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ON DIFFERENTIAL OPERATOR FOR MULTIVALENT FUNCTIONS

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Abstract: In this article we define a differential operator for multivalent functions in an open disk. Further, we define some classes containing this operator. Partial sums are considered.

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1. Introduction

Let $T(p)$ denote the class of functions f of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \quad (p \in \mathbf{N},\ z \in U), \tag{1}$$

which are analytic and p-valent (multivalent) in the open unit disk $U = \{z : z \in \mathbf{C} \text{ and } |z| < 1\}$. Let be given two functions $f, g \in T(p)$, $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p}$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$. Then their *convolution* or *Hadamard product* $f(z) * g(z)$ is defined by $f(z) * g(z) = z^p + \sum_{n=1}^{\infty} a_n b_n z^{n+p} \quad (z \in U)$. Define a function $\varphi_p(a, c; z)$ as follows

$$\varphi_p(a, c; z) := z^p + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p}, \quad c \neq 0, -1, -2, ...,$$

where $(a)_n$ is the Pochhammer symbol.

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Assuming that $a = k+p > 0$ and $c = 1$ for $k = 0, 1, 2, \ldots$ in $\varphi_p(a, c; z)$ yields

$$\varphi_p(k+p, 1; z) = z^p + \sum_{n=1}^{\infty} \frac{(k+p)_n}{(1)_n} z^{n+p}. \tag{2}$$

Next we define the following differential operator $\mathcal{D}_\lambda^k : T(p) \to T(p)$ by

$$\begin{aligned} D^0 f(z) &= f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p}, \\ D_{\lambda,p}^1 f(z) &= (1-\lambda p) f(z) + \lambda z f'(z) = z^p + \sum_{n=1}^{\infty} (1+\lambda n) a_n z^{n+p}, \\ &\vdots \\ D_{\lambda,p}^k f(z) &= z^p + \sum_{n=1}^{\infty} (1+\lambda n)^k a_n z^{n+p} \quad (z \in U), \end{aligned} \tag{3}$$

where

$$\Big(p \in \mathbf{N}, \;\; k \in \mathbf{N}_0, \, \lambda \geq 0\Big).$$

Again by applying convolution product on (2) and (3) we have the following operator

$$\begin{aligned} \mathcal{D}_{\lambda,p}^k f(z) &= \frac{z^p}{(1-z)^{k+p}} * D_{\lambda,p}^k f(z) \\ &= z^p + \sum_{n=1}^{\infty} \frac{(k+p)_n}{(1)_n} (1+\lambda n)^k a_n z^{n+p} \\ &= z^p + \sum_{n=1}^{\infty} C(n,k) (1+\lambda n)^k a_n z^{n+p} \quad (z \in U), \end{aligned} \tag{4}$$

where $C(n,k) := \frac{(k+p)_n}{(1)_n}$.

Remark 1. The symbol $\mathcal{D}_{\lambda,p}^k f(z)$, when $\lambda = 0$, $p = 1$, was introduced by Ruscheweyh [2] and when $\lambda = 0$, see Goel and Sohi [1].

A function $f \in T(p)$ is said to be *p-valent starlike of order* $\mu, 0 \leq \mu < p$ if $\Re\Big\{\frac{zf'(z)}{f(z)}\Big\} > \mu$ $(z \in U)$, this class is denoted by $S_p^*(\mu)$. A function $f \in T(p)$ is said to be *p-valent convex of order* $\mu, 0 \leq \mu < p$ if $\Re\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > \mu$ $(z \in U)$, this class is denoted by $C_p(\mu)$.

A function $f \in T(p)$ is said *to be in the class* $S_p^*(\mu, \lambda)$ *of order* μ, where $0 \leq \mu < p$ if $\Re\Big\{\frac{z[\mathcal{D}_{\lambda,p}^k f(z)]'}{\mathcal{D}_{\lambda,p}^k f(z)}\Big\} > \mu$ $(z \in U)$. A function $f \in T(p)$ is said *to be* *in the class* $C_p(\mu,\lambda)$ *of order* μ, where $0 \leq \mu < p$ if $\Re\Big\{1 + \frac{z[\mathcal{D}^k_{\lambda,p} f(z)]''}{[\mathcal{D}^k_{\lambda,p} f(z)]'}\Big\} > \mu$ $(z \in U)$.

For $0 \leq \alpha < p$ and $\beta \geq 0$, let $S^*_p(\alpha,\beta,\lambda)$ be the subclass of $T(p)$ consisting of functions of the form (1) satisfying the analytic criterion

$$\Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - \alpha\Big\} > \beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - p\Big| \quad (z \in U). \tag{5}$$

Also, for $0 \leq \alpha < p$ and $\beta \geq 0$, let $C_p(\alpha,\beta,\lambda)$ be the subclass of $T(p)$ satisfying the analytic criterion

$$\Re\Big\{1 + \frac{z[\mathcal{D}^k_{\lambda,p} f(z)]''}{[\mathcal{D}^k_{\lambda,p} f(z)]'} - \alpha\Big\} > \beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]''}{[\mathcal{D}^k_{\lambda,p} f(z)]'} - (p-1)\Big| \quad (z \in U). \tag{6}$$

The main aim of this work is to study coefficient bounds and extreme points of the general classes $S^*_p(\alpha,\beta,\lambda)$ and $C_p(\alpha,\beta,\lambda)$. Partial sums f_m of functions f in the classes $S^*_p(\alpha,\beta,\lambda)$ and $C_p(\alpha,\beta,\lambda)$ are considered.

2. The Classes $S^*_p(\alpha,\beta,\lambda)$ and $C_p(\alpha,\beta,\lambda)$

In this section we obtain a sufficient condition and extreme points for functions f in the classes $S^*_p(\alpha,\beta,\lambda)$ and $C_p(\alpha,\beta,\lambda)$.

Theorem 1. *A sufficient condition for a function f of the form (1) to be in $S^*_p(\alpha,\beta,\lambda), 0 \leq \alpha < p, \beta \geq 0$ is*

$$\sum_{n=1}^{\infty}[(1+\beta)n + (p-\alpha)]C(n,k)(1+\lambda n)^k|a_n| < p - \alpha \quad (z \in U). \tag{7}$$

Proof. It suffices to show that

$$\beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - p\Big| - \Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - p\Big\} \leq p - \alpha \quad (z \in U).$$

$$\beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - p\Big| - \Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - p\Big\} \leq (1+\beta)\Big|\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - p\Big|$$
$$\leq \frac{(1+\beta)\sum_{n=1}^{\infty} nC(n,k)(1+\lambda n)^k|a_n||z|^{n+p}}{1 - \sum_{n=1}^{\infty} C(n,k)(1+\lambda n)^k|a_n||z|^{n+p}}$$
$$\leq \frac{(1+\beta)\sum_{n=1}^{\infty} nC(n,k)(1+\lambda n)^k|a_n|}{1 - \sum_{n=p+1}^{\infty} C(n,k)(1+\lambda n)^k|a_n|}.$$

This last expression is bounded above by $(p-\alpha)$ if

$$\sum_{n=1}^{\infty}[(1+\beta)n+(p-\alpha)]C(n,k)(1+\lambda n)^k|a_n| < p-\alpha,$$

and the proof is complete. □

In the same manner we can obtain the next result.

Theorem 2. *A sufficient condition for a function f of the form (1) to be in $C_p(\alpha,\beta,\lambda), 0 \le \alpha < p, \beta \ge 0$ is*

$$\sum_{n=1}^{\infty}(n+p)[n(1+\beta)+(p-\alpha)]C(n,k)(1+\lambda n)^k|a_n| < p(p-\alpha). \tag{8}$$

Proof. It suffices to show that for $p \in \mathbf{N}$ $0 \le \alpha < p$, $\beta \ge 0$, $z \in U$

$$\beta\left|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\right| - \Re\left\{\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\right\} \le p-\alpha,$$

$$\begin{aligned}
&\beta\left|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\right| - \Re\left\{\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\right\} \\
&\le (1+\beta)\left|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\right| \\
&\le \frac{(1+\beta)\sum_{n=1}^{\infty}n(n+p)C(n,k)(1+\lambda n)^k|a_n||z|^{n+p-1}}{p|z|^{p-1}-\sum_{n=1}^{\infty}(n+p)C(n,k)(1+\lambda n)^k|a_n||z|^{n+p-1}} \\
&\le \frac{(1+\beta)\sum_{n=1}^{\infty}n(n+p)C(n,k)(1+\lambda n)^k|a_n|}{p-\sum_{n=1}^{\infty}(n+p)C(n,k)(1+\lambda n)^k|a_n|}.
\end{aligned}$$

This last expression is bounded above by $(p-\alpha)$ if

$$\sum_{n=1}^{\infty}(n+p)[n(1+\beta)+(p-\alpha)]C(n,k)(1+\lambda n)^k|a_n| < p(p-\alpha) \quad (p \in \mathbf{N}).$$

This completes the proof. □

3. Partial Sums

In this section, applying methods used by Silverman [3] and Silvia [4], we will investigate the ratio of a function defined in (1) to its sequence of partial sums

$$f_{m+p}(z) = z^p + \sum_{n=1}^{m} a_n z^{n+p} \quad (z \in U) \tag{9}$$

when the coefficients are small enough in order to satisfy either condition (7) or (8). More precisely, we will determine sharp lower bounds for

$$\Re\Big\{\frac{f(z)}{f_{m+p}(z)}\Big\}, \Re\Big\{\frac{f_{m+p}(z)}{f(z)}\Big\}, \Re\Big\{\frac{f'(z)}{f'_{m+p}(z)}\Big\} \text{ and } \Re\Big\{\frac{f'_{m+p}(z)}{f'(z)}\Big\}.$$

In the sequel, we will make use of the fact that $\Re\Big\{\frac{(1+w(z))}{(1-w(z))}\Big\} > 0$ $(z \in U)$ if and only if $w(z) = \sum_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|w(z)| < |z|$.

Theorem 3. *Let f given by (1) satisfy (7). Then*

$$\Re\Big\{\frac{f(z)}{f_{m+p}(z)}\Big\} > 1 - \frac{p-\alpha}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}, \quad (10)$$

where $z \in U$, $p > \alpha$, $m = 0, 1, 2, ...$

The result is sharp for every m with the extremal function

$$f(z) = z^p + \frac{p-\alpha}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k} \times z^{m+p+1}. \quad (11)$$

Proof. Assume that $f \in T(p)$ satisfies (7). By setting

$$w(z) = \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{p-\alpha} \times \Big\{\frac{f(z)}{f_{m+p}(z)} - \Big(1 - \frac{p-\alpha}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}\Big)\Big\} := 1 + \frac{H_{m+p+1}\sum_{n=m+1}^{\infty} a_n z^n}{1+\sum_{n=1}^{m} a_n z^n},$$

where

$$H_{m+p+1} := \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{p-\alpha}.$$

Thus we find that

$$\Big|\frac{w(z)-1}{w(z)+1}\Big| \le \frac{H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n|}{2-2\sum_{n=1}^{m}|a_n| - H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n|} \le 1 \quad (z \in U)$$

if and only if

$$2H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n| \le 2-2\sum_{n=1}^{m}|a_n|$$

which is equivalent to

$$\sum_{n=1}^{m}|a_n| + H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n| \le 1. \tag{12}$$

In order to see that

$$f(z) = z^p + \frac{z^{m+p+1}}{H_{m+p+1}} \quad (z \in U)$$

gives a sharp result, we observe that for $z = re^{\frac{\pi i}{m+p}}$ $(z \in U)$ that

$$\frac{f(z)}{f_{m+p}(z)} = 1 + \frac{z^{m+p}}{H_{m+p+1}} \to 1 - \frac{1}{H_{m+p+1}} \quad as \quad z \to 1^-.$$

This completes the proof. □

Theorem 4. *Let f given by (1) satisfy (7). Then*

$$\Re\Big\{\frac{f_{m+p}(z)}{f(z)}\Big\} > \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}, \tag{13}$$

where $z \in U$, $p > \alpha$, $m = 0,1,2,\ldots$. The result is sharp for every m with an extremal function given by (11).

Proof. Assume that $f \in T(p)$ and satisfies (7). Write

$$w(z) = \Big(1 + \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{p-\alpha}\Big) \times \Big\{\frac{f_{m+p}(z)}{f(z)} - \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}\Big\}$$

$$= 1 - \frac{\Big(1+H_{m+p+1}\Big)\sum_{n=m+1}^{\infty} a_n z^n}{1+\sum_{n=1}^{m} a_n z^n},$$

where H_{m+p+1} is defined in Theorem 3 This yields that

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{(1+H_{m+p+1})\sum_{n=m+1}^{\infty}|a_n|}{2-2\sum_{n=p+1}^{m}|a_n|-(1+H_{m+p+1})\sum_{n=m+1}^{\infty}|a_n|} \leq 1 \quad (z \in U)$$

if and only if

$$2[(1+H_{m+p+1})\sum_{n=m+1}^{\infty}|a_n|] \leq 2-2\sum_{n=2}^{m}|a_n|$$

or

$$\sum_{n=p+1}^{m}|a_n|+(1+H_{m+p+1})\sum_{n=m+1}^{\infty}|a_n| \leq 1, \tag{14}$$

which gives (13). The bound in (13) is sharp for all $m \in \mathbf{N}$ with the extremal function given by (11). This completes the proof. □

Theorem 5. *Let f given by (1) satisfy (7). Then*

$$\Re\left\{\frac{f'(z)}{f'_{m+p}(z)}\right\} \geq 1 - \frac{(m+p+1)(p-\alpha)}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}, \tag{15}$$

where $z \in U$, $p > \alpha$, $m = 0, 1, 2, ...$

Proof. Assume that $f \in T(p)$ satisfies (7). Write

$$w(z) = \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{p-\alpha} \times \left\{\frac{f'(z)}{f'_{m+p}(z)} - \left(1-\frac{(m+p+1)(p-\alpha)}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}\right)\right\}$$

$$= \frac{1+\frac{H_{m+p+1}}{(m+p+1)}\sum_{n=m+1}^{\infty}\frac{n+p}{p}a_nz^n+\sum_{n=1}^{\infty}\frac{n+p}{p}a_nz^n}{1+\sum_{n=1}^{m}\frac{n+p}{p}a_nz^n}$$

$$= 1+\frac{\frac{H_{m+p+1}}{(m+p+1)}\sum_{n=m+1}^{\infty}\frac{n+p}{p}a_nz^n}{1+\sum_{n=1}^{m}\frac{n+p}{p}a_nz^n},$$

where H_{m+p+1} is defined in Theorem 3. This implies

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{\frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}{2-2\sum_{n=1}^{m}\frac{n+p}{p}|a_n|-\frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|} \leq 1 \quad (z \in U)$$

if and only if

$$2[\frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n}{p}|a_n|] \le 2-2\sum_{n=p+1}^{m}\frac{n}{p}|a_n|,$$

i.e.

$$\sum_{n=1}^{m+p}\frac{n}{p}|a_n| + \frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n}{p}|a_n| \le 1.$$

We therefore obtain (15). The result is sharp with functions given by (11). The proof of Theorem 5 is completed. □

Theorem 6. *Let f given by (1) satisfy (7). Then*

$$\Re\Big\{\frac{f'_m(z)}{f'(z)}\Big\} \ge \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{(m+p+1)(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}. \tag{16}$$

Proof. Assume that $f \in T(p)$ satisfies (7). Consider

$$w(z) = \Big((m+p+1) + \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{p-\alpha}\Big) \times \Big\{\frac{f'_m(z)}{f'(z)} - \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{(m+p+1)(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}\Big\}$$

$$= 1 - \frac{(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}a_n z^n}{1+\sum_{n=2}^{m}\frac{n+p}{p}a_n z^n}.$$

This implies that

$$\Big|\frac{w(z)-1}{w(z)+1}\Big| \le \frac{(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}{2-2\sum_{n=1}^{m}\frac{n+p}{p}|a_n| - (1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|} \le 1 \qquad (z \in U)$$

if and only if

$$2[(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|] \le 2-2\sum_{n=1}^{m}\frac{n+p}{p}|a_n|,$$

i.e.

$$\sum_{n=1}^{m}\frac{n+p}{p}|a_n| + (1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n| \le 1.$$

We therefore obtain (16). The result is sharp with functions given by (11). The proof of Theorem 6 is complete. □

In the same manner as the proof of Theorems 3-6, we can show the following results:

Theorem 7. *Let f given by (1) satisfy (8). Then*

$$\Re\left\{\frac{f(z)}{f_{m+p}(z)}\right\} > 1 - \frac{p(p-\alpha)}{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k} \quad (z \in U). \tag{17}$$

The result is sharp for every m with the extremal function

$$f(z) = z^p + \frac{p(p-\alpha)}{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k} \times z^{m+p+1}. \tag{18}$$

Theorem 8. *Let f given by (1) satisfy (8). Then*

$$\Re\left\{\frac{f_{m+p}(z)}{f(z)}\right\} > \frac{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{p(p-\alpha)+(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}. \tag{19}$$

The result is sharp for every m with the extremal function given by (18).

Theorem 9. *Let f given by (1) satisfy (8). Then*

$$\Re\left\{\frac{f'(z)}{f'_{m+p}(z)}\right\} \geq 1 - \frac{p(m+p+1)(p-\alpha)}{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}. \tag{20}$$

Theorem 10. *Let f given by (1) satisfy (8). Then*

$$\Re\left\{\frac{f'_m(z)}{f'(z)}\right\} \geq \frac{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}{p(m+p+1)(p-\alpha)+(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1+\lambda(m+p+1))^k}. \tag{21}$$

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