International Journal of Pure and Applied Mathematics
Volume 62 No. 3 2010, 305-326

GLOBAL EXPONENTIAL STABILITY AND EXISTENCE OF PERIODIC SOLUTION AND ANTI-PERIODIC SOLUTION FOR DELAYED COHEN-GROSSBERG BAM NEURAL NETWORKS WITH IMPULSE ON TIME SCALES

Lili Zhao[1 §], Ping Liu[2]
[1,2]Department of Mathematics
Yunnan University
Kunming, Yunnan, 650091, P.R. CHINA
[1]e-mail: llzhao@ynu.edu.cn
[2]e-mail: liuping@ynu.edu.cn

Abstract: Recently, many authors have studied the existence and global exponential stability of periodic solution and anti-periodic solution of many kinds of neural networks on time scales, by using the continuation theorem of coincidence degree theory, $M-$matrix theory and constructing some suitable Lyapunov functions. But, in this work, we only use the continuation theorem of coincidence degree theory, $M-$matrix theory to study the existence and exponential stability of periodic solution and anti-periodic solutions of a class of higher-order Cohen-Grossberg type neural networks with distributed delays and impulse on time scales. The activation functions f_j, g_i, are not assumed to be bounded in this work. Finally, an example is given to illustrate the effectiveness of our main results.

AMS Subject Classification: 26A33
Key Words: Cohen-Grossberg BAM neural networks, exponential stability, periodic solution, anti-periodic solutions, impulse, time scales

1. Introduction

Arising from problems in applied sciences, such as pattern recognitions, signal

Received: June 21, 2010

§Correspondence author

processing, optimization and associative, it is well known that Cohen-Grossberg BAM neural networks have been extensively studied by many authors during the past ten yeas, see [11], [1], [20], [9] and references therein. Many results for the existence of their periodic solutions, almost periodic solutions, and the exponential convergence properties for Cohen-Grossberg neural networks have been reported in the literatures. See for instance, see [6], [10], [12]. However, very few results are available on the existence and exponential stability of anti-periodic solutions for neural networks, while the existence of anti-periodic solutions plays an important role in characterizing the behavior of nonlinear differential equations (see [5], [18], [16], [15]).

The theory of calculus on time scales (see [4], [2] and references cited therein) was initiated by Stefan Hilger in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis, and it has a tremendous potential for application and has recently received much attention since his foundational work. In fact, both continuous and discrete systems are very important in implementing and applications. Therefore, it is meaningful to study dynamic system on time scales which can unify the differential and difference system. To my best knowledge, there are few papers applying the method of coincidence degree to investigate the existence and exponential stability of anti-periodic solutions to impulsive Cohen-Grossberg BAM neural networks with distributed delays on time scales in the form of

$$\begin{cases} x_i^{\Delta}(t) &= -a_i(x_i(t))\Big[\alpha_i(x_i(t)) - \sum\limits_{j=1}^{m} p_{ji}(t) f_j(y_j(t-\tau_{ji})) + c_i(t)\Big] \\ & \quad i=1,2,\ldots,n,\ \ t\in(0,\infty)_{\mathbb{T}},\ \ t\neq t_k, \\ \Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = I_{ik}(x_i(t_k)),\quad i=1,2,\ldots,n,\ k\in\mathbb{N} \\ y_j^{\Delta}(t) &= -b_j(y_j(t))\Big[\beta_j(y_j(t)) - \sum\limits_{i=1}^{n} q_{ij}(t) g_i(x_i(t-\sigma_{ij})) + d_j(t)\Big] \\ & \quad j=1,2,\ldots,m,\ \ t\in(0,\infty)_{\mathbb{T}},\ \ t\neq t_k, \\ \Delta y_j(t_k) &= y_j(t_k^+) - y_j(t_k^-) = J_{jk}(y_j(t_k)),\quad j=1,2,\ldots,m,\ k\in\mathbb{N}, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is an $\frac{\omega}{2}$-periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. $x_i(t)$, $y_j(t)$ are the activation of i-th neuron in neural field $F_{\mathbb{X}}$ and the j-th neuron in neural field $F_{\mathbb{Y}}$. For each interval L of $\mathbb{R}$ we denote by $L_{\mathbb{T}} = L\cap\mathbb{T}$, $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, $\Delta y_j(t_k) = y_j(t_k^+) - y_j(t_k^-)$, $x_i(t_k^+)$, $y_j(t_k^+)$, $x_i(t_k^-)$, $y_j(t_k^-)(i=1,2,\ldots,n,\ j=1,2,\ldots,m)$, represent the right and left limit of $x_i(t_k)$, $y_j(t_k)$ in the sense of time scales. $\{t_l\}$ is a sequence of real numbers such that $0 < t_1 < t_2 < \ldots < t_l \to \infty$ as $l\to\infty$. There exists a positive integer q such that $t_{l+q} = t_l + \frac{\omega}{2}$, $I_{i(k+q)} = -I_{ik}$, $J_{j(k+q)} = -J_{jk}$, $l\in\mathbb{Z}$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$. Without loss of generality, we also assume that $[0,\frac{\omega}{2})_{\mathbb{T}} \cap \{t_l : l\in\mathbb{Z}\} = \{t_1,t_2,\ldots,t_q\}$. The delays satisfy:

$0 \le \tau_{ji} \le \tau$, $0 \le \sigma_{ij} \le \sigma$, for all $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, $k \in \mathbb{N}$. The system (1.1) is supplemented with initial values given by

$$\begin{cases} x_i(s) &= \phi_i(s), \quad s \in (-\infty, 0]_{\mathbb{T}}, \\ y_j(s) &= \psi_j(s), \quad s \in (-\infty, 0]_{\mathbb{T}}, \end{cases} \tag{1.2}$$

where $\phi_i(\cdot)$, $\psi_j(\cdot)$ denote continuous function defined on $(-\infty, 0]_{\mathbb{T}}$.

Throughout this paper, we assume that:

(H_1) $\tau_{ji}, \sigma_{ij} \ge 0, p_{ji}, q_{ij} \in C(\mathbb{T}, \mathbb{R})$ are $\frac{\omega}{2}$-periodic functions, and $c_i, d_j \in C(\mathbb{T}, \mathbb{R})$ are $\frac{\omega}{2}$-anti-periodic functions, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

(H_2) The activation functions $f = (f_1, \dots, f_m)^T$ and $g = (g_1, \dots, g_n)^T$ are odd functions on $\mathbb{R}$ and are Lipschitz functions, that is, there exist positive numbers l_j^f, l_i^g such that $|f_j(x) - f_j(y)| \le l_j^f |x - y|$, $|g_i(x) - g_i(y)| \le l_i^g |x - y|$ for $x, y \in \mathbb{R}$, $i = 1, \dots, n$, $j = 1, 2, \dots, m$.

(H_3) a_i, $b_j \in C(\mathbb{R}, \mathbb{R}^+)$, $a_i(-u) = a_i(u)$, $b_j(-u) = b_j(u)$ and there exist positive numbers $\underline{a}_i$, $\bar{a}_i$, $\bar{b}_j$, $\underline{b}_j$ such that $\underline{a}_i \le a_i(u) \le \bar{a}_i$, $\underline{b}_j \le b_j(u) \le \bar{b}_j$ for all $u \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

(H_4) α_i, $\beta_j \in C^1(\mathbb{R}, \mathbb{R})$, $\alpha_i(u) = -\alpha_i(-u)$, $\beta_j(u) = -\beta_j(-u)$ and there exist δ_i, δ_i^m, η_j, $\eta_j^m > 0$ such that $0 < \delta_i^m \le \frac{d\alpha_i(t)}{dt} \le \delta_i$, $0 < \eta_j^m \le \frac{d\beta_j(t)}{dt} \le \eta_j$, for all $i = 1, \dots, n$, $j = 1, 2, \dots, m$.

(H_5) I_{ik}, $J_{jk} \in C(\mathbb{R}, \mathbb{R})$ and there exist positive numbers I_{ik}^M, J_{jk}^M such that $|I_{ik}(x) - I_{ik}(y)| \le I_{ik}^M |x - y|$, $|J_{jk}(x) - J_{jk}(y)| \le J_{jk}^M |x - y|$, $x, y \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, $k \in \mathbb{N}$.

For the sake of convenience, we denote

$$\bar{h} = \max_{t \in [0, \omega]_{\mathbb{T}}} |h(t)|, \qquad \|h\|_2 = \left(\int_0^\omega |h(t)|^2 \Delta t \right)^{\frac{1}{2}},$$

where h is an ω-periodic function.

2. Preliminaries

In this section, we will cite some definitions and lemmas which will be used in the proofs of our main results.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Let $\omega \in \mathbb{R}$, $\omega > 0$, $\mathbb{T}$ is an ω-periodic time scale if $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ such that $t + \omega \in \mathbb{T}$ and $\mu(t) = \mu(t + \omega)$ whenever $t \in \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T})$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\big|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]\big| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If y is continuous, then y is right-dense continuous, and y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous. If $y^{\Delta}(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a).$$

Definition 2.1. Function $f{=}(f_1, \dots, f_n)$ is a Lipschitz if it satisfies $|f_i(x) - f_i(y)| \le l_i|x - y|$, $i = 1, \dots, n$ for any $x, y \in \mathbb{R}$.

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t.

Definition 2.2. (see [8]) If $a \in \mathbb{T}, \sup \mathbb{T} = \infty$, and f is rd-continuous on $[a, \infty)$, then we define the improper integral by

$$\int_a^{\infty} f(t)\Delta t = \lim_{b \to \infty} \int_a^b f(t)\Delta t$$

providing this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

Definition 2.3. (see [7]) For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then for $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$, define $D^+V^{\Delta}(t, x(t))$ to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N_\varepsilon \subset N$ of t such that

$$\frac{[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t,s) f(t, x(t))]}{\mu(t,s)} < D^+V^{\Delta}(t, x(t)) + \varepsilon$$

for each $s \in N_\varepsilon$, $s > t$, where $\mu(t,s) \equiv \sigma(t) - s$. If t is rd and $V(t, x(t))$ is continuous at t, this reduces to

$$D^+V^{\Delta}(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(\sigma(t)))}{\sigma(t) - t}.$$

Definition 2.4. (see [3]) We say that a time scale $\mathbb{T}$ is a periodic if there exists $p > 0$, such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale.

Definition 2.5. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is $\frac{\omega}{2}$-anti-periodic if there exists a natural number n such that $\frac{\omega}{2} = np$, $f(t+\frac{\omega}{2}) = -f(t)$ for all $t \in \mathbb{T}$ and $\frac{\omega}{2}$ is the smallest number such that $f(t + \frac{\omega}{2}) = -f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is $\frac{\omega}{2}$-anti-periodic if $\frac{\omega}{2}$ is the smallest positive number such that $f(t + \frac{\omega}{2}) = -f(t)$ for all $t \in \mathbb{T}$.

A function $r : \mathbb{T} \to R$ is called regressive if

$$1 + \mu(t) r(t) \neq 0$$

for all $t \in \mathbb{T}^k$.

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau)) \Delta\tau \right\}, \quad \text{for } s, t \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu pq, \quad p \ominus q := p \oplus (\ominus q), \quad \ominus p := \frac{p}{1 + \mu p}.$$

Then the generalized exponential function has the following properties.

Lemma 2.1. *Assume that $p,\ q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then:*

(*i*) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(*ii*) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t,s)$.

Lemma 2.2. (see [4]) *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$. Then*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

Lemma 2.3. *If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in C(\mathbb{T}, \mathbb{R})$, then:*

(i) $\displaystyle\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t;$

(ii) if $f(t) \geq 0$, for all $a \leq t < b$, then $\displaystyle\int_a^b f(t)\Delta t \geq 0;$

(iii) if $|f(t)| \leq g(t)$ on $[a, b) = \{t \in \mathbb{T} : a \leq t < b\}$, then $\displaystyle|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t.$

The proofs of the following lemmas can be found in [17], [19], [13], respectively.

Lemma 2.4. (see [17]) *Let $t_1, t_2 \in [0, \omega]_{\mathbb{T}}$. If $x : \mathbb{T} \to \mathbb{R}$ is ω-periodic, then*

$$x(t) \leq x(t_1) + \int_0^{\omega} |x^{\Delta}(s)|\Delta s, \quad x(t) \geq x(t_2) - \int_0^{\omega} |x^{\Delta}(s)|\Delta s.$$

Lemma 2.5. (Cauchy-Schwarz Inequality on Time Scale, see [19]) *Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left(\int_a^b |f(t)|^2\Delta t\right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2\Delta t\right)^{\frac{1}{2}}.$$

Lemma 2.6. (see [13]) *Assume that $\{f_n\}_{n\in N}$ is a function sequence on J such that*

(i) $\{f_n\}_{n\in N}$ is uniformly bounded on J;

(ii) $\{f_n^{\Delta}\}_{n\in N}$ is uniformly bounded on J.

Then there is a subsequence of $\{f_n\}_{n\in N}$ converges uniformly on J.

Lemma 2.7. *Let $\mathbb{T}$ be a ω-periodic time scale, then $\sigma(t + \omega) = \sigma(t) + \omega$, for all $t \in \mathbb{T}$.*

Proof. By using the definition of forward jump operator, we have $\sigma(t)+\omega \geq t+\omega$, then $\sigma(t)+\omega \geq \sigma(t+\omega)$. Now we claim that $\sigma(t)+\omega = \sigma(t+\omega)$. If it is not true, we assume that $\sigma(t+\omega) = t_1^* < \sigma(t)+\omega$. From the definition of infima (inf), we know that there exist a $t_2^* \in \mathbb{T}$, $t_2^* > t+\omega$, such that

$$t_2^* < t_1^* + \frac{\sigma(t)+\omega-t_1^*}{2} = \frac{\sigma(t)+\omega+t_1^*}{2} < \sigma(t)+\omega. \tag{2.1}$$

From (2.1), we obtain $t_2^*-\omega < \sigma(t)$, on the other hand, since $t_2^* > t+\omega$, $t_2^*-\omega \geq \sigma(t)$, which is a contradiction. The proof of Lemma 2.7 is complete. □

From Lemma 2.7, we obtain the following lemma.

Lemma 2.8. *Let $\mathbb{T}$ be a ω-periodic time scale, then $\mu(t)$ is a ω-periodic function.*

Proof.

$$\mu(t+\omega) = \sigma(t+\omega) - t - \omega = \sigma(t)+\omega-t-\omega = \sigma(t)-t = \mu(t).$$ □

Form Lemma 2.8, we know that if $\theta, \tau \in \mathbb{T}$ are constants, then $e_\theta(t,-\tau)$ is a ω-periodic function.

Definition 2.6. The anti-periodic solution $z^*(t)$ of system (1.1) is said to be exponentially stable if there exist a positive constant α such that for every $\delta \in \mathbb{T}$, there exists $N = N(\delta) \geq 1$ such that the solution of (1.1) through $(\delta, z(\delta))$ satisfies

$$\sum_{i=1}^{n} |x_i(t)-x_i^*(t)| + \sum_{j=1}^{m} |y_j(t)-y_j^*(t)| \leq N\Big[\|\phi-x^*\| + \|\psi-y^*\|\Big] e_{\Theta a}(t,\delta), \quad t \in \mathbb{T}^+,$$

where

$$\|\phi-x^*\| = \sum_{i=1}^{n} \big[\sup_{\delta\in(-\infty,0]_{\mathbb{T}}} |\phi_i(\delta)-x_i^*(\delta)|\big],\ \|\psi-y^*\| = \sum_{j=1}^{m} \big[\sup_{\delta\in(-\infty,0]_{\mathbb{T}}} |\psi_j(\delta)-y_j^*(\delta)|\big].$$

The following fixed point theorem of coincidence degree is crucial in the arguments of our main results.

Lemma 2.9. (see [14]) *Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces, $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \overline{\Omega} \neq \emptyset$ and $N : \overline{\Omega} :\to \mathbb{Y}$ is L-compact. Further, we assume that*

$(H6)$ *$Lx - Nx \neq \lambda(-Lx - N(-x))$ for all $D(L) \cap \partial\Omega$, $\lambda \in (0, 1]$.*

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \partial\Omega$.

Definition 2.7. *A* real $n \times n$ matrix $A = (a_{ij})$ is said to be a non-singular M-matrix if $a_{ij} \leq 0, i \neq j,\ i, j = 1, \ldots, n$ and all successive principal minors of A are positive.

3. Existence of Periodic Solution and Anti-Periodic Solution

In this section, we will study the existence of at least one anti-periodic solution of (1.1).

Theorem 3.1. *Assume that the assumptions (H_1)-(H_6) are satisfied and*

$$E_{(n+m)\times(n+m)} = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix},\ (E_1)_{n\times n} = \mathrm{diag}(o_1, o_2, \ldots, o_n),$$

$$E_2 = (e_{ij})_{n\times m},\ E_3 = (e^{,}_{ji})_{m\times n},\ (E_4)_{m\times m} = \mathrm{diag}(\vartheta_1, \vartheta_2, \ldots, \vartheta_m),$$

is a nonsingular M-matrix, where

$$o_i = 1 - \bar{a}_i\delta_i\omega - \sum_{k=1}^{2q} I_{ik}^M,\ \ \vartheta_j = 1 - \bar{b}_j\eta_j\omega - \sum_{k=1}^{2q} J_{jk}^M,\ \ e_{ij} = \bar{a}_i\omega\bar{p}_{ji}l_j^f,\ \ e^{,}_{ji} = \bar{b}_j\omega\bar{q}_{ij}l_i^g,$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Then system (1.1) *has at least one $\frac{\omega}{2}$-anti-periodic solution.*

Proof. Let

$$C^\kappa[0, \omega; t_1, t_2, \ldots, t_q, t_{q+1}, \ldots, t_{2q}]_{\mathbb{T}} = \Big\{ z : [0, \omega]_{\mathbb{T}} \to \mathbb{R}^{n+m} | z^\kappa(t)$$
$$\text{is a piecewise continuous map with first-class points in}$$
$$[0, \omega]_{\mathbb{T}} \cap \{t_k\}, \text{and at each discontinuity point it is continuous on the left,}$$
$$\kappa = 0, 1 \Big\}.$$

Let

$$\mathbb{X} = \Big\{ z \in C[0, \omega; t_1, t_2, \ldots, t_q, t_{q+1}, \ldots, t_{2q}]_{\mathbb{T}} : z(t + \frac{\omega}{2}) = -z(t),\ t \in [0, \frac{\omega}{2}]_{\mathbb{T}} \Big\}$$

and

$$\mathbb{Y} = \mathbb{X} \times \mathbb{R}^{(n+m)\times q}$$

be two Banach spaces with the norms

$$||z||_{\mathbb{X}} = \sum_{i=1}^{n} |x_i|_0 + \sum_{j=1}^{m} |y_j|_0, \quad ||u||_{\mathbb{Y}} = ||z||_{\mathbb{X}} + ||v||, \quad z \in \mathbb{X},\ v \in \mathbb{R}^{(n+m)\times q},$$

in which

$$|x_i|_0 = \max_{t\in[0,\omega]_{\mathbb{T}}} |x_i(t)|,\ i = 1, 2, \dots, n,$$

$$|y_j|_0 = \max_{t\in[0,\omega]_{\mathbb{T}}} |y_j(t)|,\ j = 1, 2, \dots, m,$$

$||.||$ is any norm of $\mathbb{R}^{(n+m)\times q}$. Set

$$L : \operatorname{Dom} L \cap \mathbb{X} \to \mathbb{Y},\ z \to (z^{\Delta}, \Delta z(t_1), \dots, \Delta z(t_q)).$$

Here $\operatorname{Dom} L = \Big\{ x \in C^1[0, \omega; t_1, t_2, \dots, t_{2q}]_{\mathbb{T}} : z(t + \frac{\omega}{2}) = -z(t),\ t \in [0, \frac{\omega}{2}]_{\mathbb{T}} \Big\}$, and $N : \mathbb{X} \to \mathbb{Y}$,

$$Nz = \left(\begin{pmatrix} A_1(t) \\ \vdots \\ A_n(t) \\ A_{n+1}(t) \\ \vdots \\ A_{n+m}(t) \end{pmatrix}, \begin{pmatrix} \Delta x_1(t_1) \\ \vdots \\ \Delta x_n(t_1) \\ \Delta y_1(t_1) \\ \vdots \\ \Delta y_m(t_1) \end{pmatrix}, \begin{pmatrix} \Delta x_1(t_2) \\ \vdots \\ \Delta x_n(t_2) \\ \Delta y_1(t_2) \\ \vdots \\ \Delta y_m(t_2) \end{pmatrix}, \dots, \begin{pmatrix} \Delta x_1(t_q) \\ \vdots \\ \Delta x_n(t_q) \\ \vdots \\ \Delta y_1(t_q) \\ \vdots \\ \Delta y_m(t_q) \end{pmatrix} \right),$$

where

$$A_i(t) = -a_i(x_i(t)) \Big[\alpha_i(x_i(t)) - \sum_{j=1}^{m} p_{ji}(t) f_j(y_j(t - \tau_{ji})) + c_i(t) \Big], \qquad i = 1, 2, \dots, n.$$

$$A_{n+j}(t) = -b_j(y_j(t)) \Big[\beta_j(y_j(t)) - \sum_{i=1}^{n} q_{ij}(t) g_i(x_i(t - \sigma_{ij})) + d_j(t) \Big], \qquad j = 1, 2, \dots, m.$$

Obviously,

$$\operatorname{Ker} L = \Big\{ z \in \mathbb{X} \Big| z = 0 \Big\},$$

$$\operatorname{Im} L = \Big\{ u = (f, C_1, C_2, \dots, C_q) \in \mathbb{Y} \Big| \int_0^{\omega} f(s)\Delta s = 0 \Big\} \equiv \mathbb{Y}.$$

and then

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 0.$$

So, $\operatorname{Im} L$ is closed in $\mathbb{Y}$, L is a Fredholm mapping of index zero. Define the project operators P and Q as

$$Pz = \frac{1}{\omega}\int_0^{\omega} x(t)\Delta t = 0, \quad z \in \mathbb{X},$$

$$Qu = Q(f, C_1, C_2, \ldots, C_q) = \left(\frac{1}{\omega}\int_0^{\omega} f(s)\Delta s, 0, \ldots, 0\right), \quad u \in \mathbb{Y}.$$

It is not difficult to show that P and Q are continuous projectors and satisfy

$$\operatorname{Im}\, P = \operatorname{Ker} L,\ \operatorname{Im}\, L = \operatorname{Ker}\, Q = \operatorname{Im}\, (I - Q).$$

Further, let $L_P^{-1} = L|_{\mathrm{Dom} \cap \operatorname{Ker} P}$ and the generalized inverse $K_P = L_P^{-1}$ is given by

$$(K_P u)(t) = \int_0^t f(s)\Delta s + \sum_{t>t_k} C_k - \frac{1}{2}\int^{\frac{\omega}{2}} f(s)\Delta s - \frac{1}{2}\sum_{k=1}^{q} C_k,$$

in which $C_{q+i} = -C_i$ for all $1 \le i \le q$. Thus, the expression of QNz is

$$\left(\begin{pmatrix} \frac{1}{\omega}\int_0^{\omega} A_1(t)\Delta t \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega} A_n(t)\Delta t \\ \frac{1}{\omega}\int_0^{\omega} A_{n+1}(t)\Delta t \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega} A_{n+m}(t)\Delta t \end{pmatrix}, 0, \ldots, 0\right),$$

and then

$$K_P(I-Q)Nz = \begin{pmatrix} \int_0^t A_1(s)\Delta s + \sum\limits_{t>t_k} I_{1k}(x_1(t_k)) \\ \vdots \\ \int_0^t A_n(s)\Delta s + \sum\limits_{t>t_k} I_{nk}(x_n(t_k)) \\ \int_0^t A_{n+1}(s)\Delta s + \sum\limits_{t>t_k} J_{1k}(y_1(t_k)) \\ \vdots \\ \int_0^t A_{n+m}(s)\Delta s + \sum\limits_{t>t_k} J_{mk}(y_m(t_k)) \end{pmatrix} - \begin{pmatrix} \frac{1}{2}\int_0^{\frac{\omega}{2}} A_1(s)\Delta s \\ \vdots \\ \frac{1}{2}\int_0^{\frac{\omega}{2}} A_n(s)\Delta s \\ \frac{1}{2}\int_0^{\frac{\omega}{2}} A_{n+1}(s)\Delta s \\ \vdots \\ \frac{1}{2}\int_0^{\frac{\omega}{2}} A_{n+m}(s)\Delta s \end{pmatrix}$$

$$
- \begin{pmatrix} \frac{1}{2}\sum\limits_{k=1}^{q} I_{1k}(x_1(t_k)) \\ \vdots \\ \frac{1}{2}\sum\limits_{k=1}^{q} I_{nk}(x_n(t_k)) \\ \frac{1}{2}\sum\limits_{k=1}^{q} J_{1k}(y_1(t_k)) \\ \vdots \\ \frac{1}{2}\sum\limits_{k=1}^{q} J_{mk}(y_m(t_k)) \end{pmatrix}.
$$

Clearly, QN and $K_P(I-Q)N$ are both continuous. Using Lemma 2.6, it is easy to show that $QN(\overline{\Omega})$, $K_P(I-Q)N(\overline{\Omega})$ are compact for any open bounded set $\Omega \subset \mathbb{X}$. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

In order to apply Lemma 2.9, we need to find an appropriate open bounded subset Ω in $\mathbb{X}$. Corresponding to the operator equation $Lz - Nz = \lambda(-Lz - N(-z))$, $\lambda \in (0,1]$, we have

$$
\begin{cases} x_i^{\Delta}(t) = \dfrac{1}{1+\lambda} B_i(t,z) - \dfrac{\lambda}{1+\lambda} B_i(t,-z),\ t \in (0,\infty)_{\mathbb{T}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \neq t_k,\ i = 1,2,\ldots,n \\ \Delta x_i(t_k) = \dfrac{1}{1+\lambda} I_{ik}(x_i(t_k)) - \dfrac{\lambda}{1+\lambda} I_{ik}(-x_i(t_k)), \quad i = 1,2,\ldots,n,\ k \in \mathbb{N} \\ y_j^{\Delta}(t) = \dfrac{1}{1+\lambda} B_{n+j}(t,z) - \dfrac{\lambda}{1+\lambda} B_{n+j}(t,-z),\ t \in (0,\infty)_{\mathbb{T}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \neq t_k,\ j = 1,2,\ldots,m \\ \Delta y_j(t_k) = \dfrac{1}{1+\lambda} J_{jk}(y_j(t_k)) - \dfrac{\lambda}{1+\lambda} J_{jk}(-y_j(t_k)), \quad j = 1,2,\ldots,m,\ k \in \mathbb{N}, \end{cases} \tag{3.1}
$$

where

$$
B_i(t,z) = -a_i(x_i(t))\Big[\alpha_i(x_i(t)) - \sum_{j=1}^{m} p_{ji}(t) f_j(y_j(t-\tau_{ji})) + c_i(t)\Big],\ i = 1,2,\ldots,n,
$$

$$
B_{n+j}(t,z) = -b_j(y_j(t))\Big[\beta_j(y_j(t)) - \sum_{i=1}^{n} q_{ij}(t) g_i(x_i(t-\sigma_{ij})) + d_j(t)\Big],
$$

$$
j = 1,2,\ldots,m,
$$

$$
B_i(t,-z) = -a_i(-x_i(t))\Big[\alpha_i(-x_i(t)) - \sum_{j=1}^{m} p_{ji}(t) f_j(-y_j(t-\tau_{ji})) + c_i(t)\Big],
$$

$$i = 1, 2, \ldots, n,$$

$$B_{n+j}(t,-z) = -b_j(-y_j(t))\Big[\beta_j(-y_j(t)) - \sum_{i=1}^{n} q_{ij}(t)g_i(-x_i(t-\sigma_{ij})) + d_j(t)\Big],$$

$$j = 1, 2, \ldots, m.$$

Suppose that $z = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)^T$ is a solution of system (3.1) for a certain $\lambda \in (0,1)$, set $t_0 = t_0^+ = 0$, $t_{2q+1} = \omega$, we obtain

$$\begin{aligned}
\int_0^\omega |x_i^\Delta(t)|\Delta t &= \sum_{k=1}^{2q+1}\int_{t_{k-1}^+}^{t_k} |x_i^\Delta(t)|\Delta t + \sum_{k=1}^{2q} |I_{ik}(x_i(t_k))| \\
&\le \int_0^\omega \Big|\frac{1}{1+\lambda}\Big\{ - a_i(x_i(t))\Big[\alpha_i(x_i(t)) - \sum_{j=1}^{m} p_{ji}(t)f_j(y_j(t-\tau_{ji})) + c_i(t)\Big]\Big\} \\
&\quad - \frac{\lambda}{1+\lambda}\Big\{ - a_i(-x_i(t))\Big[\alpha_i(-x_i(t)) - \sum_{j=1}^{m} p_{ji}(t)f_j(-y_j(t-\tau_{ji})) + c_i(t)\Big]\Big\}\Big|\Delta t \\
&\quad + \sum_{k=1}^{2q}\Big|\frac{1}{1+\lambda}I_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda}I_{ik}(-x_i(t_k))\Big| \\
&\le \Big[\frac{1}{1+\lambda}+\frac{\lambda}{1+\lambda}\Big]\int_0^\omega \max\Big\{\Big|-a_i(x_i(t))\Big[\alpha_i(x_i(t))-\sum_{j=1}^{m} p_{ji}(t)f_j(y_j(t-\tau_{ji}))+c_i(t)\Big]\Big|, \\
&\quad \Big| - a_i(-x_i(t))\Big[\alpha_i(-x_i(t)) - \sum_{j=1}^{m}\Big(p_{ji}(t)f_j(-y_j(t-\tau_{ji})) + c_i(t)\Big]\Big|\Big\}\Delta t \\
&\quad + \Big[\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\Big]\sum_{k=1}^{2q}\max\Big\{\Big|I_{ik}(x_i(t_k))\Big|, \Big|I_{ik}(x_i(t_k))\Big|\Big\} \\
&\le \bar{a}_i\Big[\int_0^\omega |\alpha_i(x_i(t))|\Delta t + \sum_{j=1}^{m}\Big(\bar{p}_{ji}\int_0^\omega |f_j(y_j(t-\tau_{ji}))-f_j(0)|\Delta t + \bar{p}_{ji}|f_j(0)|\omega\Big) + \bar{c}_i\omega\Big] \\
&\quad + \sum_{k=1}^{2q}\max\Big\{\Big|I_{ik}(x_i(t_k)) - I_{ik}(0)\Big|, \Big|I_{ik}(-x_i(t_k)) - I_{ik}(0)\Big|\Big\} + \sum_{k=1}^{2q}|I_{ik}(0)| \\
&\le \bar{a}_i\Big[\delta_i\omega^{\frac{1}{2}}||x_i||_2 + \sum_{j=1}^{m}\Big(\bar{p}_{ji}l_j^f\omega^{\frac{1}{2}}\|y_j\|_2 + \bar{p}_{ji}|f_j(0)|\omega\Big) + \bar{c}_i\omega\Big] \\
&\quad + \|x_i\|_0\sum_{k=1}^{2q} I_{ik}^M + \sum_{k=1}^{2q}|I_{ik}(0)|, \quad (3.2)
\end{aligned}$$

$$\int_0^\omega |y_j^\Delta(t)|\Delta t \le \bar{b}_j\Big[\eta_j\omega^{\frac{1}{2}}||y_j||_2 + \sum_{i=1}^n\Big(\bar{q}_{ij}l_i^g\|x_i\|_2 + \bar{q}_{ij}|g_i(0)|\omega\Big) + \bar{d}_i\omega\Big] + \|y_j\|_0\sum_{k=1}^{2q}J_{jk}^M + \sum_{k=1}^{2q}|J_{jk}(0)|. \quad (3.3)$$

Since every $x_i(t)$ is an arbitrary $\frac{\omega}{2}$-anti-periodic function, there exist constants $\bar{\iota}_i, \underline{\iota}_i \in [0,\omega]_{\mathbb{T}}$ such that $x_i(\bar{\iota}_i) \ge 0$, $x_i(\underline{\iota}_i) \le 0$. By using Lemma 2.4, we have

$$x_i(t) \le x_i(\underline{\iota}_i) + \int_0^\omega |x_i^\Delta(s)|\Delta s,\ x_i(t) \ge x_i(\bar{\iota}_i) - \int_0^\omega |x_i^\Delta(s)|\Delta s, \quad i = 1,2,\dots,n, \quad t \in [0,\omega]_{\mathbb{T}}. \quad (3.4)$$

In view of (3.4), we obtain

$$|x_i(t)| \le \int_0^\omega |x_i^\Delta(s)|\Delta s,\ t \in [0,\omega]_{\mathbb{T}},\ i = 1,2,\dots,n,$$

that is

$$\|x_i\|_0 \le \int_0^\omega |x_i^\Delta(s)|\Delta s,\ \ i = 1,2,\dots,n. \quad (3.5)$$

Similarly

$$\|y_j\|_0 \le \int_0^\omega |x_j^\Delta(s)|\Delta s,\ \ j = 1,2,\dots,m. \quad (3.6)$$

In addition, we have that

$$\|x_i\|_2 = \Big(\int_0^\omega |x_i(s)|^2\Delta s\Big)^{\frac{1}{2}} \le \omega^{\frac{1}{2}}\max_{t\in[0,\omega]_{\mathbb{T}}}|x_i(t)| = \omega^{\frac{1}{2}}\|x_i\|_0,$$

$$\|y_j\|_2 = \Big(\int_0^\omega |y_j(s)|^2\Delta s\Big)^{\frac{1}{2}} \le \omega^{\frac{1}{2}}\max_{t\in[0,\omega]_{\mathbb{T}}}|y_j(t)| = \omega^{\frac{1}{2}}\|y_j\|_0.$$

From (3.2), (3.3), (3.5), and (3.6), we obtain

$$\Big[1 - \bar{a}_i\delta_i\omega - \sum_{k=1}^{2q}I_{ik}^M\Big]\|x_i\|_0 - \bar{a}_i\omega\sum_{j=1}^m\bar{p}_{ji}l_j^f\|y_j\|_0 \le \bar{a}_i\omega\Big[\sum_{j=1}^m\bar{p}_{ji}|f_j(0)| + \bar{c}_i\Big] + \sum_{k=1}^{2q}|I_{ik}(0)| := D_i,\ \ i = 1,2,\dots,n, \quad (3.7)$$

and

$$\Big[1 - \bar{b}_j\eta_j\omega - \sum_{k=1}^{2q}J_{jk}^M\Big]\|y_j\|_0 - \bar{b}_j\omega\sum_{i=1}^n(\bar{q}_{ij}l_i^g\|x_i\|_0$$

$$\leq \bar{b}_j\omega\Big[\sum_{i=1}^{n}\bar{q}_{ij}|g_i(0)|+\bar{d}_j\Big]+\sum_{k=1}^{2q}|J_{jk}(0)|:=D_j^{'},\ j=1,2,\ldots,m. \tag{3.8}$$

Denote

$$\|z\|_0=(\|x_1\|_0,\|x_2\|_0,\ldots,\|x_n\|_0,\|y_1\|_0,\|y_2\|_0,\ldots,\|y_m\|_0)^T,$$

$$D=(D_1,D_2,\ldots,D_n,D_1^{'},D_2^{'},\ldots,D_m^{'})^T.$$

Thus, (3.7), (3.8) can be rewritten in the matrix form

$$E||z||_2\leq D.$$

From the conditions of Theorem 3.1, E is a nonsingular M matrix, therefore

$$||z||_0\leq E^{-1}D:=(C_1,C_2,\ldots,C_n,C_1^{'},C_2^{'},\ldots,C_m^{'})^T,$$

that is

$$||x_i||_0\leq C_i,\quad ||y_j||_0\leq C_j^{'},\quad i=1,2,\ldots,n,\quad j=1,2,\ldots,m.$$

Let $C=\sum\limits_{i=1}^{n}C_i+\sum\limits_{j=1}^{m}C_j^{'}+C_0$, where C_0 is any positive constant. It is clear that ζ is independent of λ. Take $\Omega=\Big\{z\in\mathbb{X}\Big|\|z\|_{\mathbb{X}}<C\Big\}$. Obviously, Ω satisfies all the requirement in Lemma 2.9 and the condition (H) is satisfied. In view of all the discussions above, we conclude from Lemma 2.9 that system (1.1) has at least one $\frac{\omega}{2}$-anti-periodic solution. This completes the proof. □

Corollary 3.1. *If all conditions of Theorem 3.1 are satisfied, then system (1.1) has at least one ω-periodic solution.*

Proof. It is easy to see that if $x(t)$ is a $\frac{\omega}{2}$-anti-periodic solution of system (1.1), then $x(t)$ is also an ω-periodic solution of (1.1). □

4. Global Exponential Stability of Anti-Periodic Solutions

Suppose that $z^*(t)=(x_1^*(t),x_2^*(t),\ldots,x_n^*(t),y_1^*(t),y_2^*(t),\ldots,y_m^*(t))^T$ is a anti-periodic solution of system (1.1). In this section, we study the exponential stability of $z^*(t)$.

Theorem 4.1. *Assume that (H_1)-(H_5) hold. Suppose further that:*

(H_6) *There exist $l_i^{a\alpha}$, $l_j^{b\beta}>0$, such that*

$$|(a_i\alpha_i)(u)-(a_i\alpha_i)(v)|\geq l_i^{a\alpha}|u-v|,\ [(a_i\alpha_i)(u)-(a_i\alpha_i)(v)](u-v)\geq 0,$$

$$|(b_j\beta_j)(u)-(b_j\beta_j)(v)|\geq l_j^{b\beta}|u-v|,\ [(b_j\beta_j)(u)-(b_j\beta_j)(v)](u-v)\geq 0,$$

for $u,v\in\mathbb{R}$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

(H_7) *There exist* l_i^a, $l_j^b > 0$, *such that*

$$|a_i(u) - a_i(v)| \leq l_i^a |u - v|, \quad |b_j(u) - b_j(v)| \leq l_j^b |u - v|,$$

$$\sum_{j=1}^{m} \bar{p}_{ji} F_j < \frac{1}{l_i^a}, \quad \sum_{i=1}^{n} \bar{q}_{ij} G_i < \frac{1}{l_j^b},$$

where:

$$F_j = \max_{t \in [0,\omega]_{\mathbb{T}}} \{|f_j(y_j^*(t))|\}, \quad G_i = \max_{t \in [0,\omega]_{\mathbb{T}}} \{|g_i(x_i^*(t))|\},$$

$$H_j = \max_{t \in [0,\omega]_{\mathbb{T}}} \{|h_j(y_j^*(t))|\}, \quad U_i = \max_{t \in [0,\omega]_{\mathbb{T}}} \{|u_i(x_i^*(t))|\},$$

$u, v \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

(H_8) *For* $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$

$$-l_i^{a\alpha} + l_i^a \bar{c}_i + \bar{a}_i \sum_{j=1}^{m} \bar{p}_{ji} l_j^f < -1, \quad -l_j^{b\beta} + l_j^b \bar{d}_j + \bar{b}_j \sum_{i=1}^{n} \bar{q}_{ij} l_i^g < -1.$$

(H_9) *Impulsive operators* $I_{ik}(x_i(t))$, $J_{jk}(y_j(t))$ *satisfy:*

$$\begin{cases} I_{ik}(x_i(t_k)) & = & -\gamma_{ik} x_i(t_k),\ 0 < \gamma_{ik} < 2,\ i = 1, 2, \ldots, n,\ k \in \mathbb{N}, \\ J_{jk}(y_j(t_k)) & = & -\hat{\gamma}_{jk} y_j(t_k),\ 0 < \hat{\gamma}_{jk} < 2,\ j = 1, 2, \ldots, m,\ k \in \mathbb{N}. \end{cases}$$

Then the $\frac{\omega}{2}$*-anti-periodic solution of system (1.1) is globally exponentially stable.*

Proof. According to Theorem 3.1, we know that (1.1) has an $\frac{\omega}{2}$-anti-periodic solution

$$z^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t), y_1^*(t), y_2^*(t), \ldots, y_m^*(t))^T.$$

Suppose that

$$z(t) = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$$

is an arbitrary solution of (1.1), then by (1.1) we have

$$\begin{aligned} (x_i(t) - x_i^*(t))^{\Delta} = & -\Big[a_i(x_i(t))\alpha_i(x_i(t)) - a_i(x_i^*(t))\alpha_i(x_i^*(t))\Big] \\ & + \Big[a_i(x_i(t)) - a_i(x_i^*(t))\Big] \sum_{j=1}^{m} p_{ji}(t) f_j(y_j^*(t - \tau_{ji})) \\ & + a_i(x_i(t)) \sum_{j=1}^{m} p_{ji}(t) \Big[f_j(y_j(t - \tau_{ji})) - f_j(y_j^*(t - \tau_{ji}))\Big] \\ & - \Big[a_i(x_i(t)) - a_i(x_i^*(t))\Big] c_i(t),\ i = 1, 2, \ldots, n,\ t \in (0, \infty)_{\mathbb{T}}, \end{aligned}$$

$$t \neq t_k,\ k \in \mathbb{N},$$

$$\begin{aligned}(y_j(t) - y_j^*(t))^\Delta = &-\Big[b_j(y_j(t))\beta_j(y_j(t)) - b_j(y_j^*(t))\beta_j(y_j^*(t))\Big] \\ &+ \Big[b_j(y_j(t)) - b_j(y_j^*(t))\Big]\sum_{i=1}^{n} q_{ij}(t) g_i(x_i^*(t-\sigma_{ij})) \\ &+ b_j(y_j)\sum_{i=1}^{n} q_{ij}(t)\Big[g_i(x_i(t-\sigma_{ij})) - g_i(x_i^*(t-\sigma_{ij}))\Big] \\ &- \Big[b_j(y_j(t)) - b_j(y_j^*(t))\Big] d_j(t),\ j = 1,2,\ldots,m,\ t \in (0,\infty)_{\mathbb{T}}, \\ &t \neq t_k,\ k \in \mathbb{N},\end{aligned}$$

Hence from the condition $(H_7), (H_8)$ and (H_9), we have

$$\begin{aligned}D^+|x_i(t) - x_i^*(t)|^\Delta \le &-l_i^{a\alpha}|x_i(t) - x_i^*(t)| + l_i^a \sum_{j=1}^{m} \bar{p}_{ji} F_j |x_i(t) - x_i^*(t)| \\ &+ \bar{a}_i \sum_{j=1}^{m} \bar{p}_{ji} l_j^f |y_j(t-\tau_{ji}) - y_j^*(t-\tau_{ji})| \\ &+ l_i^a |x_i(t) - x_i^*(t)| \bar{c}_i,\ i = 1,2,\ldots,n,\ t \in (0,\infty)_{\mathbb{T}},\ t \neq t_k,\ k \in \mathbb{N},\end{aligned} \quad (4.1)$$

$$\begin{aligned}D^+|y_j(t) - y_j^*(t)|^\Delta \le &-l_j^{b\beta}|y_j(t) - y_j^*(t)| + l_j^b \sum_{i=1}^{n} \bar{q}_{ij} G_i |y_j(t) - y_j^*(t)| \\ &+ \bar{b}_j \sum_{i=1}^{n} \bar{q}_{ij} l_i^g |x_i(t-\sigma_{ij}) - x_i^*(t-\sigma_{ij})| \\ &+ l_j^b |y_j(t) - y_j^*(t)| \bar{d}_j,\ j = 1,2,\ldots,m,\ t \in (0,\infty)_{\mathbb{T}},\ t \neq t_k,\ k \in \mathbb{N}.\end{aligned} \quad (4.2)$$

Also, for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, $k \in \mathbb{N}$,

$$\begin{aligned}x_i(t_k^+) - x_i^*(t_k^+) &= x_i(t_k) + I_{ik}(x_i(t_k)) - x_i^*(t_k) - I_{ik}(x_i^*(t_k)) \\ &= (1-\gamma_{ik})(x_i(t_k) - x_i^*(t_k)),\end{aligned}$$

$$\begin{aligned}y_j(t_k^+) - y_j^*(t_k^+) &= y_j(t_k) + J_{jk}(y_j(t_k)) - y_j^*(t_k) - J_{jk}(y_j(t_k)) \\ &= (1-\hat{\gamma}_{jk})(y_j(t_k) - y_j^*(t_k)).\end{aligned}$$

Hence

$$|x_i(t_k^+) - x_i^*(t_k^+)| = |1-\gamma_{ik}||x_i(t_k) - x_i^*(t_k)| < |x_i(t_k) - x_i^*(t_k)|, \quad i = 1,2,\ldots,n,\ k \in \mathbb{N},$$

$$|y_j(t_k^+) - y_j^*(t_k^+)| = |1 - \hat{\gamma}_{jk}||y_j(t_k) - y_j^*(t_k)| < |y_j(t_k) - y_j^*(t_k)|,$$
$$j = 1, 2, \ldots, m,\ k \in \mathbb{N}.$$

Let Q_i, $Q_j^{,}$ be defined by

$$\begin{aligned} Q_i(\xi) &= l_i^{a\alpha} - \xi - l_i^a \bar{c}_i - \bar{a}_i \sum_{j=1}^m \bar{p}_{ji} l_j^f e_\xi(\frac{\omega}{2}, -\tau),\ \xi \in [0, \infty),\ i = 1, 2, \ldots, n, \\ Q_j^{,}(\epsilon) &= l_j^{b\beta} - \epsilon - l_j^b \bar{d}_j - \bar{b}_j \sum_{i=1}^n \bar{q}_{ij} l_i^g e_\epsilon(\frac{\omega}{2}, -\sigma),\ \epsilon \in [0, \infty),\ j = 1, 2, \ldots, m. \end{aligned}$$

It is clear that

$$Q_i(0) = l_i^{a\alpha} - l_i^a \bar{c}_i - \bar{a}_i \sum_{j=1}^m \bar{p}_{ji} l_j^f > 1,\ i = 1, 2, \ldots, n,$$

$$Q_j^{,}(0) = l_j^{b\beta} - l_j^b \bar{d}_j - \bar{b}_j \sum_{i=1}^n \bar{q}_{ij} l_i^g > 1,\ j = 1, 2, \ldots, m.$$

Since Q_i, $Q_j^{,}$ are continuous on $[0, \infty)$ and $Q_i(\xi) \to -\infty$, $Q_j^{,}(\epsilon) \to -\infty$ as $\xi \to +\infty$, $\epsilon \to +\infty$, there exist $\xi_i^* > 0$, $\epsilon_j^* > 0$ such that $Q_i(\xi_i^*) = 1$, $Q_j^{,}(\epsilon_j^*) = 1$ and $Q_i(\xi) > 1$, $Q_j^{,}(\varepsilon) > 1$ for $\xi \in (0, \xi_i^*)$, $\epsilon \in (0, \epsilon_j^*)$. By choosing $\varepsilon^* = \min\{\min_{1\le i\le n}\{\xi_i^*\}, \min_{1\le j\le m}\{\epsilon_j^*\}, \frac{\alpha_0}{2}, \frac{\beta_0}{2}\}$, we have

$$Q_i(\varepsilon^*) = l_i^{a\alpha} - \varepsilon^* - l_i^a \bar{c}_i - \bar{a}_i \sum_{j=1}^m \bar{p}_{ji} l_j^f e_{\varepsilon^*}(\frac{\omega}{2}, -\tau) \ge 1,\ i = 1, 2, \ldots, n,$$

$$Q_j^{,}(\varepsilon^*) = l_j^{b\beta} - \varepsilon^* - l_j^b \bar{d}_j - \bar{b}_j \sum_{i=1}^n \bar{q}_{ij} l_i^g e_{\varepsilon^*}(\frac{\omega}{2}, -\sigma) \ge 1,\ j = 1, 2, \ldots, m.$$

Now, we define

$$\begin{cases} w_i(t) = e_{\varepsilon^*}(t, \delta)|x_i(t) - x_i^*(t)|,\ t \in \mathbb{T},\ \delta \in (0, \infty)_{\mathbb{T}},\ i = 1, 2, \ldots, n, \\ v_j(t) = e_{\varepsilon^*}(t, \delta)|y_j(t) - y_j^*(t)|,\ t \in \mathbb{T},\ \delta \in (0, \infty)_{\mathbb{T}},\ j = 1, 2, \ldots, m. \end{cases} \tag{4.3}$$

For $t > 0$, $t \neq t_k$, $k \in \mathbb{N}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, by (4.1)-(4.3) and Lemma 2.8, we have

$$\begin{aligned} D^+ w_i^\Delta(t) \le\ & \varepsilon^* e_{\varepsilon^*}(t, \delta)|x_i(t) - x_i^*(t)| + e_{\varepsilon^*}(\sigma(t), \delta)\Big\{ - l_i^{a\alpha}|x_i(t) - x_i^*(t)| \\ & + l_i^a |x_i(t) - x_i^*(t)| \sum_{j=1}^m \bar{p}_{ji} F_j + \bar{a}_i \sum_{j=1}^m \bar{p}_{ji} l_j^f |y_j(t - \tau_{ji}) - y_j^*(t - \tau_{ji})| \\ & + l_i^a |x_i(t) - x_i^*(t)| \bar{c}_i \Big\} \end{aligned}$$

$$\leq \quad [1+\mu(t)\varepsilon^*]\bigg\{\bigg(-l_i^{a\alpha}+\varepsilon^*+l_i^a\sum_{j=1}^{m}(\bar{p}_{ji}F_j)+l_i^a\bar{c}_i\bigg)w_i(t)$$

$$+\bar{a}_i\sum_{j=1}^{m}\bar{p}_{ji}l_j^f e_{\varepsilon^*}(t,t-\tau_{ji})v_j(t-\tau_{ji})\bigg\}$$

$$\leq \quad [1+\mu\varepsilon^*]\bigg\{\bigg(-l_i^{a\alpha}+\varepsilon^*+l_i^a\sum_{j=1}^{m}(\bar{p}_{ji}F_j)+l_i^a\bar{c}_i\bigg)w_i(t)$$

$$+\bar{a}_i\sum_{j=1}^{m}\bar{p}_{ji}l_j^f e_{\varepsilon^*}(\frac{\omega}{2},-\tau)v_j(t-\tau_{ji})\bigg\} \tag{4.4}$$

$$D^+v_j^{\Delta}(t) \quad \leq \quad [1+\mu\varepsilon^*]\bigg\{\bigg(-l_j^{b\beta}+\varepsilon^*+l_j^b\sum_{i=1}^{n}(\bar{q}_{ij}G_i)+l_j^b\bar{d}_j\bigg)v_j(t)$$

$$+\bar{b}_j\sum_{i=1}^{n}\bar{q}_{ij}l_i^g e_{\varepsilon^*}(\frac{\omega}{2},-\sigma)w_i(t-\sigma_{ij}), \tag{4.5}$$

where $\mu = \sup_{t\in[0,\frac{\omega}{2}]_{\mathbb{T}}} \mu_i(t)$. Also

$$\begin{cases} w_i(t_k^+) &= |1-\gamma_{ik}|w_i(t_k)| < w_i(t_k),\ i=1,2,\ldots,n,\ k\in\mathbb{N}, \\ v_j(t_k^+) &= |1-\hat{\gamma}_{jk}|v_j(t_k)| < v_j(t_k),\ j=1,2,\ldots,m,\ k\in\mathbb{N}. \end{cases} \tag{4.6}$$

Let $W_i(t)=w_i(t)$, $i=1,2,\ldots,n$, $W_{n+j}(t)=v_j(t)$, $j=1,2,\ldots,m$. It is easy to see that

$$\max_{1\leq k\leq n+m}\{W_k(s)\} \leq [1+e_{\varepsilon^*}(0,\delta)]\bigg[\|\phi-x^*\|+\|\psi-y^*\|\bigg],\ s\in(-\infty,0]_{\mathbb{T}}.$$

Next, we claim that

$$\max_{1\leq k\leq n+m}\{W_k(t)\} \leq [1+e_{\varepsilon^*}(0,\delta)]\bigg[\|\phi-x^*\|+\|\psi-y^*\|\bigg],\ t\in(0,\infty)_{\mathbb{T}}. \tag{4.7}$$

If (4.7) is not true, there exists a $t^*>0$ and some k_0 such that $W_{k_0}(t^*) = [1+e_{\varepsilon^*}(0,\delta)][\|\phi-x^*\|+\|\psi-y^*\|]$, $D^+[W_{k_0}^{\Delta}(t^*)]\geq 0$, (if $t^*\neq t_k$), $\Delta W_{k_0}(t^*)\geq 0$ (if $t^*=t_k$), and $W_k(t)\leq[1+e_{\varepsilon^*}(0,\delta)][\|\phi-x^*\|+\|\psi-y^*\|]$ for $t\in(-\infty,t^*)_{\mathbb{T}}$, $k=1,2,\ldots,n,n+1,\ldots,n+m$.

If $k_0\in\{1,2,\ldots,n\}$, from (4.4), we have

$$D^+W_{k_0}^{\Delta}(t^*) \leq [1+\mu\varepsilon^*]\bigg(-Q_{k_0}(\varepsilon^*)+1\bigg)[1+e_{\varepsilon^*}(0,\delta)]\bigg[\|\phi-x^*\|+\|\psi-y^*\|\bigg] < 0. \tag{4.8}$$

If $k_0=n+j_0,\ j_0\in\{1,2,\ldots,m\}$, from (4.5), we obtain

$$D^+W_{k_0}^{\Delta}(t^*) \leq [1+\mu\varepsilon^*]\Big(-Q_{j_0}^{'}(\varepsilon^*)+1\Big)[1+e_{\varepsilon^*}(0,\delta)]\Big[\|\phi-x^*\|+\|\psi-y^*\|\Big] < 0. \quad (4.9)$$

From (4.6) (4.8) and (4.9), we get a contradiction, so

$$\begin{cases} w_i(t) \leq [1+e_{\varepsilon^*}(0,\delta)]\Big[\|\phi-x^*\|+\|\psi-y^*\|\Big], \ t\in(0,\infty)_{\mathbb{T}}, \ i=1,2,\ldots,n, \\ v_j(t) \leq [1+e_{\varepsilon^*}(0,\delta)]\Big[\|\phi-x^*\|+\|\psi-y^*\|\Big], \ t\in(0,\infty)_{\mathbb{T}}, \ j=1,2,\ldots,m, \end{cases}$$

This means that

$$\sum_{i=1}^{n}|x_i(t)-x_i^*(t)|+\sum_{j=1}^{m}|y_j(t)+y_j^*(t)| \leq N(\delta)e_{\Theta\varepsilon^*}(t,\delta)\Big[\|\phi-x^*\|+\|\psi-y^*\|\Big],$$

where $N(\delta)=(n+m)[1+e_{\varepsilon^*}(0,\delta)] \geq 1$. This completes the proof. □

Similar to Corollary 3.1, we can obtain the following Corollary.

Corollary 4.1. *If all conditions of Theorem 3.1 and Theorem 4.1 hold, then the system (1.1) has at least one ω-periodic solution, which is globally exponentially stable.*

5. An Example

Let $\mathbb{T}$ is a $\frac{1}{2}$-periodic time scale, and

$$(a_i(x))_{2\times1} = \begin{pmatrix} \frac{3}{4}(\frac{1}{\pi}\arctan|x|+1) \\ \frac{3}{8}(\frac{1}{\pi}\arctan|x|+1) \end{pmatrix}, \quad (b_j(x))_{2\times1} = \begin{pmatrix} \frac{3}{8}(\frac{1}{\pi}\arctan|x|+\frac{5}{2}) \\ \frac{3}{16}(\frac{1}{\pi}\arctan|x|+\frac{5}{2}) \end{pmatrix},$$

$$(\alpha_i(x))_{2\times1} = \begin{pmatrix} \frac{1}{6}x \\ \frac{1}{12}x \end{pmatrix}, \quad (\beta_j(x))_{2\times1} = \begin{pmatrix} \frac{1}{8}x \\ \frac{1}{16}x \end{pmatrix},$$

$$(p_{ji}(t))_{2\times2} = \begin{pmatrix} \frac{4}{9\pi}\cos(4\pi t) & 0 \\ 0 & \frac{2}{9\pi}\sin(4\pi t) \end{pmatrix},$$

$$(f_j(x))_{2\times1} = (g_i(x))_{2\times1} = \begin{pmatrix} \frac{1}{2}x \\ x \end{pmatrix},$$

$$(q_{ij}(t))_{2\times2} = \begin{pmatrix} \frac{4}{9\pi}\sin^2(2\pi t) & 0 \\ 0 & \frac{2}{9\pi}\cos(4\pi t) \end{pmatrix}, \ (\sigma_{ij})_{2\times2} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$(c_i(t))_{2\times1} = \begin{pmatrix} \frac{1}{2}\sin(4\pi t) \\ \frac{1}{2}\cos(4\pi t) \end{pmatrix}, \ d_j(t))_{2\times1} = \begin{pmatrix} \frac{1}{4}\sin(4\pi t) \\ \frac{1}{4}\cos(4\pi t) \end{pmatrix},$$

$$(\tau_{ji})_{2\times 2} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix},$$

then system (1.1), has at least one $\frac{1}{2}$−anti-periodic solution and it is global exponential stable.

By calculating, we have

$$\omega = 1,\ \bar{a}_1 = \bar{b}_1 = \frac{9}{8},\ \bar{a}_2 = \bar{b}_2 = \frac{9}{16},\ \underline{a}_1 = \frac{3}{4},\ \underline{a}_2 = \frac{3}{8},\ \underline{b}_1 = \frac{15}{16},\ \underline{b}_2 = \frac{15}{32},$$

$$\delta_1 = \delta_1^m = \frac{1}{6},\ \delta_2 = \delta_2^m = \frac{1}{12},$$

$$\eta_1 = \eta_1^m = \frac{1}{8},\ \eta_2 = \eta_2^m = \frac{1}{16},\ \bar{p}_{11} = \bar{q}_{11} = \frac{4}{9\pi},\ \bar{p}_{22} = \bar{q}_{22} = \frac{2}{9\pi},$$

$$l_1^a = \frac{3}{4\pi},\ l_2^a = \frac{3}{8\pi},\ l_1^b = \frac{3}{8\pi},\ l_2^b = \frac{3}{16\pi},\ l_1^f = l_1^g = \frac{1}{2},\ l_2^f = l_2^g = 1,\ \bar{c}_1 = \bar{c}_2 = \frac{1}{2},$$

$$\bar{d}_1 = \bar{d}_2 = \frac{1}{4},\ \sigma = 2,\ \tau = 4,\ I_{ik}^M = \gamma_{ik} = 0.1,\ J_{jk}^M = \hat{\gamma}_{jk} = 0.6,\ \bar{p}_{ji} = \bar{q}_{ij} = 0,\ i \neq j.$$

It is not difficult to see that (H_1)-(H_9), are satisfied, and $E_{4\times 4}$ is an nonsingular M matrix. From Theorems 3.1 and 4.1, we know that the above system has at least one $\frac{1}{2}$-anti-periodic solution and it is global exponential stable.

Acknowledgments

This work is supported by the Natural Sciences Foundation of the education department of Yun Nan Province under Grant k1050559.

References

[1] C.H. Bai, Stability analysis of Cohen-Grossberg BAM neural networks with delays and impulses, *Chaos, Solitons and Fractalas*, **35** (2008), 263-267.

[2] L. Bi, M. Bohner, M. Fan, Periodic solutions of functional dynamic equations with infinite delay, *Nonlinear Analysis: Theory, Methods and Applications*, **68** (2008), 1226-1245.

[3] M. Bohner, M. Fan, J.M. Zhang, Existence of periodic solutions in predator-prey and competetion dynamic systems, *Nonlinear Anal: Real Word Appl.*, **330**, No. 1 (2007), 1-9.

[4] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston (2003).

[5] Y.Q. Cheng, Anti-periodic solutions for semilinear evolution equations, *J. Math. Anal. Appl.*, **315** (2006), 37-348.

[6] T. Huang, C. Li, G. Chen, Stability of Cohen-Grossberg neural networks with unbounded distributed delays, *Chaos, Solitons and Fractalas*, **34** (2007), 992-996.

[7] E.R. Kaufmann, Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamic equation on a time scale, *J. Math. Anal. Appl.*, **319** (2006), 315-25.

[8] V. Lakshmikantham, A.S. Vatsala, Hybrid systems on time scales, *J. Comput. Appl. Math.*, **141** (2002), 227-235.

[9] Y. Li, Existence and stability of periodic solutions for Cohen-Grossberg neural networks with multipie delays, *Chaos, Solitons and Frctals*, **20** (2004), 459-466.

[10] Y. Li, Global exponential stability of BAM neural networks with delays and impulses, *Chaos, Solitons and Fractals*, **24** (2005), 279-285.

[11] Y. Li, X. Chen, L. Zhao, Stability and existence of periodic solutions to delayed Cohen-Grossberg BAM neural networks with impulses on time scales, *Neurocomputing*, **72** (2009), 1624-1630.

[12] Y. Li, C. Yang, Global exponential stability analysis on impulsive BAM neural networks with distributed delays, *J. Math. Anal. Appl.*, **324** (2006), 1125-1139.

[13] J.L. Mawhin, *Topological Degree Methods in Noninear Boundary Value Problems*, CBMS Regional Conference Series in Mathematics, No. 40, American Mathematical Society, Provedence, RI (1979).

[14] D. O'Regan, Y.J. Chao, Y.Q. Chen, *Topological Degree Theory and Application*, Taylor and Francis Group, Boca Raton-London-New York (2006).

[15] C. Ou, Anti-periodic solutions for high-order Hopfield neural networks, *Comput. Math. Appl.*, **56** (2008), 1838-1844.

[16] J.Y. Shao, Anti-periodic solutions for shunting inhibitory cellular neural networks with time-varying delays, *Phys. Lett. A*, **372** (2008), 5011-5016.

[17] F.H. Wang, C.C. Yeh, S.L. Yu, C.H. Hong, Youngs inequality and related results on time scales, *Appl. Math. Lett.*, **18** (2005), 983-988.

[18] R. Wu, An anti-periodic LaSalle oscillation theorem, *Appl. Math. Lett.*, **21** (2008), 928-933.

[19] Y. Xing, M. Han, G. Zheng, Initial value problem for first-order intrgro-differential equation of Volterra type on time scale, *Nonlinear Anal.*, **60** (2005), 429-442.

[20] F.G. Yang, C.L. Zhang, Global stability nanlysis of impulsive BAM type Cohen-Grossberg neural networks with delays, *Appl. Math. Comput.*, **186** (2007), 932-940.