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GENERALIZATION OF β-POWER-MEAN FOR OPERATORS

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Abstract: Motivated by Hayashi's mapping and characterization of r-mean [3], we present a generalization of the mapping and the β-power-mean which was originated in [4]. Instead of two operators we generalize the β-power-mean to $2n+1$ operators. Some applications are given about the Furuta-type operator inequalities.

Dedicated to Professor Showhwa Lin
with admiration and respects.

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1. Introduction

Let H be a (finite or infinite dimensional) complex Hilbert space, and the set of all bounded positive and invertible linear operators on H be denoted by $B(H)^+$. According to [3] Hayashi gave the r-mean as follows: *For* $r > 0$ *and* $A, B \in B(H)^+$ *the map* $M_r(\cdot,\cdot) : B(H)^+ \times B(H)^+ \to B(H)^+$ *defined by*

$$M_r(A, B) = B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2}$$

is called the r*-mean*. The map was characterized in the same paper [3, Theorem 2.1] in terms of the another map. In contrast we should mention that the β-power-mean introduced by Kubo and Ando [4] was given by

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$A \natural_\beta B = A^{1/2}(A^{-1/2}BA^{-1/2})^\beta A^{1/2}$ for any real number β and $A, B \in B(H)^+$.

In particular, the binary operation $\sharp_\alpha$ for $\alpha \in [0,1]$ is called the α-power-mean, i.e., $A \sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$.

α, β-power-means are used to express alternatively the Furuta-type inequalities as we see in the literature, e.g., [5] and references therein.

Instead of two operators we generalize the β-power-mean (i.e., the map $M_r(\cdot,\cdot)$) to $2n+1$ operators for any natural number n in this section.

In what follows we assume that $A, B, A_i \in B(H)^+$, $i = 1, 2, \cdots, 2n$.

Definition 1.1. Let r_i be any real numbers, $i = 1, 2, \cdots, 2n-1$. The map $M_{r_{2n-1}} : B(H)^+ \times B(H)^+ \times \cdots \times B(H)^+ \to B(H)^+$ (i.e., $M_{r_{2n-1}}$ maps $2n+1$ copies of $B(H)^+$ into $B(H)^+$) is defined by

$$M_{r_{2n-1}}(B, A_1, A_2, \cdots, A_{2n-1}, A_{2n})$$
$$= A_{2n}^{1/2}[A_{2n-1}^{-1/2}[A_{2n-2}^{1/2} \cdots [A_4^{1/2}[A_3^{-1/2}[A_2^{1/2}(A_1^{-1/2}BA_1^{-1/2})^{r_1}A_2^{1/2}]^{r_2}A_3^{-1/2}]^{r_3}$$
$$A_4^{1/2}]^{r_4} \cdots A_{2n-3}^{-1/2}]^{r_{2n-3}}A_{2n-2}^{1/2}]^{r_{2n-2}}A_{2n-1}^{-1/2}]^{r_{2n-1}}A_{2n}^{1/2}.$$

For $n = 1$ and $A_1 = A_2 = A$ in particular:

$$M_{r_1}(B, A, A) = A^{1/2}(A^{-1/2}BA^{-1/2})^{r_1}A^{1/2} = M_{r_1}(B, A) = A\natural_\beta B$$

if $r_1 = \beta$.

Immediately we have.

Corollary 1.2. *The following relations hold.*

(1) $M_{r_1}(B, A_1, A_2) = A_2^{1/2}(A_1^{-1/2}BA_1^{-1/2})^{r_1}A_2^{1/2}$
$$= A_2^{1/2}A_1^{-1/2}M_{r_1}(B, A_1)A_1^{-1/2}A_2^{1/2} = A_2^{1/2}A_1^{-1/2}(A_1\natural_{r_1}B)A_1^{-1/2}A_2^{1/2}.$$

(2) $M_{-r_1}(B, A_1, A_2) = M_{r_1}(B^{-1}, A_1^{-1}, A_2) = [M_{r_1}(B, A_1, A_2^{-1})]^{-1}$.

We see that $M_{r_1}(\cdot,\cdot)$ can be expressed in terms of $M_{r_1}(\cdot,\cdot,\cdot)$, and vice versa.

Corollary 1.3. *For each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$, if $C \leq B \leq A$ for $A, B \in B(H)^+$ and C is a bounded positive linear operators, then:*

(1) $A^{1-t+r} \geq [M_s(C^p, B^t, A^r)]^{\frac{1-t+r}{(p-t)s+r}}$
$$= [A^{r/2}B^{-t/2}(B^t \natural_s C^p)B^{-t/2}A^{r/2}]^{\frac{1-t+r}{(p-t)s+r}}.$$

(2) $A \geq A^{\frac{t-r}{2}}[A^{r/2}B^{-t/2}(B^t \natural_s C^p)B^{-t/2}A^{r/2}]^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$
$$= M_{\frac{1-t+r}{(p-t)s+r}}(B^{-t/2}(B^t\natural_s C^p)B^{-t/2}, A^{-r}, A^{t-r})$$

$$= A^{\frac{t-r}{2}}[A^{\frac{r-t}{2}}A^{t/2}B^{-t/2}(B^t\natural_s C^p)B^{-t/2}A^{t/2}A^{\frac{r-t}{2}}]^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$
$$= A^{t-r}\sharp_{\frac{1-t+r}{(p-t)s+r}}[A^{t/2}B^{-t/2}(B^t\natural_s C^p)B^{-t/2}A^{t/2}].$$

Proof. With the conditions in above the next inequality holds.

$$A^{1-t+r} \geq [A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{1-t+r}{(p-t)s+r}},$$

and it is called extension of the grand Furuta inequality, see [6].

(1) The proof is easy by (1) in Corollary 1.2.

(2) The inequality is trivial due to (1), and the equality follows by (1) in Corollary 1.2. □

The purpose of this paper is to express the map $M_{r_{2n-1}}$ (i.e., a generalization of the β-power-mean) alternatively in Section 2. In Section 3 two special cases of Theorem 2.1 are given, and consequently, we obtain some applications about the Furuta-type operator inequalities in Section 4.

2. Alternative Expression of the Map $M_{r_{2n-1}}$

The formula (1) in Corollary 1.2 and its similar forms will be used in the proof of the next result.

Theorem 2.1. *The following relations hold.*

$$M_{r_{2n-1}}(B, A_1, A_2, \cdots, A_{2n-1}, A_{2n})$$
$$= A_{2n}^{1/2}[A_{2n-1}^{-1/2}[A_{2n-2}^{1/2}\cdots[A_4^{1/2}[A_3^{-1/2}[A_2^{1/2}(A_1^{-1/2}BA_1^{-1/2})^{r_1}A_2^{1/2}]^{r_2}A_3^{-1/2}]^{r_3}$$
$$A_4^{1/2}]^{r_4}\cdots A_{2n-3}^{-1/2}]^{r_{2n-3}}A_{2n-2}^{1/2}]^{r_{2n-2}}A_{2n-1}^{-1/2}]^{r_{2n-1}}A_{2n}^{1/2}$$
$$= M_{r_{2n-1}}\{[M_{r_{2n-3}}[[M_{r_{2n-5}}\cdots[[M_{r_5}[[M_{r_3}[[M_{r_1}(B, A_1, A_2)]^{r_2}, A_3, A_4]]^{r_4},$$
$$A_5, A_6,]]^{r_6}\cdots]]^{r_{2n-4}}, A_{2n-3}, A_{2n-2}]]^{r_{2n-2}}, A_{2n-1}, A_{2n}\}.$$

Proof. The first equality is by Definition 1.1 and we shall use the induction process to give the proof of the second equality.

For $n = 1$:

$$M_{r_1}(B, A_1, A_2) = A_2^{1/2}(A_1^{-1/2}BA_1^{-1/2})^{r_1}A_2^{1/2} = M_{r_1}(B, A_1, A_2).$$

For $n = 2$:

$$M_{r_3}(B, A_1, A_2, A_3, A_4)$$

$$= A_4^{1/2}[A_3^{-1/2}[A_2^{1/2}(A_1^{-1/2}BA_1^{-1/2})^{r_1}A_2^{1/2}]^{r_2}A_3^{-1/2}]^{r_3}A_4^{1/2}$$
$$= A_4^{1/2}[A_3^{-1/2}[M_{r_1}(B,A_1,A_2)]^{r_2}A_3^{-1/2}]^{r_3}A_4^{1/2}$$
$$= M_{r_3}\{[M_{r_1}(B,A_1,A_2)]^{r_2},A_3,A_4\}.$$

For $n = 3$:

$$M_{r_5}(B,A_1,A_2,A_3,A_4,A_5,A_6)$$
$$= A_6^{1/2}[A_5^{-1/2}[A_4^{1/2}[A_3^{-1/2}[A_2^{1/2}(A_1^{-1/2}BA_1^{-1/2})^{r_1}$$
$$A_2^{1/2}]^{r_2}A_3^{-1/2}]^{r_3}A_4^{1/2}]^{r_4}A_5^{-1/2}]^{r_5}A_6^{1/2}$$
$$= A_6^{1/2}[A_5^{-1/2}[M_{r_3}[[M_{r_1}(B,A_1,A_2)]^{r_2},A_3,A_4]^{r_4}]A_5^{-1/2}]^{r_5}A_6^{1/2}$$
$$= M_{r_5}\{[M_{r_3}[[M_{r_1}(B,A_1,A_2)]^{r_2},A_3,A_4]]^{r_4},A_5,A_6\}.$$

For $n = 4$:

$$M_{r_7}(B,A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8)$$
$$= A_8^{1/2}[A_7^{-1/2}[A_6^{1/2}[A_5^{-1/2}[A_4^{1/2}[A_3^{-1/2}[A_2^{1/2}(A_1^{-1/2}BA_1^{-1/2})^{r_1}A_2^{1/2}]^{r_2}$$
$$A_3^{-1/2}]^{r_3}A_4^{1/2}]^{r_4}A_5^{-1/2}]^{r_5}A_6^{1/2}]^{r_6}A_7^{-1/2}]^{r_7}A_8^{1/2}$$
$$= A_8^{1/2}[A_7^{-1/2}[M_{r_5}[[M_{r_3}[[M_{r_1}(B,A_1,A_2)]^{r_2},A_3,A_4]]^{r_4},A_5,A_6]^{r_6}]A_7^{-1/2}]^{r_7}A_8^{1/2}$$
$$= M_{r_7}\{[M_{r_5}[[M_{r_3}[[M_{r_1}(B,A_1,A_2)]^{r_2},A_3,A_4]]^{r_4},A_5,A_6]]^{r_6},A_7,A_8\}.$$

Suppose that the relation holds for $n = k$ for any positive integer k, i.e.,

$$M_{r_{2k-1}}(B,A_1,A_2,\cdots,A_{2k-1},A_{2k})$$
$$= A_{2k}^{1/2}[A_{2k-1}^{-1/2}[A_{2k-2}^{1/2}\cdots[A_4^{1/2}[A_3^{-1/2}[A_2^{1/2}(A_1^{-1/2}BA_1^{-1/2})^{r_1}A_2^{1/2}]^{r_2}$$
$$A_3^{-1/2}]^{r_3}A_4^{1/2}]^{r_4}\cdots A_{2k-3}^{-1/2}]^{r_{2k-3}}A_{2k-2}^{1/2}]^{r_{2k-2}}A_{2k-1}^{-1/2}]^{r_{2k-1}}A_{2k}^{1/2}$$
$$= M_{r_{2k-1}}\{[M_{r_{2k-3}}[[M_{r_{2k-5}}\cdots[[M_{r_5}[[M_{r_3}[[M_{r_1}(B,A_1,A_2)]^{r_2},A_3,A_4]]^{r_4},$$
$$A_5,A_6,]]^{r_6}\ \cdots]]^{r_{2k-4}},A_{2k-3},A_{2k-2}]]^{r_{2k-2}},A_{2k-1},A_{2k}\}.$$

Then, for $n = k+1$:

$$M_{r_{2k+1}}(B,A_1,A_2,\cdots,A_{2k+1},A_{2k+2})$$
$$= A_{2k+2}^{1/2}[A_{2k+1}^{-1/2}[A_{2k}^{1/2}\cdots[A_4^{1/2}[A_3^{-1/2}[A_2^{1/2}(A_1^{-1/2}BA_1^{-1/2})^{r_1}A_2^{1/2}]^{r_2}$$
$$A_3^{-1/2}]^{r_3}A_4^{1/2}]^{r_4}\cdots A_{2k-1}^{-1/2}]^{r_{2k-1}}A_{2k}^{1/2}]^{r_{2k}}A_{2k+1}^{-1/2}]^{r_{2k+1}}A_{2k+2}^{1/2}$$
$$= A_{2k+2}^{1/2}[A_{2k+1}^{-1/2}[M_{r_{2k-1}}([M_{r_{2k-3}}[[M_{r_{2k-5}}\cdots[[M_{r_5}[[M_{r_3}[[M_{r_1}(B,A_1,A_2)]^{r_2}$$
$$,A_3,A_4]]^{r_4},A_5,A_6]^{r_6}\cdots,A_{2k-1},A_{2k}]^{r_{2n}}A_{2k+1}^{-1/2}]^{r_{2km+1}}A_{2k+2}^{1/2}$$
$$= M_{r_{2k+1}}\{[M_{r_{2k-1}}[[M_{r_{2k-3}}\cdots[[M_{r_5}[[M_{r_3}[[M_{r_1}(B,A_1,A_2)]^{r_2},A_3,A_4]]^{r_4},$$

$$A_5, A_6,]]^{r_6} \cdots, A_{2k-3}, A_{2k-2}]]^{r_{2k-2}}, A_{2k-1}, A_{2k}]]^{r_{2k}}, A_{2k+1}, A_{2k+2}\}.$$

This completes the induction process and the proof is finished. □

3. Special Cases of Theorem 2.1

Consequences of Theorem 2.1 are the next two special cases. The conditions in both corollaries below are not much of significant meaning, but only to be used in applications in the Section 4.

Corollary 3.1. *The following equality holds for $t \in [0,1]$, and any real numbers $p_i \geq 1$, $i = 1, 2, \cdots, 2n$.*

$$[A^{-t/2}[A^{t/2} \cdots [A^{t/2}[A^{-t/2}[A^{t/2}(A^{-t/2}B^{p_1}A^{-t/2})^{p_2}A^{t/2}]^{p_3}A^{-t/2}]^{p_4}A^{t/2}]^{p_5} \cdots A^{-t/2}]^{p_{2n-2}}A^{t/2}]^{p_{2n-1}}]A^{-t/2}]^{p_{2n}}$$

$$= A^{-t/2}[A^t \natural_{p_{2n}} (A^t \natural_{p_{2n-2}} (A^t \natural_{p_{2n-4}} (A^t \cdots (A^t \natural_{p_8} (A^t \natural_{p_6} (A^t \natural_{p_4} (A^t \natural_{p_2} B^{p_1})^{p_3})^{p_5})^{p_7} \cdots)^{p_{2n-3}})^{p_{2n-1}}]A^{-t/2}.$$

Proof. In Theorem 2.1 replace B by B^{p_1}. Also let $r_i = p_{i+1}$, $i = 1, 2, \ldots, 2n-1$, and $A_j = A^t$, $j = 1, 2, \cdots, 2n$. Then, by Definition 1.1,

$$A^{-t/2}\{M_{p_{2n}}(B^{p_1}, A^t, A^t, \cdots, A^t, A^t)\}A^{-t/2}$$

(there are $2n$ copies of A^t)

$$= [A^{-t/2}[A^{t/2} \cdots [A^{t/2}[A^{-t/2}[A^{t/2}(A^{-t/2}B^{p_1}A^{-t/2})^{p_2}A^{t/2}]^{p_3}A^{-t/2}]^{p_4}A^{t/2}]^{p_5} \cdots A^{-t/2}]^{p_{2n-2}}A^{t/2}]^{p_{2n-1}}]A^{-t/2}]^{p_{2n}}$$

(i.e., the left side of of the equality),

and by Theorem 2.1,

$$A^{-t/2}\{M_{p_{2n}}(B^{p_1}, A^t, A^t, \cdots, A^t, A^t)\}A^{-t/2}$$

(there are $2n$ copies of A^t)

$$= A^{-t/2}\{M_{p_{2n}}[[M_{p_{2n-2}}[[M_{p_{2n-4}} \cdots [[M_{p_6}[[M_{p_4}[[M_{p_2}(B^{p_1}, A^t, A^t)]^{p_3}, A^t, A^t]]^{p_5}, A^t, A^t,]]^{p_7} \cdots, A^t, A^t]]^{p_{2n-3}}, A^t, A^t]]^{p_{2n-1}}, A^t, A^t]\}A^{-t/2}.$$

Now, applying (1) in Corollary 1.2 we obtain inductively from above.

For $n = 1$: $A^{-t/2}\{M_{p_2}(B^{p_1}, A^t, A^t)\}A^{-t/2} = A^{-t/2}(A^t \natural_{p_2} B^{p_1})A^{-t/2}$.

For $n = 2$:

$$A^{-t/2}\{M_{p_4}[[M_{p_2}(B^{p_1}, A^t, A^t)]^{p_3}, A^t, A^t]\}A^{-t/2}$$

$$= A^{-t/2}[M_{p_4}[(A^t \natural_{p_2} B^{p_1})^{p_3}, A^t, A^t]]A^{-t/2}$$
$$= A^{-t/2}(A^t \natural_{p_4} (A^t \natural_{p_2} B^{p_1})^{p_3})A^{-t/2}.$$

For $n = 3$:

$$A^{-t/2}\{M_{p_6}[[M_{p_4}[[M_{p_2}(B^{p_1}, A^t, A^t)]^{p_3}, A^t, A^t]]^{p_5}, A^t, A^t,]\}A^{-t/2}$$
$$= A^{-t/2}[M_{p_6}[(A^t \natural_{p_4}(A^t\natural_{p_2}B^{p_1})^{p_3})^{p_5}, A^t, A^t]]A^{-t/2}$$
$$= A^{-t/2}[A^t\natural_{p_6}(A^t\natural_{p_4}(A^t\natural_{p_2}B^{p_1})^{p_3})^{p_5}]A^{-t/2}.$$

It leads to

$$A^{-t/2}\{M_{p_{2n}}[[M_{p_{2n-2}}[[M_{p_{2n-4}} \cdots [M_{p_6}[[M_{p_4}[[M_{p_2}(B^{p_1}, A^t, A^t)]^{p_3}, A^t, A^t]]^{p_5}$$
$$, A^t, A^t,]]^{p_7} \cdots, A^t, A^t]]^{p_{2n-3}}, A^t, A^t]]^{p_{2n-1}}, A^t, A^t]\}A^{-t/2}$$
$$= A^{-t/2}[A^t\natural_{p_{2n}}(A^t\natural_{p_{2n-2}}(A^t\natural_{p_{2n-4}}(A^t \cdots (A^t\natural_{p_8}(A^t\natural_{p_6}(A^t\natural_{p_4}$$
$$(A^t\natural_{p_2}B^{p_1})^{p_3})^{p_5})^{p_7} \cdots)^{p_{2n-3}})^{p_{2n-1}}]A^{-t/2},$$

which is the right side of the equality and this completes the proof. □

Corollary 3.2. *The following equality holds for $t \in [0, 1]$, and any real numbers $p_i \geq 1$, $i = 1, 2, \cdots, 2n$.*

$$A^{\frac{1-t}{2}}[A^{t/2}[A^{-t/2}[A^{t/2} \cdots [A^{t/2}[A^{-t/2}(A^{\frac{t-1}{2}}BA^{\frac{t-1}{2}})^{p_2}A^{-t/2}]^{p_3}A^{t/2}]^{p_4}A^{-t/2}]^{p_5}$$
$$\cdots A^{-t/2}]^{p_{2n-1}}A^{t/2}]^{p_{2n}}A^{\frac{1-t}{2}}$$
$$= A^{1-t}\natural_{p_{2n}}\{A\natural_{p_{2n-1}}\{A^{1-t}\natural_{p_{2n-2}} \cdots \{A^{1-t}\natural_{p_4}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\}\} \cdots\},$$

and there are $2n - 3$ terms of the bracksts "}" at the end for $n \geq 2$.

Proof. In Theorem 2.1 let $A_1 = A_{2n} = A^{1-t}$, $A_j = A^{-t}$, $j = 2, 3, ..., 2n - 1$, and $r_i = p_{i+1}$, $i = 1, 2, \cdots, 2n - 1$. Then, by Definition 1.1,

$$M_{p_{2n}}(B, A^{1-t}, A^{-t}, \cdots, A^{-t}, A^{1-t})$$
$$\text{(there are } 2n - 2 \text{ copies of } A^{-t})$$
$$= A^{\frac{1-t}{2}}[A^{t/2}[A^{-t/2}[A^{t/2} \cdots [A^{-t/2}[A^{t/2}[A^{-t/2}(A^{\frac{t-1}{2}}BA^{\frac{t-1}{2}})^{p_2}A^{-t/2}]^{p_3}A^{t/2}]^{p_4}$$
$$A^{-t/2}]^{p_5} \cdots A^{-t/2}]^{p_{2n-1}}A^{t/2}]^{p_{2n}}A^{\frac{1-t}{2}}$$

(i.e., the left side of the equality), and by Theorem 2.1:

$$M_{p_{2n}}(B, A^{1-t}, A^{-t}, \cdots, A^{-t}, A^{1-t})$$
$$\text{(there are } 2n - 2 \text{ copies of } A^{-t})$$
$$= M_{p_{2n}}\{[M_{p_{2n-2}}[[M_{p_{2n-4}} \cdots [[M_{p_6}[[M_{p_4}[[M_{p_2}(B, A^{1-t}, A^{-t})]^{p_3}, A^{-t}, A^{-t}]]^{p_5}$$
$$, A^{-t}, A^{-t},]]^{p_7} \cdots A^{-t}, A^{-t}]]^{p_{2n-3}}, A^{-t}, A^{-t}]]^{p_{2n-1}}, A^{-t}, A^{1-t}\}.$$

First note that

$$[M_{p_2}(B, A^{1-t}, A^{-t})]^{p_3} = [A^{-1/2}(A^{1-t}\natural_{p_2}B)A^{-1/2}]^{p_3}$$
$$= A^{-1/2}[A\natural_{p_3}(A^{1-t}\natural_{p_2}B)]A^{-1/2}.$$

Now, again, applying (1) in Corollary 1.2 we obtain inductively from above

For $n = 2$:

$$M_{p_4}\{[M_{p_2}(B, A^{1-t}, A^{-t})]^{p_3}, A^{-t}, A^{1-t}\}$$
$$= M_{p_4}\{[A^{-1/2}(A^{1-t}\natural_{p_2}B)A^{-1/2}]^{p_3}, A^{-t}, A^{1-t}\}$$
$$= A^{\frac{1-t}{2}}[A^{t/2}A^{-1/2}[A\natural_{p_3}(A^{1-t}\natural_{p_2}B)]A^{-1/2}A^{t/2}]^{p_4}A^{\frac{1-t}{2}}$$
$$= A^{\frac{1-t}{2}}\{A^{\frac{t-1}{2}}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\}A^{\frac{t-1}{2}}\}^{p_4}A^{\frac{1-t}{2}} = A^{1-t}\natural_{p_4}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\},$$

and there is $2 \times 2 - 3 = 1$ term of "}".

For $n = 3$:

$$M_{p_6}\{[M_{p_4}[[M_{p_2}(B, A^{1-t}, A^{-t})]^{p_3}, A^{-t}, A^{-t}]]^{p_5}, A^{-t}, A^{1-t}, \}$$
$$= A^{\frac{1-t}{2}}\{A^{\frac{t-1}{2}}\{A\natural_{p_5}\{A^{1-t}\natural_{p_4}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\}\}\}A^{\frac{t-1}{2}}\}^{p_6}A^{\frac{1-t}{2}}$$
$$= A^{1-t}\natural_{p_6}\{A\natural_{p_5}\{A^{1-t}\natural_{p_4}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\}\}\},$$

and there are $2 \times 3 - 3 = 3$ terms of "}".

For $n = 4$:

$$M_{p_8}([M_{p_6}[[M_{p_4}[[M_{p_2}(B, A^{1-t}, A^{-t})]^{p_3}, A^{-t}, A^{-t}]]^{p_5}, A^{-t}, A^{-t}]]^{p_7}, A^{-t}, A^{1-t},)$$
$$= A^{\frac{1-t}{2}}\{A^{\frac{t-1}{2}}\{A\natural_{p_7}\{A^{1-t}\natural_{p_6}\{A\natural_{p_5}\{A^{1-t}\natural_{p_4}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)$$
$$\}\}\}\}\}\}A^{\frac{t-1}{2}}\}^{p_8}A^{\frac{t-1}{2}}]^{p_6}A^{\frac{1-t}{2}}$$
$$= A^{1-t}\natural_{p_8}\{A\natural_{p_7}\{A^{1-t}\natural_{p_6}\{A\natural_{p_5}\{A^{1-t}\natural_{p_4}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\}\}\}\}\},$$

and there are $2 \times 4 - 3 = 5$ terms of "}".

Inductively it leads to the required conclusion. □

4. Applications

Recently Furuta gave and proved the next two theorems; one is the further extension of an order preserving operator inequality, and the other the log majorization.

Theorem A. (see [1], Theorem 3.3) *Let* $A \geq B \geq 0$ *with* $A > 0$, $t \in [0, 1]$, *and* $p_1, p_2, \cdots, p_{2n} \geq 1$ *for a natural number* n*. Then the following inequality holds for* $r \geq t$*.*

$$A^{1-t+r} \geq \{A^{r/2}[A^{-t/2}[A^{t/2} \cdots A^{t/2}[A^{-t/2}[A^{t/2}(A^{-t/2}B^{p_1}A^{-t/2})^{p_2}A^{t/2}]^{p_3}$$
$$A^{-t/2}]^{p_4}A^{t/2}]^{p_5} \cdots A^{-t/2}]^{p_{2n-2}}A^{t/2}]^{p_{2n-1}}]A^{-t/2}]^{p_{2n}}A^{r/2}\}^q,$$

where $q = \frac{1-t+r}{(\cdots((((p_1-t)p_2+t)p_3-t)p_4+t)p_5-\cdots-t)p_{2n}+r}$.

In Theorem A, there are n terms of $A^{-t/2}$ and $n-1$ terms of $A^{t/2}$ alternatively arranged on the left side of B^{p_1}, and the same arrangement on the right side of B^{p_1}. As for the denominator of q, there are n terms of $-t$ and $n-1$ terms of t alternatively arranged.

Theorem B. (see [2], Theorem 2.1) *Let* $A > 0$*,* $B \geq 0$*,* $t \in [0,1]$*, and* $p_1, p_2, \cdots, p_{2n} \geq 1$ *for a natural number* n*. Then the following log majorization holds for* $r \geq t$*.*

$$(A\sharp_{\frac{1}{p_1}}B)^h \succ_{(\log)} A^{1-t+r}\sharp_\alpha\{A^{1-t}\natural_{p_{2n}}\{A\natural_{p_{2n-1}}\{A^{1-t}\natural_{p_{2n-2}} \cdots$$
$$\{A^{1-t}\natural_{p_4}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\}\} \cdots\}.$$

where $h = \frac{p_1p_2\cdots p_{2n}(1-t+r)}{(\cdots((((p_1-t)p_2+t)p_3-t)p_4+t)p_5-\cdots-t)p_{2n}+r}$*, and* $\alpha = \frac{h}{p_1p_2\cdots p_{2n}}$.

In the right side of the log majorization in Theorem B, there are n terms of A^{1-t} and $n-1$ terms of A alternatively arranged on the left side of B. Also, for $n \geq 2$, there are $2(n-1)$ terms of the brackets $\}$ at the end.

Theorem 4.1.*Let* $A \geq B \geq 0$ *with* $A > 0$*,* $t \in [0,1]$*, and for a natural number* n*,* $p_i \geq 1$*,* $i = 1, 2, \cdots, 2n$*. Then the following inequality holds for* $r \geq t$*.*

$$(1)\ A^{1-t+r} \geq \{A^{r/2}[A^{-t/2}[A^{t/2} \cdots A^{t/2}[A^{-t/2}[A^{t/2}(A^{-t/2}B^{p_1}A^{-t/2})^{p_2}A^{t/2}]^{p_3}$$
$$A^{-t/2}]^{p_4}A^{t/2}]^{p_5} \cdots A^{-t/2}]^{p_{2n-2}}A^{t/2}]^{p_{2n-1}}]A^{-t/2}]^{p_{2n}}A^{r/2}\}^q$$
$$= \{A^{\frac{r-t}{2}}[A^t\natural_{p_{2n}}(A^t\natural_{p_{2n-2}}(A^t\natural_{p_{2n-4}}(A^t \cdots (A^t\natural_{p_8}(A^t\natural_{p_6}(A^t\natural_{p_4}$$
$$(A^t \natural_{p_2} B^{p_1})^{p_3})^{p_5})^{p_7} \ldots)^{p_{2n-3}}]^{p_{2n-1}}A^{\frac{r-t}{2}}\}^q.$$

$$(2)\ A \geq A^{\frac{t-r}{2}}\{A^{\frac{r-t}{2}}[A^t\natural_{p_{2n}}(A^t\natural_{p_{2n-2}}(A^t\natural_{p_{2n-4}}(A^t \cdots (A^t\natural_{p_8}(A^t\natural_{p_6}(A^t\natural_{p_4}$$
$$(A^t\natural_{p_2}B^{p_1})^{p_3})^{p_5})^{p_7} \ldots)^{p_{2n-3}}]^{p_{2n-1}}A^{\frac{r-t}{2}}\}^qA^{\frac{t-r}{2}}$$
$$= A^{t-r}\sharp_q\ \{A^t\natural_{p_{2n}}(A^t\natural_{p_{2n-2}}(A^t\natural_{p_{2n-4}}(A^t \cdots (A^t\natural_{p_8}(A^t\natural_{p_6}(A^t\natural_{p_4}$$
$$(A^t\natural_{p_2}B^{p_1})^{p_3})^{p_5})^{p_7} \ldots)^{p_{2n-3}}\}^{p_{2n-1}},$$

where $q = \frac{1-t+r}{(\cdots((((p_1-t)p_2+t)p_3-t)p_4+t)p_5-\cdots-t)p_{2n}+r} \leq 1$.

Proof. (1) The inequality is due to Theorem A, and the equality is by Corollary 3.1.

(2) The inequality is trivial from (1), and the equality is due to α-power mean. ☐

Corollary 4.2. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$, and for a natural number n, $p_i \geq 1$, $i = 1, 2, \cdots, 2n$. Then the following inequality holds.*

$$A \geq \{A^{t/2}[A^{-t/2}[A^{t/2} \cdots A^{t/2}[A^{-t/2}[A^{t/2}(A^{-t/2}B^{p_1}A^{-t/2})^{p_2}A^{t/2}]^{p_3} A^{-t/2}]^{p_4}A^{t/2}]^{p_5} \cdots A^{-t/2}]^{p_{2n-2}}A^{t/2}]^{p_{2n-1}}]A^{-t/2}]^{p_{2n}}A^{t/2}\}^{q}$$
$$= [A^t \natural_{p_{2n}}(A^t \natural_{p_{2n-2}}(A^t \natural_{p_{2n-4}}(A^t \cdots (A^t \natural_{p_8}(A^t \natural_{p_6}(A^t \natural_{p_4} (A^t \natural_{p_2} B^{p_1})^{p_3})^{p_5})^{p_7} \cdots)^{p_{2n-3}}]^{q p_{2n-1}},$$

where $q = \frac{1}{(\cdots((((p_1-t)p_2+t)p_3-t)p_4+t)p_5-\cdots-t)p_{2n}+t} \leq 1.$

Proof. The inequality comes from [1, Theorem 3.1], and this is nothing but a special case of Theorem A when $r = t$. The equality is a special case of (2) in Theorem 4.1. when $r = t$, and note that $I \sharp_\alpha B = B^\alpha$. ☐

Corollary 4.3. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$, and $p_i \geq 1$ for $i = 1, 2, 3, 4$. Then the following inequality holds for $r \geq t$.*

(1) $A^{1-t+r} \geq \{A^{r/2}[A^{-t/2}[A^{t/2}(A^{-t/2}B^{p_1}A^{-t/2})^{p_2}A^{t/2}]^{p_3}A^{-t/2}]^{p_4} A^{r/2}\}^{\frac{1-t+r}{(((p_1-t)p_2+t)p_3-t)p_4+r}} = \{A^{\frac{r-t}{2}}[A^t \natural_{p_4}(A^t \natural_{p_2} B^{p_1})^{p_3}]A^{\frac{r-t}{2}}\}^{\frac{1-t+r}{(((p_1-t)p_2+t)p_3-t)p_4+r}}.$

(2) $A \geq A^{\frac{t-r}{2}}\{A^{\frac{r-t}{2}}[A^t \natural_{p_4}(A^t \natural_{p_2} B^{p_1})]^{p_3}A^{\frac{r-t}{2}}\}^{\frac{1-t+r}{(((p_1-t)p_2+t)p_3-t)p_4+r}}A^{\frac{t-r}{2}}$
$= A^{t-r} \sharp_{\frac{1-t+r}{(((p_1-t)p_2+t)p_3-t)p_4+r}} [A^t \natural_{p_4}(A^t \natural_{p_2} B^{p_1})]^{p_3}.$

Proof. The inequality in (1) comes from [1, Corollary 3.4], and this is a special case of Theorem A when $n = 2$. Thus, Corollary 4.3 is just a special case of Theorem 4.1 when $n = 2$. ☐

Theorem 4.4. *Let $A > 0$, $B \geq 0$, $t \in [0,1]$, and for a natural number n, $p_i \geq 1$, $i = 1, 2, \cdots, 2n$. Then the following log majorization holds for $r \geq t$.*

$$(A \sharp_{\frac{1}{p_1}} B)^h \succ_{(\log)} A^{1-t+r} \sharp_\alpha \{A^{1-t} \natural_{p_{2n}} \{A \natural_{p_{2n-1}} \{A^{1-t} \natural_{p_{2n-2}} \cdots \{A^{1-t} \natural_{p_4} \{A \natural_{p_3} (A^{1-t} \natural_{p_2} B)\}\} \cdots\}$$

(*there are* $2(n-1)$ *terms of the brackets* $\}$ *on the right of* B *for* $n \geq 2$)

$$= A^{1-t+r} \sharp_\alpha \{A^{\frac{1-t}{2}}[A^{t/2}[A^{-t/2}[A^{t/2} \cdots [A^{t/2}[A^{-t/2}(A^{\frac{t-1}{2}}BA^{\frac{t-1}{2}})^{p_2}A^{-t/2}]^{p_3}$$

$A^{t/2}]^{p_4} \cdots A^{-t/2}]^{p_{2n-1}} A^{t/2}]^{p_{2n}} A^{\frac{1-t}{2}}\}$,
where $h = \frac{p_1 p_2 \cdots p_{2n}(1-t+r)}{(\cdots((((p_1-t)p_2+t)p_3-t)p_4+t)p_5-\cdots-t)p_{2n}+r}$, $\alpha = \frac{h}{p_1 p_2 \cdots p_{2n}}$.

Proof. Clearly this is due to Theorem B and Corollary 3.2. □

Corollary 4.5. *Let* $A > 0$, $B \geq 0$, *and for a natural number* n, $p_i \geq 1$, $i = 1, 2 \cdots, 2n$. *Then the following log majorization holds for* $r \geq 1$.

$$(A\sharp_{\frac{1}{p_1}} B)^h \succ_{(\log)} A^r \sharp_\alpha (A\natural_{p_{2n-1}}(A\natural_{p_{2n-3}}(A\natural_{p_{2n-5}} \cdots (A\natural_{p_5}(A\natural_{p_3} B^{p_2})^{p_4})^{p_6})^{p_8} \cdots)^{p_{2n}}$$
$$= A^r \sharp_\alpha [A^{1/2}[A^{-1/2}[A^{1/2} \cdots [A^{1/2}[A^{-1/2} B^{p_2} A^{-1/2}]^{p_3} A^{1/2}]^{p_4} \cdots A^{-1/2}]^{p_{2n-1}} A^{1/2}]^{p_{2n}},$$

where $h = \frac{p_1 p_2 \cdots p_{2n} r}{(\cdots((((p_1-1)p_2+1)p_3-1)p_4+1)p_5-\cdots-1)p_{2n}+r}$, $\alpha = \frac{h}{p_1 p_2 \cdots p_{2n}}$.

Proof. Let $t = 1$ in Theorem 4.4 to get the result. □

Corollary 4.6. *Let* $A > 0$, $B \geq 0$, $t \in [0, 1]$, *and* $p_i \geq 1$, $i = 1, 2, 3, 4$. *Then the following log majorization holds for* $r \geq t$.

$$(A\sharp_{\frac{1}{p_1}} B)^h \succ_{(\log)} A^{1-t+r} \sharp_\alpha \{A^{1-t} \natural_{p_4} \{A \natural_{p_3} (A^{1-t} \natural_{p_2} B)\}\}$$
$$= A^{1-t+r} \sharp_\alpha \{A^{\frac{1-t}{2}}[A^{t/2}[A^{-t/2}(A^{\frac{t-1}{2}} B A^{\frac{t-1}{2}})^{p_2} A^{-t/2}]^{p_3} A^{t/2}]^{p_4} A^{\frac{1-t}{2}}\},$$

where $h = \frac{p_1 p_2 p_3 p_4 (1-t+r)}{(((p_1-t)p_2+t)p_3-t)p_4+r}$, *and* $\alpha = \frac{h}{p_1 p_2 p_3 p_4}$.

Proof. Let $n = 2$ in Theorem 4.4. □

Corollary 4.7. *Let* $A > 0$, $B \geq 0$, $t \in [0, 1]$, *and* $\delta \in [0, 1]$. *Then the following log majorization holds for* $s \geq 1$ *and* $r \geq t$.

$$(A\sharp_\delta B)^h \succ_{(\log)} A^{1-t+r} \sharp_\alpha (A^{1-t} \natural_s B),$$

where $h = \frac{(1-t+r)s}{(1-\delta t)s+\delta r}$, *and* $\alpha = \frac{h}{s}\delta$.

Proof. Let $\frac{1}{p_1} = \delta \in [0, 1]$, $p_2 = p_3 = 1$, and $p_4 = s$ in Corollary 4.6, and note, in this case, that

$$A^{\frac{1-t}{2}}[A^{t/2}[A^{-t/2}(A^{\frac{t-1}{2}} B A^{\frac{t-1}{2}})^{p_2} A^{-t/2}]^{p_3} A^{t/2}]^{p_4} A^{\frac{1-t}{2}} = A^{1-t} \natural_s B. \qquad \square$$

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