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ON A CLASS OF MEROMORPHIC MULTI-VALENT FUNCTIONS

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Abstract: In this paper, we define a new differential operator for a new class of multi-valent meromorphic functions in punctured unit disk. Some subordination relations are established and some known results are extended. Moreover, an application of fractional integral operator is illustrated.

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1. Introduction

Assume the meromorphic functions, which are analytic in the punctured unit disk $U := \{z \in \mathbf{C},\, 0 < |z| < 1\}$, take the structure

$$F(z) = \frac{1}{z^{p+\alpha}(1-z)^{\alpha}} \quad (z \in U),$$

where $\alpha \geq 0$. Then we obtain

$$\begin{aligned} F(z) &= \frac{1}{z^{p+\alpha}(1-z)^{\alpha}} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^{n-p-\alpha} \\ &= \frac{1}{z^{p+\alpha}} + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{n!} z^{n-p-\alpha} = \frac{1}{z^{p+\alpha}} + \sum_{n=1-p}^{\infty} \frac{(\alpha)_{n+p}}{(n+p)!} z^{n-\alpha}. \end{aligned}$$

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Let $\Sigma_{p,\alpha}$ denote the class of functions of the form

$$f(z) = \frac{1}{z^{p+\alpha}} + \sum_{n=1-p}^{\infty} a_n z^{n-\alpha} \quad \Big(\alpha \geq 0,\, p \in \mathbf{N}\Big), \tag{1}$$

which are analytic in the punctured unit disk U. The convolution of two power series f, given by (1) and

$$g(z) = \frac{1}{z^{p+\alpha}} + \sum_{n=1-p}^{\infty} b_n z^{n-\alpha}$$

is defined as the following power series:

$$f(z) * g(z) = \frac{1}{z^{p+\alpha}} + \sum_{n=1-p}^{\infty} a_n b_n z^{n-\alpha}.$$

Remark 1. In view of the Uniformization Theorem: for every simply connected Riemann surface X there exists a conformal homeomorphism $\phi : X_0 \to X$, where X_0 is one of the three standard regions, the Riemann sphere $\overline{C}$, the complex plane $\mathbf{C}$ or the unit disc U. The conformal type of X is elliptic, parabolic or hyperbolic, respectively. The map ϕ is called the uniformizing map. The case which was studied most is that $X \subset \mathbf{C}$ is a simply connected region, $X \neq \mathbf{C}$. Then X is of hyperbolic type and ϕ is a univalent function in U (see [10]).

By using the convolution product, we define a linear operator as follows

$$D_{\alpha}^{k+p-1} f(z) = \frac{z^{-p-\alpha}}{(1-z)^{k+p}} * f(z) = \frac{1}{z^{p+\alpha}} + \sum_{n=1-p}^{\infty} \frac{(k+p)_{n+p}}{(n+p)!} a_n z^{n-\alpha}, \tag{2}$$

where k is any integer greater than $-p$. when $p = 1$ and $\alpha = 0$ operator (2) was introduced by Ganigi and Uralegaddi [3] and then generalized by Yang [16] and others (see [14], [15]). Finally, operator (2) posses Ruscheweyh operator for meromorphic functions when $\alpha = 0, p = 1$.

In this recent paper we shall define new classes of the operator (2) and discuss the sufficient conditions for subordination containing this operator. For this reason we need the following preliminaries. Recalling the principle of subordination between analytic functions, let the functions f and g be analytic in $\triangle := \{z \in \mathbf{C}, |z| < 1\}$. Then we say that the function f is *subordinate* to g if there exists a Schwarz function $w(z)$, analytic in $\triangle$ such that $f(z) = g(w(z)),\ z \in \triangle$. We denote this subordination by $f \prec g$ or $f(z) \prec g(z)$, $z \in \triangle$. If the function g is univalent in $\triangle$ the above subordination is equivalent to $f(0) = g(0)$ and $f(\triangle) \subset g(\triangle)$. Let $\phi : \mathbf{C}^3 \times \triangle \to \mathbf{C}$ and let h be univalent in $\triangle$. Assume that p, ϕ are analytic and univalent in $\triangle$ if p satisfies the differential superordination

$$h(z) \prec \phi(p(z)), zp'(z), z^2p''(z); z), \tag{3}$$

then p is called a solution of the differential superordination. (If f is subordinate to g, then g is called to be superordinate to f). An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (3). An univalent function q such that $p \prec q$ for all subordinants p of (3) is said to be the best subordinant.

A function $f(z) \in \Sigma_{p,\alpha}$ belongs to the class $\mathcal{S}_{p,\alpha}(\mu)$ the class of meromorphically $p+\alpha$-valent starlike functions of order μ where $0 \le \mu < p+\alpha$, if and only if $f \ne 0$, and $-\Re\{\frac{zf'(z)}{f(z)}\} > \mu$ $(z \in U)$. A function $f(z) \in \Sigma_{p,\alpha}$ belongs to the class $\mathcal{C}_{p,\alpha}(\mu)$ the class of meromorphically $p+\alpha$-valent convex functions of order μ where $0 \le \mu < p+\alpha$, if and only if if $f' \ne 0$, and $-\Re\{1+\frac{zf''(z)}{f'(z)}\} > \mu$ $(z \in U)$. Now we define new classes containing the differential operator (2).

Definition 1. Let $f \in \Sigma_{p,\alpha}$. Then f belongs to $\mathcal{MS}_{p,\alpha}(\mu)$ if and only if

$$-\Re\Big\{\frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)}\Big\} > \mu \quad (z \in U).$$

Definition 2. Let $f \in \Sigma_{p,\alpha}$. Then f belongs to $\mathcal{MC}_{p,\alpha}(\mu)$ if and only if

$$-\Re\Big\{1+\frac{\Big(z(D_\alpha^{k+p-1}f(z))'\Big)'}{\Big(D_\alpha^{k+p-1}f(z)\Big)'}\Big\} > \mu \quad (z \in U).$$

In the present paper, we establish some sufficient conditions for functions $f \in \Sigma_{p,\alpha}$ to satisfy

$$-[\frac{\mu}{p+\alpha}\frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)}] \prec q(z) \quad (z \in U), \tag{4}$$

where $q(z)$ is a given univalent function in U. Moreover, we give applications for these results in fractional calculus. Note that in [12], Ravichandran et al studied sufficient conditions for subordination for class $(\Sigma_{1,0}), k=0$, of meromorphic functions $-\frac{zf'(z)}{f(z)} \prec q(z)$ $(z \in U)$. We need the following results in the sequel.

Lemma 1. (see [11]) *Let γ, δ be any complex numbers, $\delta \ne 0$. Let $q(z) = 1+q_1z+q_2z^2+...$ be univalent in $\triangle$, $q(z) \ne 0$. Let $Q(z) = \frac{\delta zq'(z)}{q(z)}$ be starlike and $\Re\{\frac{\gamma}{\delta}q(z)+\frac{zQ'(z)}{Q(z)}\} > 0$ $(z \in \triangle)$. If $\psi(z) = 1+c_1z+c_2z^2+...$ is analytic in $\triangle$ and satisfies $\gamma\psi(z)+\delta\frac{z\psi'(z)}{\psi(z)} \prec \gamma q(z)+\delta\frac{zq'(z)}{q(z)}$ $(z \in \triangle)$, then $\psi(z) \prec q(z)$ and q is the best dominant.*

Lemma 2. (see [8]) *Let $q(z)$ be univalent in $\triangle$, and let $\varphi(z)$ be analytic* *in a domain containing* $q(\triangle)$. *If* $zq'(z)\varphi(q(z))$ *is starlike and* $z\psi'(z)\varphi(\psi(z)) \prec zq'(z)\varphi(q(z))$ $(z \in \triangle)$, *then* $\psi(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Remark 2. Note that the authors defined and studied various type of analytic functions of fractional order (see [2]-[7]).

2. Subordination Theorems

In this section, we present some sufficient conditions for subordination of analytic functions belong to the class $\Sigma_{p,\alpha}$.

Theorem 3. *Let* q *satisfies the conditions of Lemma 1. If* $f \in \Sigma_{p,\alpha}$ *satisfies*

$$\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{zq'(z)}{q(z)} - (p+\alpha-\mu),$$

then $-\frac{z^{p+\alpha-\mu}f'(z)}{(p+\alpha-\mu)f(z)} \prec q(z)$ *and* q *is the best dominant.*

Proof. Set $\psi(z) := -\frac{z^{p+\alpha-\mu}f'(z)}{(p+\alpha-\mu)f(z)}$ $(z \in U)$. Then a computation shows

$$\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{z\psi'(z)}{\psi(z)} - (p+\alpha-\mu),$$

thus we receive the relation $\frac{z\psi'(z)}{\psi(z)} \prec \frac{zq'(z)}{q(z)}$. By an application of Lemma 1, with $\gamma = 0$ and $\delta = 1$, it follows that, $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant. □

Corollary 4. *Let* $-1 \le B < A \le 1$. *If* $f \in \Sigma_{p,\alpha}$ *satisfies*

$$\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec -(p+\alpha-\mu) + \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then $-\frac{z^{p+\alpha-\mu}f'(z)}{(p+\alpha-\mu)f(z)} \prec \frac{1+Az}{1+Bz}$ *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. Let $q(z) := \frac{1+Az}{1+Bz}$ $(z \in U)$. Then a computation shows that

$$Q(z) := \frac{zq'(z)}{q(z)} = \frac{(A-B)z}{(1+Az)(1+Bz)}.$$

Since $-1 \le B < A \le 1$ yields that $\Re\{Q(z)\} > 0$. Hence $Q(z)$ is starlike in U. The result now follows from Theorem 3. □

The next result can be found in [12].

Corollary 5. *Let the hypothesis of Theorem 3 hold. Then* $-[\frac{z(f(z))'}{f(z)}] \prec q(z)$ $(z \in U)$, *and* q *is the best dominant.*

Proof. Assume $p = 1$ and $\mu = \alpha$. □

The next results show sufficient conditions of subordinations involving the differential operator (2).

Theorem 6. *Let q satisfies the conditions of previous lemma. If $f \in \Sigma_{p,\alpha}$ satisfies*

$$\frac{z(D_\alpha^{k+p-1}f(z))''}{(D_\alpha^{k+p-1}f(z))'} - \frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)} \prec \frac{zq'(z)}{q(z)} - 1,$$

then $-[\frac{\mu}{p+\alpha}\frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)}] \prec q(z)$ *and q is the best dominant.*

Proof. Assume $\psi(z) := -[\frac{\mu}{p+\alpha}\frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)}]$. A computation gives

$$\frac{z(D_\alpha^{k+p-1}f(z))''}{(D_\alpha^{k+p-1}f(z))'} - \frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)} = \frac{z\psi'(z)}{\psi(z)} - 1,$$

which implies $\frac{z\psi'(z)}{\psi(z)} \prec \frac{zq'(z)}{q(z)}$. By an application of Lemma 2, with $\varphi(w) = \frac{1}{w}$, it follows that, $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant. □

Corollary 7. *Let $-1 \le B < A \le 1$. If $f \in \Sigma_{p,\alpha}$ satisfies*

$$\frac{z(D_\alpha^{k+p-1}f(z))''}{(D_\alpha^{k+p-1}f(z))'} - \frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)} \prec \frac{(A-B)z}{(1+Az)(1+Bz)} - 1,$$

then

$$-[\frac{\mu}{p+\alpha}\frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)}] \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 8. *If $f \in \Sigma_{p,\alpha}$ satisfies*

$$\frac{z(D_\alpha^{k+p-1}f(z))''}{(D_\alpha^{k+p-1}f(z))'} - \frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)} \prec \frac{2z}{(1+z)(1-z)} - 1,$$

then

$$-[\frac{\mu}{p+\alpha}\frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)}] \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

In general we have the following result

Corollary 9. *Let w be analytic function in U, such that $0 < |w(z)| < 1$,*

$\forall z \in U$. If $f \in \Sigma_{p,\alpha}$ satisfies

$$\frac{z(D_\alpha^{k+p-1}f(z))''}{(D_\alpha^{k+p-1}f(z))'} - \frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)} \prec \frac{2z}{(1+w(z))(1-w(z))} - 1,$$

then

$$-[\frac{\mu}{p+\alpha}\frac{z(D_\alpha^{k+p-1}f(z))'}{D_\alpha^{k+p-1}f(z)}] \prec \frac{1+w(z)}{1-w(z)}$$

and $\frac{1+w(z)}{1-w(z)}$ is the best dominant.

Corollary 10. *Let the hypothesis of Theorem 6 hold. Then $-[\frac{z(f(z))'}{f(z)}] \prec q(z)$ $(z \in U)$, and q is the best dominant.*

Proof. Assume $\mu = p = 1$ and $k = \alpha = 0$. □

The next result can be found in [1].

Corollary 11. *Let the hypothesis of Theorem 6 be hold. Then*

$$-[\frac{1}{p}\frac{z(D^{k+p-1}f(z))'}{D^{k+p-1}f(z)}] \prec q$$

and q is the best dominant.

Proof. Assume $\mu = 1$ and $\alpha = 0$. □

3. Applications

In this section, we introduce some applications of Section 2 containing fractional integral operators. Assume that

$$f(z) = \sum_{n=1-p}^{\infty} \varphi_n z^{n-2\alpha} \quad (z \in U \cup \{0\}), \tag{5}$$

where $p \in \mathbf{N}$ and let us begin with the following definitions

Definition 12. (see [13]) The fractional integral of order α is defined, for a function f by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta)(z-\zeta)^{\alpha-1}d\zeta, \quad \alpha > 0,$$

where the function f is analytic in simply-connected region of the complex z-plane ($\mathbf{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Note that (see [13], [9])

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha} \quad (\mu > -1).$$

Thus we have

$$I_z^\alpha f(z) = \sum_{n=1-p}^{\infty} a_n z^{n-\alpha} \text{ and } \frac{1}{z^{p+\alpha}} + I_z^\alpha f(z) \in \Sigma_{p,\alpha}.$$

Then we have the following results.

Theorem 13. *Let the assumptions of Theorem 3 hold. Then for f defined in (5)*

$$-\frac{z^{p+\alpha-\mu}(\frac{1}{z^{p+\alpha}} + I_z^\alpha f(z))'}{(p+\alpha-\mu)(\frac{1}{z^{p+\alpha}} + I_z^\alpha f(z))} \prec q(z) \quad (z \in U)$$

and q is the best dominant.

Proof. Let the function F be defined by

$$F(z) := \frac{1}{z^{p+\alpha}} + I_z^\alpha f(z) \quad (z \in U).$$

Then the result comes directly from Theorem 3. □

Theorem 14. *Let the assumptions of Theorem 6 hold. Then for f defined in (5)*

$$-\Big[\frac{\mu}{p+\alpha} \frac{z\Big(D_\alpha^{k+p-1}(\frac{1}{z^{p+\alpha}} + I_z^\alpha f(z))\Big)'}{D_\alpha^{k+p-1}\Big(\frac{1}{z^{p+\alpha}} + I_z^\alpha f(z)\Big)}\Big] \prec q(z) \quad (z \in U)$$

and q is the best dominant.

Proof. Let the function F be defined by

$$F(z) := \frac{1}{z^{p+\alpha}} + I_z^\alpha f(z) \quad (z \in U).$$

Then Theorem 6 gives the result. □

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