x) = \sum_{k=-\infty}^{+\infty} \varphi(wx-k) \left[w \int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)du \right], \quad x \in I\!R.$$

Under the above assumptions there holds $L^\infty(I\!R) \subset \mathrm{Dom}\, S := \bigcap_{w>0} \mathrm{Dom}\, S_w^\varphi$ where $\mathrm{Dom}\, S_w^\varphi$ is the space of all functions $f : I\!R \to I\!R$ for which the series defining $S_w^\varphi f$ is absolutely convergent for every $x \in I\!R$.

We have the following Voronovskaya formula for $S_w^\varphi f$.

Theorem 1. *Let $f \in L^\infty(I\!R)$ be a function such that $f'(x)$ exists at a point $x \in I\!R$. Under the above assumptions there holds*

$$\lim_{w \to +\infty} w[(S_w^\varphi f)(x) - f(x)] = \frac{f'(x)}{2}.$$

Proof. We have that

$$(S_w^\varphi f)(x) - f(x) = \sum_{k=-\infty}^{+\infty} \varphi(wx-k) \left[w \int_{\frac{k}{w}}^{\frac{k+1}{w}} (f(u) - f(x))du \right].$$

Since f is differentiable at the point x there exists a bounded function h such that $\lim_{t \to 0} h(t) = 0$ and

$$f(u) = f(x) + f'(x)(u-x) + h(u-x)(u-x).$$

Thus we have

$$(S_w^\varphi f)(x) - f(x) = f'(x) \sum_{k=-\infty}^{+\infty} \varphi(wx-k) \left[w \int_{\frac{k}{w}}^{\frac{k+1}{w}} (u-x)du \right]$$

$$+ \sum_{k=-\infty}^{+\infty} \varphi(wx-k)\Big[w\int_{\frac{k}{w}}^{\frac{k+1}{w}} h(u-x)(u-x)du\Big] = I_1 + I_2.$$

We immediately have

$$\begin{aligned} I_1 = f'(x) \sum_{k=-\infty}^{+\infty} \varphi(wx-k)\Big[w\int_{\frac{k}{w}}^{\frac{k+1}{w}} (u-x)du\Big] \\ = \frac{f'(x)}{2} \sum_{k=-\infty}^{+\infty} \varphi(wx-k)w\Big((\frac{k+1}{w}-x)^2 - (\frac{k}{w}-x)^2\Big) \\ = \frac{f'(x)}{2} \sum_{k=-\infty}^{+\infty} \varphi(wx-k)w(\frac{k+1}{w}-x)^2 \\ - \frac{f'(x)}{2} \sum_{k=-\infty}^{+\infty} \varphi(wx-k)w(\frac{k}{w}-x)^2 \\ = \frac{f'(x)}{2w} \sum_{k=-\infty}^{+\infty} \varphi(wx-k) + f'(x) \sum_{k=-\infty}^{+\infty} \varphi(wx-k)(\frac{k}{w}-x) \\ = \frac{f'(x)}{2w} + \frac{f'(x)}{w} m_1(\varphi) = \frac{f'(x)}{2w}. \end{aligned}$$

Now we estimate

$$I_2 = \sum_{k=-\infty}^{+\infty} \varphi(wx-k)\Big[w\int_{\frac{k}{w}}^{\frac{k+1}{w}} h(u-x)(u-x)du\Big].$$

In oder to do that let $\varepsilon > 0$ be fixed. There exists $\eta > 0$ such that $|h(t)| \le \varepsilon$ for every $|t| \le \eta$.

Moreover let $\overline{w} > 0$ such that for every $w > \overline{w}$, $1/w < \eta/2$. Then we have

$$\begin{aligned} |I_2| \le \sum_{|k-wx|<\eta w/2} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} |h(u-x)||u-x|du \\ + \sum_{|k-wx|\ge\eta w/2} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} |h(u-x)||u-x|du \\ = I_2' + I_2''. \end{aligned}$$

Let us consider I_2'. Since for every $u \in [k/w, (k+1)/w]$ there holds

$$|u-x| \le |u-\frac{k}{w}| + |\frac{k}{w}-x| \le \eta,$$

we have

$$|I_2'| \leq \varepsilon \sum_{|k-wx|<\eta w/2} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} |u-x| du$$
$$\leq \frac{\varepsilon}{2} \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| w \Big((\frac{k+1}{w} - x)^2 + (\frac{k}{w} - x)^2 \Big)$$
$$\leq \frac{\varepsilon}{w} \Big(\frac{M_0(\varphi)}{2} + M_2(\varphi) + M_1(\varphi) \Big).$$

Now we consider I_2''. Firstly let $R > 0$ be such that

$$\sum_{|u-k|>R} |\varphi(u-k)|(u-k)^2 < \varepsilon$$

uniformly with respect to $u \in I\!R$. Moreover let $\overline{w}$ be such that $\eta w/2 > R$, for every $w > \overline{w}$. Then

$$\sum_{|k-wx|>\eta w/2} |\varphi(wx-k)|(wx-k)^2 < \varepsilon$$

for every $x \in I\!R$ and $w > \overline{w}$. Analogously the same inequality holds also for the series

$$\sum_{|k-wx|>\eta w/2} |\varphi(wx-k)||wx-k|^j < \varepsilon$$

for $j = 0, 1$. So we have

$$|I_2''| \leq \|h\|_\infty \sum_{|k-wx|\geq \eta w/2} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} |u-x| du$$
$$\leq \frac{\|h\|_\infty}{2} \sum_{|k-wx|\geq \eta w/2} |\varphi(wx-k)| w \Big((\frac{k+1}{w} - x)^2 + (\frac{k}{w} - x)^2 \Big)$$
$$\leq \frac{5\varepsilon \|h\|_\infty}{2w}.$$

So we obtain the assertion. □

Remark. We can relax the boundedness assumption on f assuming that there are two positive constants a, b such that

$$|f(x)| \leq a + b|x|, \quad \text{for every } x \in I\!R.$$

At first let us remark that in this instance $f \in \mathrm{Dom}\, S$. Indeed

$$|(S_w^\varphi f)(x)| \leq \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} |f(u)| du$$

$$\leq \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} (a+b|u|)du$$

$$\leq M_0(\varphi)(a + \frac{b}{2w} + bwx^2 + b|x|) + M_1(\varphi)(2b|x| + \frac{b}{w}) + M_2(\varphi)\frac{b}{w}$$

and so the series defining $S_w^\varphi f$ is absolutely convergent for every $x \in I\!R$. Moreover, putting for a fixed $x_0 \in I\!R$,

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

the Taylor polynomial of first order centered at the point x_0, by the Taylor formula we can write

$$\frac{f(x) - P_1(x)}{(x - x_0)} = h(x - x_0),$$

where h is a function such that $\lim_{t \to 0} h(t) = 0$. Then h is bounded in a neighbourhood of x_0, say $[x_0 - \delta, x_0 + \delta]$, while for $|x - x_0| > \delta$, we have

$$|h(x - x_0)| \leq \frac{a + b|x|}{|x - x_0|} + \frac{|P_1(x)|}{|x - x_0|},$$

and the second right-hand side of the above inequality is bounded for $|x - x_0| > \delta$. Thus $h(\cdot - x_0)$ is bounded on $I\!R$, and we can proceed as in the proof of Theorem 1, obtaining the same Voronovskaya formula.

3. A Quantitative Estimate

Our aim is to determine the order of the convergence in Theorem 1 using a suitable modulus of continuity.

For a given $\varepsilon > 0$ we define

$$\omega(f, \varepsilon) := \sup_{|x-y|<\varepsilon} |f(x) - f(y)|.$$

In [16], for $f \in C^m$, it is stated the following version of the Taylor formula

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_m(f; x_0, x),$$

for $x_0, x \in I\!R$, $m \geq 1$ and the remainder $R_m(f; x_0, x)$ is estimated by

$$|R_m(f; x_0, x)| \leq \frac{|x - x_0|^m}{m!} \omega(f^{(m)}; |x - x_0|).$$

Note that, since $f^{(m)}$ is continuous, there holds

$$\omega(f^{(m)}; |x - x_0|) = o(1)$$

for $x \to x_0$.

In what follows, we will need the following K-functional, introduced by J. Peetre ([25], see also [26]) and defined by

$$K(\varepsilon, f, C^0, C^1) := \inf\{\|f-g\|_\infty + \varepsilon\|g'\|_\infty : g \in C^1\}$$

for $f \in C^0$ and $\varepsilon \geq 0$. In order to relate the K-functional to a modulus of continuity, we will quote the following lemma (see [26] Corollary 2.1 and [24] Lemma 12.1)

Lemma 1. *For every $f \in C^0$ there holds*

$$K(\varepsilon/2, f, C^0, C^1) = \frac{1}{2}\widetilde{\omega}(f, \varepsilon), \ \varepsilon \geq 0.$$

Here $\widetilde{\omega}(f, \cdot)$ denotes the least concave majorant of $\omega(f, \cdot)$ (see e.g. [3]).

As in [16], we have the following estimate of the remainder $R_m(f; x_0, x)$ in terms of $\widetilde{\omega}$

Lemma 2. *For $m \in I\!N_0$ let $f \in C^m$ and $x, x_0 \in I\!R$. Then we have*

$$|R_m(f; x_0, x)| \leq \frac{|x-x_0|^m}{m!}\widetilde{\omega}\left(f^{(m)}, \frac{|x-x_0|}{m+1}\right).$$

We study an estimate of the convergence in Theorem 1 in terms of the modulus $\widetilde{\omega}$, in case $m = 1$. We have the following

Theorem 2. *Let $f \in C^1$ be fixed and let $x \in I\!R$. Then there holds*

$$|w[(S_w^\varphi f)(x) - f(x)] - \frac{f'(x)}{2}| \leq \frac{A}{2}\widetilde{\omega}\left(f', \frac{1}{w}\frac{B}{A}\right),$$

where $A = M_0(\varphi) + M_1(\varphi) + 2M_2(\varphi)$ and $B = M_0(\varphi)/3 + M_1(\varphi) + M_2(\varphi)$.

Proof. For a given $f \in C^1$, as in Theorem 1, we can write

$$\begin{aligned}
&\left|(S_w^\varphi f)(x) - f(x) - \frac{f'(x)}{2w}\right| \\
= &\left|\sum_{k=-\infty}^{+\infty} \varphi(wx-k) f'(x) w \int_{\frac{k}{w}}^{\frac{k+1}{w}} (u-x)du\right. \\
+ &\left.\sum_{k=-\infty}^{+\infty} \varphi(wx-k) w \int_{\frac{k}{w}}^{\frac{k+1}{w}} h(u-x)(u-x)du - \frac{f'(x)}{2w}\right| \\
\leq &\sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} |h(u-x)||u-x|du.
\end{aligned}$$

Putting $R_1(f, x, u) = h(u-x)(u-x)$, by Lemma 1 and Lemma 2 for $m = 1$, we have

$$\begin{aligned}
&\left|(S_w^{\varphi} f)(x) - f(x) - \frac{f'(x)}{2w}\right| \\
\leq & \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} |u-x| \widetilde{\omega}\left(f', \frac{|x-u|}{2}\right) du \\
= & \; 2 \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} |u-x| K\left(\frac{|u-x|}{4}, f', C^0, C^1\right) du := J.
\end{aligned}$$

Let now $g \in C^2$. We have

$$\begin{aligned}
&J \leq 2 \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| w \int_{\frac{k}{w}}^{\frac{k+1}{w}} |u-x| \left[\|(f-g)'\|_\infty + \frac{|x-u|}{4}\|g''\|_\infty\right] du \\
= & \; 2\|(f-g)'\|_\infty w \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| \int_{\frac{k}{w}}^{\frac{k+1}{w}} |u-x| du \\
+ & \; \frac{\|g''\|_\infty}{2} w \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| \int_{\frac{k}{w}}^{\frac{k+1}{w}} (u-x)^2 du \\
\leq & \; w\|(f-g)'\|_\infty \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| (\frac{k+1}{w} - x)^2 \\
+ & \; w\|(f-g)'\|_\infty \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| (\frac{k}{w} - x)^2 \\
+ & \; \frac{\|g''\|_\infty}{6} w \sum_{k=-\infty}^{+\infty} |\varphi(wx-k)| \left(\frac{1}{w^3} + \frac{3}{w^2}(\frac{k}{w} - x) + \frac{3}{w}(\frac{k}{w} - x)^2\right) \\
\leq & \; A\frac{\|(f-g)'\|_\infty}{w} + B\frac{\|g''\|_\infty}{2w^2} = \frac{A}{w}\left[\|(f-g)'\|_\infty + \frac{\|g''\|_\infty}{2w}\frac{B}{A}\right].
\end{aligned}$$

Taking the infimum over $g \in C^2$ we obtain

$$J \leq \frac{A}{2w}\widetilde{\omega}\left(f', \frac{1}{w}\frac{B}{A}\right)$$

and so the assertion follows. □

Remark. As a consequence of Theorem 2, under the above assumptions we get the uniform convergence for $w((S_w^{\varphi} f)(x) - f(x))$ to $\frac{f'(x)}{2}$.

Remark. Note that when φ has a compact support $I = [-R, R]$, $R > 0$ we can obtain a slightly different estimate

$$\left| w[(S_w^\varphi f)(x) - f(x)] - \frac{f'(x)}{2} \right| \leq \frac{M_0(\varphi)(1+R+2R^2)}{2} \widetilde{\omega} \left(f', \frac{1}{w} \frac{1/3+R+R^2}{1+R+2R^2} \right).$$

Indeed we have easily

$$J \leq A \frac{\|(f-g)'\|_\infty}{w} + B \frac{\|g''\|_\infty}{2w^2} \leq M_0(\varphi)(1+R+2R^2) \frac{\|(f-g)'\|_\infty}{w} + M_0(\varphi)(1/3+R+R^2) \frac{\|g''\|_\infty}{2w^2}.$$

4. Examples

In this section we will consider some particular examples of kernels φ for which the theory developed before can be applied. Note that the Voronovskaya formula stated in Theorem 1 doesn't depend on the kernel φ and the order of pointwise approximation to a differentiable function is always the same and it cannot be improved also when the function f is more regular.

I) The Bochner-Riesz Kernel. Let us consider the one-dimensional Bochner-Riesz kernel defined by (see e.g. [10] and [30])

$$\varphi(x) \equiv b^\gamma(x) = 2^\gamma \Gamma(\gamma+1)(|x|)^{-1/2-\gamma} J_{(1/2)+\gamma}(|x|)$$

for $\gamma > 0$, where J_λ is the Bessel function of order λ. It is well known that

$$\widehat{b^\gamma}(v) = \begin{cases} (1-v^2)^\gamma, & |v| \leq 1, \\ 0, & |v| > 1. \end{cases}$$

Using the Poisson summation formula

$$(-i)^j \sum_{k=-\infty}^{+\infty} \varphi(u-k)(u-k)^j \sim \sum_{k=-\infty}^{+\infty} \widehat{\varphi}^{(j)}(2\pi k) e^{i2\pi ku},$$

there holds for $u \in \mathbb{R}$

$$\sum_{k=-\infty}^{+\infty} b^\gamma(u-k) = \widehat{b^\gamma}(0) = 1$$

and

$$m_1(b^\gamma) = \sum_{k=-\infty}^{+\infty} b^\gamma(u-k)(u-k) = 0.$$

From Proposition 1 in [6], for every $\gamma > 5/2$ we have $M_2(b^\gamma) < +\infty$ and

$$\lim_{r\to+\infty} \sum_{|k-u|>r} |b^\gamma(u-k)|(u-k)^2 = 0,$$

uniformly with respect to $u \in I\!R$. Thus all the assumptions are satisfied for this kernel.

II) A Particular Blackman-Harris Kernel. Let us define the kernel, for $x \in I\!R$

$$\varphi(x) \equiv B(x)$$
$$= \frac{1}{2}\mathrm{sinc}(x) + \frac{9}{32}(\mathrm{sinc}(x+1) + \mathrm{sinc}(x-1)) - \frac{1}{32}(\mathrm{sinc}(x+3) + \mathrm{sinc}(x-3)),$$

where $\mathrm{sinc}(x) := \frac{\sin \pi x}{\pi x}$. From [18], there holds that $B(x) = \mathcal{O}(|x|^{-5})$ as $|x| \to +\infty$. Hence from Remark 3.2(d) and Lemma 3.1 in [4] it follows that $M_2(B)$ is finite and

$$\lim_{r\to+\infty} \sum_{|k-u|>r} |B(u-k)|(u-k)^2 = 0.$$

Indeed, there exists $N > 0$ such that $|B(x)| < M/|x|^5$ for $|x| > N$. So, denoting by $[t]$ the greatest integer less or equal to t, we have for $r > N$

$$\sum_{|k-u|>r} |B(u-k)|(u-k)^2 \le M \sum_{|k-u|>r} \frac{1}{|u-k|^3} \le \frac{M}{r} \sum_{|k-u|>r} \frac{1}{[u-k]^2}$$
$$\le \frac{2M}{r} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Moreover

$$\widehat{B}(v) = \frac{1}{\sqrt{2\pi}} \lambda(\frac{v}{\pi}),$$

where

$$\lambda(v) = (\frac{1}{2} + \frac{9}{16}\cos(\pi v) - \frac{1}{16}\cos(3\pi v))\chi_{[-1,1]}(v)$$

and χ_I denotes the characteristic function of the set I. Thus using Lemma 3 in [12] we have

$$m_1(B) = \sum_{k=-\infty}^{+\infty} B(u-k)(u-k) = 0.$$

Thus all the assumptions are satisfied for this kernel.

III) The Generalized Jackson Kernels. For every $n \in I\!N$ let us consider the function (see [4])

$$\varphi(x) \equiv G_n(x) = c_n \mathrm{sinc}^{2n}(\frac{x}{2n\pi\alpha}), \quad x \in I\!R$$

where $\alpha \geq 1$ and c_n is a normalization constant. For $n \geq 2$ from Remark 3.2(d) and Lemma 3.1 in [4] it follows that $M_2(G_n)$ is finite and as in the previous example we have

$$\lim_{r\to+\infty} \sum_{|k-u|>r} |G_n(u-k)|(u-k)^2 = 0.$$

Moreover taking into account that G_n is bandlimited to the interval $[-1/\alpha, 1/\alpha]$ and using again the Poisson summation formula we get

$$m_1(G_n) = \sum_{k=-\infty}^{+\infty} G_n(u-k)(u-k) = 0.$$

Thus all the assumptions are satisfied for this kernel.

IV) Combinations of Spline Functions. Here, using a method developed in [12], we give an explicit example of kernel φ with compact support, satisfying all the previous assumptions. In order to do that, let us define the central B-splines of order $h \in I\!N$ as

$$B_h(x) := \frac{1}{(h-1)!} \sum_{j=0}^{h} (-1)^j \begin{pmatrix} h \\ j \end{pmatrix} (\frac{h}{2} + x - j)_+^{h-1},$$

where $x_+^r := \max\{x^r, 0\}$. It is well known that the Fourier transform of the functions B_h is given by

$$\widehat{B}_h(v) = \left(\frac{\sin v/2}{v/2}\right)^h, \quad v \in I\!R,\ h \in I\!N$$

(see [29] and [12]). Given real numbers $\varepsilon_0,\ \varepsilon_1$ with $\varepsilon_0 < \varepsilon_1$ we will construct a linear combination of translates of B_h, with $h \geq 2$, of type

$$\varphi(x) = a_0 B_h(x - \varepsilon_0) + a_1 B_h(x - \varepsilon_1)$$

in such a way that i) and iii) are satisfied (note that in this instance assumption ii) is automatically satisfied). Using the Poisson summation formula, we have to find the constants a_0 and a_1 such that

$$\widehat{\varphi}(2\pi k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0, \end{cases}$$

$$\widehat{\varphi}'(2\pi k) = 0 \text{ for every } k \in \boldsymbol{Z}.$$

The Fourier transform of φ is given by

$$\widehat{\varphi}(v) = \widehat{B}_h(v)(a_0 e^{-i\varepsilon_0 v} + a_1 e^{-i\varepsilon_1 v}).$$

Since

$$\widehat{B}_h'(2k\pi) = 0 \quad \text{for every } k \in \boldsymbol{Z}$$

we obtain the system

$$\widehat{\varphi}(0) = a_0 + a_1 = 1$$

and

$$\widehat{\varphi}'(0) = -i(\varepsilon_0 a_0 + \varepsilon_1 a_1) = 0$$

while for $k \neq 0$ we obtain identities $0 = 0$. Solving the above linear system we obtain the unique solution

$$a_0 = \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_0}, \; a_1 = -\frac{\varepsilon_0}{\varepsilon_0 - \varepsilon_1}.$$

Moreover it is easy to see that the support of the function φ is contained in the interval $[\varepsilon_0 - \frac{h}{2}, \varepsilon_1 + \frac{h}{2}]$.

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