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COMMON FIXED POINT AND BEST SIMULTANEOUS APPROXIMATIONS FOR CIRIC TYPE (f, g)-WEAK CONTRACTION AND WEAK ASYMPTOTIC CONTRACTION

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Abstract: We introduces a new conception of Ciric type (f, g)-weak contractive mapping and the existence of common fixed points is established for three mappings, where T is either Ciric type $(f, g)-$weakly contractive mapping or weakly asymptotically (f, g)-nonexpansive mapping. As applications, the invariant best simultaneous approximation results are proved in this paper.

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1. Introduction

The study of common fixed points of mappings satisfying certain contractive

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condition has been at the center of vigorous research activity. The well-known Banach's contraction mapping principle is one of the most useful results in nonlinear analysis. In a metric space setting it can briefly be stated as follows:

Theorem A. *Let (X, d) be a complete metric space and $T : X \to X$ a strict contraction, i.e., a map satisfying*

$$d(Tx, Ty) \leq ad(x, y), \quad \text{for all } x, y \in X, \tag{1.1}$$

where $0 \leq a < 1$ is constant. Then T has a unique fixed point in X.

Theorem A, has many applications in solving nonlinear functional equations but suffers from one drawback – the contractive condition (1.1) forces T to be continuous throughout X. In order to remove this drawback, in 1968 Kannan has obtained a fixed point theorem for mappings T that need not be continuous by considering insted of (1.1) the following contractive condition:

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)], \quad \text{for all } x, y \in X. \tag{1.2}$$

Following Kannan's Fixed Point Theorem, a lot of paper were devoted to obtaining fixed point theorem for various types of contractive type conditions that do not require the continuity of T, see for example the one obtained by Chatterjea. One of the most general contraction condition obtained in this way, for which the picard iteration still converge to the unique fixed point, was given by Ciric in 2010, by considering the contractive condition

$$d(Tx, Ty) \leq h. \max\{d(x, y).d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1.3}$$

for all $x, y \in X$ and some constant $0 \leq h < 1$.

In 1963, Meinardus [16] employed the Schauder Fixed Point Theorem to prove a result regarding invariant approximation. Further generalizations of the result of Meinardus were obtained by Habiniak [5], Hicks and Humphries [6], Jungck and Sessa [12], Singh [27], Smoluk [28] and Subrahmanyam [29]. Recently, Al-Thagafi [1] extended the work of Singh [27], Smoluk [28], Subrahmanyam [29] and proved some results on invariant approximations for commuting maps. Shahzad [26], Hussain and Jungck [7], Hussain et al [8], Hussain and Rhoades [9], Jungck and Hussain [11], O'Regan and Hussain [18], O'Regan and Shahzad [19] and Pathak et al [20] extended the work of Al-Thagafi [1] for R-subweakly commuting, compatible and C_q-commuting maps.

The applications of fixed point theorems are remarkable in diverse disciplines of mathematics, statistics, engineering and economics in dealing with problems arising in approximation theory, potential theory, game theory, theory of differential equations, theory of integral equations, etc., see [21].

2. Preliminaries and Definition

Let (X, d) be a metric space and D be the subset of X. We shall use $\mathbb{N}$ to denote the set of positive integers, $cl(D)$ to denote the closure of a set D and $wcl(D)$ to denote the weak closure of a set D.

Definition 2.1. Let $f, g, T : D \to D$ be mappings. Then mapping T is called an (f, g)-contraction (see [7]) if, there exists $0 \le k < 1$ such that

$$d(Tx, Ty) \le kd(fx, gy),$$

for any $x, y \in D$. If $k = 1$, then T is called (f, g)-nonexpansive.

Definition 2.2. Let X be a metric space. A mapping $T : X \to X$ is called weakly contractive (see [23]) if, for each $x, y \in X$,

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is lower semi continuous function from the right such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$.

We now introduce the notion of various weak contractions:

Definition 2.3. The mapping $T : X \to X$ is (f, g)-weakly contractive if, for each $x, y \in X$,

$$d(Tx, Ty) \le d(fx, gy) - \varphi(d(fx, gy)),$$

where $f, g : X \to X$ is a self mapping and and $\varphi : [0, \infty) \to [0, \infty)$ is lower semi continuous function from the right such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$.

If $\varphi(t) = (1 - k)t$, $0 < k < 1$, then a (f, g)-weakly contractive mapping is called a (f, g)-contraction. Note that if $f = g = I$ and φ is continuous non-decreasing, then the definition of (f, g)-weakly contractive mapping is the same as appeared in [23].

Let Φ denote the collection of all function φ from $\mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ satisfying the properties: (i) φ is continuous (ii) $\varphi(t) < t$ for all $t > 0$. Then :

Definition 2.4. A mapping T from a metric space (X, d) into itself is said to be asymptotic (f, g)-contraction if, for each integer $n \ge 1$, there exist a function $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that $d(T^n x, T^n y) \le \varphi_n(d(fx, gy))$ for all $x, y \in X$ and if $\varphi_n \to \varphi \in \Phi$ uniformly on the range of d. If $f = g = I$, then T is called asymptotic contraction, see [15].

Definition 2.5. A continuous mapping T from a metric space (X, d) into itself is said to be weak asymptotic (f, g)-contraction if for an arbitrary $\epsilon > 0$, there is an integer $n_\epsilon \ge 1$ such that as $\epsilon \to 0$, $n_\epsilon \to \infty$ and $d(T^{n_\epsilon} x, T^{n_\epsilon} y) \le$

$\varphi(d(fx, gy)) + \epsilon$ for all $x, y \in X$, where $\varphi \in \Phi$ is given (i.e. independent of ϵ).

It is easily seen that an asymptotic contraction in the sense of Definition 2.4 is, due to the requirement that $\varphi_n \to \varphi \in \Phi$ uniformly on the range of d, a weakly asymptotic contraction in the sense of Definition 2.5. If $f = g = I$, then T is called weak asymptotic contraction, see [31].

The map T is called uniformly asymptotically regular on D (see [30]) if for each $\eta > 0$, there exists $N(\eta) = N$ such that $d(T^n x, T^{n+1}x) < \eta$ for all integer $\eta \geq N$ and all $x \in D$. The set of fixed points of T (resp. I) is denoted by $F(T)$(resp. $F(I)$). A point $x \in D$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ ($x = fx = Tx$). The set of coincidence points of f and T is denoted by $C(f, T)$. The pair f, T is called commuting if $Tfx = fTx$ for all $x \in D$. The map $T : D \to X$ is said to be demiclosd at 0 if, for every sequence x_n in D converging weakly to x and Tx_n converges to $0 \in X$, then $0 = Tx$. The set $P_D(u) = \{x \in D : d(x, u) = \text{dist}\,(u, D)\}$ is called the set of best approximants to $u \in X$ out of D, where $\text{dist}\,(u, D) = \inf\{d(y, u) : y \in D\}$. Suppose A, G are bounded subsets of X, then we write

$$r_G(A) = \inf_{g \in G} \sup_{a \in A} d(a, g),$$

$$cent_G(A) = \{g_0 \in G : \sup_{a \in A} d(a, g_0) = r_G(A)\}.$$

The number $r_G(A)$ is called the *Chebyshev radius* of A w.r.t. G and an element $y_0 \in cent_G(A)$ is called a *best simultaneous approximation* of A w.r.t. G. If $A = \{u\}$, then $r_G(A) = d(u, G)$ and $cent_G(A)$ is the set of all best approximations, $P_G(u)$, of u out of G. We also refer the reader to Milman [17], Sahney and Singh [25] and Vijayaraju [30] for further details.

We next give results which generalized several known results including the recent result of Khan and Akbar [14].

3. Main Results

The following result is a particular case of Theorem 2.1 of Ciric, Hussain and Cakic [2].

Lemma 3.1. *Let D be a nonempty subset of a metric space $X := (X, d)$ and $T : D \to D$. If $cl(T(D)) \subseteq D$, $clT(D)$ is complete, and*

$$d(Tx, Ty) \leq m(x, y) - \varphi(m(x, y)), \tag{3.1}$$

for all $x, y \in D$, *where*

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

and $\varphi : [0, \infty) \to [0, \infty)$ *is real function such that:*

(i) $\varphi(t) > 0$, *for all* $t > 0$,

(ii) $\lim_{s \to t+} \varphi(t) > 0$, *for all* $t > 0$,

(iii) $t - \varphi(t)$ *is non-decreasing,*

(iv) $\lim_{t \to \infty} \varphi(t) = +\infty$.

Then $D \cap F(T)$ *is singleton.*

Theorem 3.2. *Let* D *be a nonempty subset of a metric space* X *and* T, f, g *be self-maps of* D. *If* $F(f) \cap F(g)$ *is nonempty,* $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, $clT(D)$ *is complete, and* T, f *and* g *satisfy the following condition:*

$$d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)), \tag{3.2}$$

for all $x, y \in D$, *where*

$$M(x, y) = \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(fx, Ty), d(Tx, gy)\},$$

and $\varphi : [0, \infty) \to [0, \infty)$ *is real function such that* φ *is lower semi continuous function from the right such that* φ *is positive on* $(0, \infty)$ *and* $\varphi(0) = 0$.

Then $D \cap F(T) \cap F(f) \cap F(g)$ *is singleton.*

Proof. $cl(F(f) \cap F(g))$ being subset of $clT(D)$ is complete. Moreover, for all $x, y \in F(f) \cap F(g)$, we have by inequality (3.1),

$$\begin{aligned} d(Tx, Ty) &\leq M(x, y) - \varphi(M(x, y)) \\ &= \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(Ty, fx), d(Tx, gy)\} \\ &\quad - \varphi(\max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(Ty, fx), d(Tx, gy)\}) \\ &= \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\} \\ &\quad - \varphi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\}) \\ &= m(x, y) - \varphi(m(x, y)) \end{aligned}$$

for all $x, y \in F(f) \cap F(g)$. It follows from Lemma 3.1 that $F(f) \cap F(g) \cap F(T) \cap D \neq \phi$. □

Corollary 3.3. *Let* D *be a nonempty subset of a metric space* X *and* T, f, g *be self-maps of* D. *If* $F(f) \cap F(g)$ *is nonempty,* $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, $clT(D)$ *is complete, and* T, f *and* g *satisfy the following condition:*

$$d(Tx, Ty) \leq \alpha(M(x, y))M(x, y), \tag{3.3}$$

for all $x, y \in D$, *where* $\alpha : [0, \infty) \to [0, \infty)$ *is real function such that* φ *is lower semi continuous function from the right such that* φ *is positive on* $(0, \infty)$ *and*

$\varphi(0) = 0$.

Then $D \cap F(T)$ is singleton.

Proof. Set $\varphi(t) = (1 - \alpha(t))t$, then inequality implies

$$d(Tx, Ty) \leq \alpha(M(x, y))M(x, y),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is real function such that φ is lower semi continuous function from the right such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$. The result follows from Theorem 3.2. □

Corollary 3.4. (see [14], Lemma 2.2) *Let D be a nonempty subset of a metric space X and T, f, g be self-maps of D. If $F(f) \cap F(g)$ is nonempty, $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, $clT(D)$ is complete, and T, f and g satisfy for all $x, y \in D$ and $0 \leq h < 1$,*

$$d(Tx, Ty) \leq h. \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(fx, Ty), d(Tx, gy)\}, \quad (3.4)$$

then $D \cap F(f) \cap F(g) \cap F(T)$ is singleton.

Proof. If $\varphi(t) = (1 - h)t$ for a constant h with $0 \leq h < 1$, the result follows from Theorem 3.2. □

Remark 1. In our Theorem 3.2, the assumptions of Lemma 2.1 of O'Regan and Shazad [18], that "D is closed, $TD \subseteq ID$, either I or T is continuous on D and (I, T) is R-weakly commuting pair on D" are replaced with "$F(f) \cap F(g)$ is nonempty and $cl(F(f) \cap F(g)) \subset F(f) \cap F(g)$." So our Theorem 2.2 is extension of their Lemma 2.1 and Theorem 3.2.

Theorem 3.5. *Let D be a nonempty subset of a normed (resp. Banach) space X and T, f and g be self-map of D. Suppose that $F(f) \cap F(g)$ is p-starshaped, $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ (resp, $wclT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$), $clT(D)$ is compact (resp, $wclT(M)$ is weakly compact), T is continuous on D (resp, $I - T$ is demiclosed at 0, where I stands for identity map) and*

$$\|Tx - Ty\| \leq M(x, y) - \varphi(M(x, y)), \quad (3.5)$$

for all $x, y \in D$, where

$$M(x, y) = \max\{\|fx - gy\|, \operatorname{dist}(fx, [p, Tx]), \operatorname{dist}(gy, [p, Ty]), \operatorname{dist}(gy, [p, Tx]), \operatorname{dist}(fx, [p, Ty])\}.$$

Then $D \cap F(T) \cap F(f) \cap F(g) \neq \phi$.

Proof. Let $\{k_n\}$ be a sequence in$(0, 1)$ such that $k_n \to 1$. For each $n \geq 1$, let $T_n x = (1 - k_n)p + k_n Tx$ for all $x \in F(f) \cap F(g)$. As $F(f) \cap F(g)$ is p-starshaped, $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, then $clT_n(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ for each $n \geq 1$. Notice that

$$\begin{aligned}\|T_n x - T_n y\| &\leq k_n \|Tx - Ty\| \leq k_n(M(x,y) - \varphi(M(x,y)))\\ &\leq \limsup_{n\to\infty} k_n(M(x,y) - \varphi(M(x,y)))\\ &\leq M(x,y) - \varphi(M(x,y)),\end{aligned}$$

where

$$\begin{aligned}M(x,y) = \max\{&\|fx - gy\|, \operatorname{dist}(fx,[p,Tx]), \operatorname{dist}(gy,[p,Ty]),\\ &\operatorname{dist}(gy,[p,Tx]), \operatorname{dist}(fx,[p,Ty])\}\\ \leq \max\{&\|fx-gy\|, \|fx - T_n x\|, \|gy - T_n y\|, \|gy - T_n x\|, \|fx - T_n y\|\},\end{aligned}$$

for all $x, y \in F(f) \cap F(g)$. Thus

$$\begin{aligned}\|T_n x - T_n y\| \leq \max\{&\|fx-gy\|, \|fx - T_n x\|, \|gy - T_n y\|,\\ &\|gy - T_n x\|, \|fx - T_n y\|\}\\ - \varphi(\max\{&\|fx-gy\|, \|fx - T_n x\|, \|gy - T_n y\|,\\ &\|gy - T_n x\|, \|fx - T_n y\|\}),\end{aligned}$$

for all $x, y \in F(f) \cap F(g)$.

As $clT(D)$ is compact, then $cl(T_n D)$ is compact for each $n \geq 1$. Thus $clT_n(D)$ is complete for each $n \geq 1$. It follows from Theorem 3.2 that for each $n \geq 1$, there exist $x_n \in F(f) \cap F(g)$ such that $F(f) \cap F(g) \cap F(T_n) = x_n$. Now, compactness of $cl(T(D))$ implies that there exists a subsequence Tx_{n_i} of Tx_n such that $Tx_{n_i} \to z \in clT(D)$ as $n_i \to \infty$. Since $\{Tx_{n_i}\}$ is a sequence in $T(F(f) \cap F(g))$, and $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, therefore $z \in F(f) \cap F(g)$. Moreover,

$$x_{n_i} = T_{n_i} x_{n_i} = (1 - k_{n_i})p + k_{n_i} T x_{n_i} \to z.$$

Letting $i \to \infty$ and using the continuity of T, we get $z = Tz$. Thus $D \cap F(T) \cap F(f) \cap F(g) \neq \phi$.

The weak compactness of $wcl(T(D))$ implies that $wcl(T_n(D))$ is weakly compact and hence complete due to completeness of X. From Theorem 3.2, for each $n \geq 1$, there exist $x_n \in F(f) \cap F(g)$ such that $x_n = fx_n = gx_n = T_n x_n$. From above analysis, we have $\|(x_n - Tx_n)\| \to 0$ as $n \to \infty$. The weak compactness of $wcl(T(D))$ implies that there is a subsequence Tx_m of Tx_n converging weakly to $y \in wclT(D)$ is a sequence in $(F(f) \cap F(g)$, therefore $y \in wcl(T((F(f) \cap F(g))) \subseteq (F(f) \cap F(g)$. Also we have $x_m - Tx_m \to \infty$. If $I - T$ is demiclosed at 0, then $y = Ty$. Thus $D \cap F(T) \cap F(f) \cap F(g) \neq \phi$. □

Remark 2. (1) In our Theorem 3.5, the assumptions of Theorem 2.2(i) of Hussain and Jungck [7] that, "M is complete, q-starshaped, f and g are affine and continuous on M, $T(M) \subset f(M) \cap g(M)$, $q \in f(M) \cap g(M)$ and (T, f) and (T, g) are R-subweakly commuting on M satisfying $\|T(x) - T(y)\| \leq \max\{\|fx - gy\|, \text{dist}\,(fx, [q, Tx]), \text{dist}\,(gy, [q, Ty]),$

$$\frac{1}{2}[\text{dist}\,(gy, [q, Tx]), \text{dist}\,(fx, [q, Ty])]\}\text{''} \tag{3.6}$$

are replaced with "M is nonempty subset, $F(f) \cap F(g)$ is q-starshaped, $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ and (T, f, g) satisfies (3.5) for all $x, y \in M$." (2). In our Theorem 3.5, the assumptions of Theorem 2.2(ii) of Hussain and Jungck [7] that, "M is weakly compact, q-starshaped, f and g are affine and continuous on M, $T(M) \subset f(M) \cap g(M)$, $q \in f(M) \cap g(M)$, $f - T$ is demiclosed at 0 and (T, f) and (T, g) are R-subweakly commuting on M satisfying (3.6) are replaced with "$wclT(M)$ is weakly compact, $F(f) \cap F(g)$ is q-starshaped, $wclT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, $id - T$ is demiclosed at 0 and (T, f, g) satisfies (3.5) for all $x, y \in M$." (3). In our Theorem 3.5 the assumptions of Theorem 3.2 of O'Regan and Shahzad that [19], "$q \in F(I)$, D is closed and q-starshaped, I and T are continuous on D, $T(D) \subseteq I(D)$, (I, T) is an R-subweakly commuting pair on D, and I is affine", are replaced with " $F(f) \cap F(g)$ is p-starshaped, $cl(F(f) \cap F(g)) \subseteq (F(f) \cap F(g)$ and T is continuous on D". (4). Theorem 3.5 properly extend and contains Theorem 3.3 of [1], Theorem 2.2 of [21], Theorem 2.3 of [14], result of Hicks and Humphries [6] Theorem 2.5 of O'Regan and Hussain [18].

We next show that a weakly asymptotic contraction on a nonempty set has a common fixed point provided it has a bounded orbit.

Theorem 3.6. *Let f, g, T be self-map of a subset D of a metric space (X, d). Assume that $F(f) \cap F(g)$ is p-starshaped, T is uniformly asypmptotically regular and weakly asypmptotic (f, g)-contraction on D. If $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g))$ (resp. $wclT(F(f) \cap F(g)) \subseteq F(f) \cap F(g))$), $clT(D)$ is compact (resp. $wcl(T(D))$ is weakly compact and $I - d$ is demiclosed at 0) then $F(T) \cap F(f) \cap F(g) \neq 0$.*

Proof. Let for an arbitrary $\epsilon > 0$, there is an integer $n_\epsilon \geq 1$ such that as $\epsilon \to 0$, $n_\epsilon \to \infty$ and $\{k_{n_\epsilon}\}_{n_\epsilon=1}^{\infty}$ be a real sequence with $0 \leq k_{n_\epsilon} < 1$ such that $\limsup_{n_\epsilon \to \infty} k_{n_\epsilon} = 1$. Define T_{n_ϵ} Self map on $F(f) \cap F(g)$ by

$$T_{n_\epsilon} x = (1 - k_{n_\epsilon})p - k_{n_\epsilon} T^{n_\epsilon} x,$$

for all $x \in F(f) \cap F(g)$ and $n_\epsilon \geq 1$. Since p is star-center of $F(f) \cap F(g)$ and $T(F(f) \cap F(g)) \in F(f) \cap F(g)$ it follows that T_{n_ϵ} maps $F(f) \cap F(g)$ to itself for each n_ϵ. Now applying condition (I), we obtain

$$d(T_{n_\epsilon} x, T_{n_\epsilon} y) \leq k_{n_\epsilon} d(T^{n_\epsilon} x, T^{n_\epsilon} y)$$

$$\leq k_{n_\epsilon}(\varphi(d(fx,gy))+\epsilon)$$
$$\leq \limsup_{n_\epsilon\to\infty} k_{n_\varepsilon}(\varphi(d(fx,gy))+\epsilon)=\varphi(d(fx,gy))+\epsilon.$$

for all $\varphi \in \Phi$ and, thereby, implying that T_{n_ϵ} is a weakly asymptotic contraction for each $n_\epsilon \geq 1$. As $F(f)\cap F(g)$ is p-starshaped and $clT(F(f)\cap F(g)) \subseteq F(f)\cap F(g)$ so $clT_{n_\epsilon}(F(f)\cap F(g)) \subseteq F(f)\cap F(g)$ for each $n_\epsilon \geq 1$. Notice that compactness of $clT(M)$ implies that $clT_{n_\epsilon}(M)$ is compact and hence complete. It follows, by Theorem 3.2, for each $n_{\epsilon\geq 1}$, there exist x_{n_ϵ} such that $x_{n_\epsilon} = f(x_{n_\epsilon}) = g(x_{n_\epsilon}) = T_{n_\epsilon}(x_{n_\epsilon})$. Now

$$\|x_{n_\epsilon}-Tx_{n_\epsilon}\| = \|x_{n_\epsilon}-T^{n_\epsilon}x_{n_\epsilon}\| + \|T^{n_\epsilon}x_{n_\epsilon}-T^{n_\epsilon+1}x_{n_\epsilon}\| + \|T^{n_\epsilon+1}x_{n_\epsilon}-Tx_{n_\epsilon}\|$$
$$\leq \|x_{n_\epsilon}-T^{n_\epsilon}x_{n_\epsilon}\| + \|T^{n_\epsilon}x_{n_\epsilon}-T^{n_\epsilon+1}x_{n_\epsilon}\| + k_1\|fT^{n_\epsilon}x_{n_\epsilon}-gx_{n_\epsilon}\|.$$

Since for each $n_\epsilon \geq 1$, $T^{n_\epsilon}(F(f)\cap F(g)) \subseteq F(f)\cap F(g)$ and $x_{n_\epsilon} \in F(f)\cap F(g)$, therefore $T^{n_\epsilon}x_{n_\epsilon} \in F(f)\cap F(g)$. Thus $fT^{n_\epsilon}x_{n_\epsilon} = T^{n_\epsilon}x_{n_\epsilon}$. As T is uniformly asymptotically regular, we have

$$d(x_{n_\epsilon},Tx_{n_\epsilon}) \leq d(x_{n_\epsilon},T^{n_\epsilon}x_{n_\epsilon}) + d(T^{n_\epsilon}x_{n_\epsilon},T^{n_\epsilon+1}x_{n_\epsilon}) + k_1 d(T^{n_\epsilon}x_{n_\epsilon},x_{n_\epsilon}) \to 0,$$

as $n_\epsilon \to \infty$. Thus $x_{n_\epsilon} - Tx_{n_\epsilon} \to 0$ as $n_\epsilon \to \infty$. As $clT(D)$ is compact, so there exist a subsequence $\{Tx_{m_\epsilon}\}$ of $\{Tx_{n_\epsilon}\}$ such that $Tx_{m_\epsilon} \to z \in clT(D)$ as $m \to \infty$. Since $\{Tx_{m_\epsilon}\}$ is a sequence in $T(F(f)\cap F(g))$ and $clT(F(f)\cap F(g)) \subseteq F(f)\cap F(g)$, therefore $z \in F(f)\cap F(g)$. Moreover,

$$d(Tx_{m_\epsilon},Tz) \leq k_1 d(fx_{m_\epsilon},gz) = k_1 d(x_{m_\epsilon},Tx_{m_\epsilon}) + k_1 d(Tx_{m_\epsilon},z).$$

Taking the limit as $m_\epsilon \to \infty$, we get $z = Tz$. Thus $D\cap F(T)\cap F(f)F(g) \neq \phi$.

The weak compactness of $wcl(T(D))$ implies that $wcl(T_{n_\epsilon}(D))$ is weakly compact and hence complete due to completeness of X. From Lemma 3.1, for each $n_\epsilon \geq 1$, there exist $x_{n_\epsilon} \in F(f)\cap F(g)$ such that $x_{n_\epsilon} = fx_{n_\epsilon} = gx_{n_\epsilon} = (1-k_{n_\epsilon})p - k_{n_\epsilon}T^{n_\epsilon}x$. From above analysis, we have $d(x_{n_\epsilon},Tx_{n_\epsilon}) \to 0$ as $n_\epsilon \to \infty$. The weak compactness of $wcl(T(D))$ implies that there is a subsequence $Tx_{m_{\epsilon_i}}$ of Tx_{n_ϵ} converging weakly to $y \in wclT(D)$ is a sequence in $(F(f)\cap F(g)$, therefore $y \in wcl(T((F(f)\cap F(g))) \subseteq (F(f)\cap F(g)$. Also we have $x_{m_\epsilon} - Tx_{m_\epsilon} \to \infty$. If $I-T$ is demiclosed at 0, then $y = Ty$. Thus $D\cap F(T)\cap F(f)\cap F(g) \neq \phi$. □

Remark 3. By weaken the notion of asymptotic contraction, our Theorem 3.6 is a generalization of Theorem 2.10 of Khan and Akbar [14].

Corollary 3.7. *Let (X,d) be a metric space and T,f,g be self-maps of X. Suppose that $y_1,y_2 \in X$, $D \subseteq cent_K(y_1,y_2)$, where $cent_K(A)$ is the set of best simultaneous approximations of A w.r.t. K. Assume that $D_0 := D\cap F(f)\cap F(g)$ is p-starshaped, $cl(T(D_0) \subseteq D_0$ (resp. $wcl(T(D_0) \subseteq D_0$), $clT(D)$*

is compact (resp. $wclT(D)$ is weakly compact and $I-d$ is demiclosed at 0), T is uniformaly asymptotic regular and weak asymptotic (f,g)-contraction on D, then $cent_K(y_1,y_2)\cap F(T)\cap F(f)\cap F(g)\neq\phi$.

Remark 4. (1) In second case of our Corollary 3.7, the assumptions of Theorem 4.1 of Chen and Li that "$p\in F(I,T)$, $F(I)$ is q-starshaped, $B_M(p)$ is q-starshaped and weakly compact, $cl(T(M))$ is complete, (I,T) is a Banach operator pair on $B_M(p)$, I is weakly or strongly continuous on $B_M(p)$, T is I-nonexpansive on $B_M(p)\cup p$, and $I(B_M(p))\subseteq B_M(p)$" are replaced with "$y_1,y_2\in X$, $D\subseteq cent_K(y_1,y_2)$, $D\cap F(f)\cap F(g)$ is p-starshaped $wclT(F(f)\cap F(g))\subseteq F(f)\cap F(g)$, $wcl(T(D))$ is weakly compact, and T is weak asymptotic (f,g)-contraction and uniformly asymptoticly regular on D".

(2) In the first case of our Corollary 3.7, the assumptions of Theorem 4.2 of Chen and Li, "$p\in F(I,T)$, $D_M^I(p)\cap F(I)$ is q-starshaped, $clT(D_M^I(p))$ is compact, (I,T) is a Banach operator pair on $D_M^I(p)$, T is I-nonexpansive on $D_M^I(p)\cup p$ and I is continuous on $cl(T(D_M^I(p)))$" are replaced with "$y_1,y_2\in X$, $D\subseteq cent_K(y_1,y_2)$, $D_0:=D\cap F(f)\cap F(g)$ is p-starshaped $clT(D_0)\subseteq D_0$, T is weak asymptotic (f,g)-contraction and uniformly asymptoticly regular on D and $cl(T(D))$ is compact".

(3) Corollary 3.7 is extension of Theorems 2.2 and 2.7 of Khan and Akbar [13], Corollary 2.13 of Khan and Akbar [14] and Theorem 2.3 of Vijayaraju [30] in convex structure.

(4) In the first case of our Corollary 3.7, the assumptions of Corollary 2.13 of Khan and Akbar [14], "T is asymptotically (f,g) nonexpansive on D" is replaced by "weak asymptotic (f,g)-contraction on D".

Corollary 3.8. *Let K be a nonempty subset of a metric space (X,d) and $y_1,y_2\in X$. Suppose that T and f are self-maps of K such that T is weak asymptotic f-contraction. Suppose that the set $F(f)$ is nonempty. Let the set D, of best simultaneous K-approximants to y_1 and y_2, is nonempty, compact and starshaped with respect to an element q in $F(f)$ and D is invariant under T and f. Assume further that T and f are commuting, T is uniformly asymptotically regular on D and f is an affine continuous mapping on D with $f(D)=D$. Then D contain a T- and f-invariant point.*

Proof. As f is continuous and D is closed, therefore $F(f)$ is closed. Using the commutativity of T with f, we obtain $T(F(f))\subseteq F(f)$. Thus $clT(F(f))\subseteq cl(F(f))=F(f)$. Since f is affine and $p\in F(f)$, so $F(f)$ is p-starshaped. The desired conclusion follows now from Theorem 3.6 above. □

The following example shows that the contractive condition (3.2) is substantially more general than the condition (3.4), even if (X, d) is compact and convex Euclidean space.

Example 3.9. Let $K = [0, \frac{1}{2}]$ be a closed interval with usual metric and let $f, g, T : K \to K$ and $\varphi : [0, +\infty) \to [0, +\infty)$ be mappings defined as follows:

$$\begin{aligned} f(x) &= x^2, &&\text{for all} \quad 0 \le x \le \frac{1}{2}, \\ g(x) &= 2x^2, &&\text{for all} \quad 0 \le x \le \frac{1}{2}, \\ T(x) &= x^2 - x^4, &&\text{for all} \quad 0 \le x \le \frac{1}{2}, \\ \varphi(t) &= \frac{1}{2}t^2, &&\text{for all} \quad 0 \le t \le \frac{1}{2}, \\ \varphi(t) &= \frac{1}{4}t, &&\text{for all} \quad t > \frac{1}{2}. \end{aligned}$$

Let $x, y \in K$ be arbitrary. Without loss of generality we may suppose that $x \le y$. Then we have

$$\begin{aligned} M(x,y) &\max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(fx, Ty), d(Tx, gy)\} \\ &= d(g(y), T(x)), d(g(y), T(x)) = 2y^2 - x^2(1 - x^2). \end{aligned}$$

Since $2y^2 \ge 2y^2 - x^2(1 - x^2)$ for all $x \in [0, \frac{1}{2}]$, it follows that

$$-4y^4 \le -(2y^2 - x^2(1 - x^2))^2.$$

Thus we have

$$\begin{aligned} d(T(x), T(y)) &= y^2 - y^4 - x^2 + x^4 = (y^2 - x^2(1 - x^2)) - y^4 \\ &\le (2y^2 - x^2(1 - x^2)) - 2y^4 \le (2y^2 - x^2(1 - x^2)) - \frac{1}{2}(4y^4) \\ &\le (2y^2 - x^2(1 - x^2)) - \frac{1}{2}(2y^2 - x^2(1 - x^2))^2 \\ &\le M(x,y) - \varphi(M(x,y)). \end{aligned}$$

Therefore, f, g and T satisfy (3.2). Also it is easy to see that the mapping $\varphi(t)$ satisfies all hypotheses (i)-(iv) in Theorem 3.1. Thus we can apply our Theorem 3.1 and Theorem 3.2. On the other hand, for any fixed k; $0 < k < 1$, we have, for $x = 0$ and each $y \in X$ with $0 < y < \sqrt{1-k}$,

$$d(T(0), Y(y)) = y^2 - y^4 = (1 - y^2)y^2 > k.y^2 = k_1.d(g(y), T(0)) = k_1.M(0, y).$$

Thus, T does not satisfy (3.4). Therefore, the theorems of Khan and Akbar [14], Jungck and Hussain [11], Das and Naik [3] as well as the theorem of Al-Thagafi [1] cannot be applied.

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