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REAL RANK OF THE GROUP C^*-ALGEBRAS OF LIE GROUPS

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Abstract: We estimate the real rank of the group C^*-algebras of Lie groups. We consider separately the case of connected solvable Lie groups of type I, the case of non-compact, locally compact CCR groups such as semi-simple or reductive Lie groups, and the case of certain connected or disconnected, solvable Lie groups of non type I.

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1. Introduction

The real rank for C^*-algebras viewed as noncommutative spaces is introduced by Brown and Pedersen [2] as a real analogue to the (topological) stable rank of Rieffel [10] and as an algebraic real dimension, in the commutative case, corresponding to real dimension for spaces. In particular, C^*-algebras with real rank zero have been of great interest in the classification theory of C^*-algebras by K-theory (see, for instance, [12]).

On the other hand, the stable rank of group C^*-algebras of some Lie groups has been estimated by Sheu [13], Takai and the author [20], [21], and the author

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[14], [15], [16], [17]. Recently, the author [19] has obtained a useful formula (an upper bound given below in the list) of real rank for extensions of C^*-algebras, which is an analogue to the formula of stable rank for the extensions in [10].

In this paper we estimate the real rank of the group C^*-algebras of Lie groups. We consider separately the case of connected solvable Lie groups of type I, the case of non-compact, locally compact CCR groups such as semi-simple or reductive Lie groups, and the case of certain connected or disconnected, solvable Lie groups of non type I, using some known structure theorems for those group C^*-algebras. These results generalize those in [18] for the real rank of CCR C^*-algebras by a different method in computation.

Notation. For a C^*-algebra $\mathfrak{A}$, we denote by $\mathfrak{A}_{sa}$ the set of all self-adjoint elements of $\mathfrak{A}$. For a unital C^*-algebra $\mathfrak{A}$, denote by $L_n(\mathfrak{A})$ the set of all elements $(a_j)_{j=1}^n$ of $\mathfrak{A}^n$ such that $\sum_{j=1}^n b_j a_j = 1$ for some $(b_j)_{j=1}^n$ of $\mathfrak{A}^n$. Set $L_n(\mathfrak{A})_{sa} = L_n(\mathfrak{A}) \cap (\mathfrak{A}_{sa})^n$. The real rank of a unital C^*-algebra $\mathfrak{A}$ is defined to be the smallest non-negative integer $n = \mathrm{RR}(\mathfrak{A}) \geq 0$ such that $L_{n+1}(\mathfrak{A})_{sa}$ is dense in $(\mathfrak{A}_{sa})^{n+1}$. If no such integer exists, then set $\mathrm{RR}(\mathfrak{A}) = \infty$. The real rank of a non-unital C^*-algebra $\mathfrak{A}$ is defined by $\mathrm{RR}(\mathfrak{A}) = \mathrm{RR}(\mathfrak{A}^+)$, where $\mathfrak{A}^+$ is the unitization of $\mathfrak{A}$. The connected stable rank of a unital C^*-algebra $\mathfrak{A}$ is defined to be the smallest positive integer $n = \mathrm{csr}(\mathfrak{A}) \geq 1$ such that $L_m(\mathfrak{A})$ is connected for any $m \geq n$. If no such integer exists, then set $\mathrm{csr}(\mathfrak{A}) = \infty$. The connected stable rank of a non-unital C^*-algebra $\mathfrak{A}$ is defined by $\mathrm{csr}(\mathfrak{A}) = \mathrm{csr}(\mathfrak{A}^+)$.

List of Formulae. For $x \in \mathbb{R}$, set $\lceil x \rceil = \lfloor x \rfloor + 1$ if x is not an integer, where $\lfloor x \rfloor$ means the maximum integer $\leq x$, and $\lceil x \rceil = x$ if x is an integer.

Let $C(X)$ be the C^*-algebra of continuous functions on a compact Hausdorff space X. Then

$$\mathrm{RR}(C(X)) = \dim X \quad \text{(see [2, Proposition 1.1]);}$$
$$\mathrm{csr}(C(X)) \leq \lfloor (\dim X + 1)/2 \rfloor + 1 \quad \text{(see [7, Corollary 2.5]).}$$

As for tensor products with the $n \times n$ matrix algebra $M_n(\mathbb{C})$ and the C^*-algebra $\mathbb{K}$ of all compact operators, by [1],

$$\mathrm{RR}(C(X) \otimes M_n(\mathbb{C})) = \lceil \dim X/(2n-1) \rceil \quad \text{and} \quad \mathrm{RR}(\mathfrak{A} \otimes \mathbb{K}) \leq 1$$

for any C^*-algebra $\mathfrak{A}$. For a C^*-algebra $\mathfrak{A}$, by [11, Theorem 4.7],

$$\mathrm{csr}(\mathfrak{A} \otimes M_n(\mathbb{C})) \leq \lceil (\mathrm{csr}(\mathfrak{A}) - 1)/n \rceil + 1,$$

and $\mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq 2$ by [7, Corollary 2.5] or [13, Theorem 3.10].

For a short exact sequence of C^*-algebras: $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$,

$$\max\{\mathrm{RR}(\mathfrak{I}), \mathrm{RR}(\mathfrak{A}/\mathfrak{I})\} \leq \mathrm{RR}(\mathfrak{A}), \quad \text{and}$$
$$\mathrm{RR}(\mathfrak{A}) \leq \max\{\mathrm{RR}(\mathfrak{I}), \mathrm{RR}(\mathfrak{A}/\mathfrak{I}), \mathrm{csr}(\mathfrak{A}/\mathfrak{I}) - 1\}$$

by [4] and [19] respectively. On the other hand, by [13, Theorem 3.9]

$$\mathrm{csr}(\mathfrak{A}) \leq \max\{\mathrm{csr}(\mathfrak{I}), \mathrm{csr}(\mathfrak{A}/\mathfrak{I})\}.$$

Let $\mathfrak{A}$ be a C^*-algebra which has a composition series of closed ideals $\mathfrak{I}_j$. Then we have

$$\sup_j \mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) \leq \mathrm{RR}(\mathfrak{A}) \leq \sup_j \max\{\mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}), \mathrm{csr}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) - 1\}.$$

Sketch of Proof. Using the real rank estimates for C^*-algebra extensions above, we have

$$\mathrm{RR}(\mathfrak{I}_2/\mathfrak{I}_1) \leq \mathrm{RR}(\mathfrak{I}_2) \leq \max\{\mathrm{RR}(\mathfrak{I}_1), \mathrm{RR}(\mathfrak{I}_2/\mathfrak{I}_1), \mathrm{csr}(\mathfrak{I}_2/\mathfrak{I}_1) - 1\},$$
$$\max\{\mathrm{RR}(\mathfrak{I}_2), \mathrm{RR}(\mathfrak{I}_3/\mathfrak{I}_2)\} \leq \mathrm{RR}(\mathfrak{I}_3)$$
$$\leq \max\{\mathrm{RR}(\mathfrak{I}_2), \mathrm{RR}(\mathfrak{I}_3/\mathfrak{I}_2), \mathrm{csr}(\mathfrak{I}_3/\mathfrak{I}_2) - 1\}, \quad \cdots\cdots$$

so that we inductively obtain for each j, with $\mathfrak{I}_0 = \{0\}$,

$$\max_{1\leq k\leq j} \mathrm{RR}(\mathfrak{I}_k/\mathfrak{I}_{k-1}) \leq \mathrm{RR}(\mathfrak{I}_j) \leq \max_{1\leq k\leq j} \max\{\mathrm{RR}(\mathfrak{I}_k/\mathfrak{I}_{k-1}), \mathrm{csr}(\mathfrak{I}_k/\mathfrak{I}_{k-1}) - 1\}.$$

If the union of $\mathfrak{I}_j$ is dense in $\mathfrak{A}$, we obtain the formula above using the density, and if not, we apply for transfinite induction, from which the same estimate is concluded. □

2. The Case of Lie Groups of Type I

Theorem 2.1. *Let $C^*(G)$ be the group C^*-algebra of a connected solvable Lie group G of type I. Then*

$$\mathrm{RR}(C_0(G_1^\wedge)) \leq \mathrm{RR}(C^*(G)) \leq \max\{1, \mathrm{RR}(C_0(G_1^\wedge)), \mathrm{csr}(C_0(G_1^\wedge) - 1\},$$

where $G_1^\wedge$ is the space of all 1-dimensional representations of G, and $C_0(G_1^\wedge)$ is the C^*-algebra of all continuous functions on $G_1^\wedge$ vanishing at infinity. This implies the following estimate:

$$\dim G_1^\wedge \leq \mathrm{RR}(C^*(G)) \leq \max\{1, \dim G_1^\wedge, \lfloor(\dim G_1^\wedge + 1)/2\rfloor\}$$
$$\leq \max\{1, \dim G_1^\wedge\}.$$

Proof. Since G is a connected solvable Lie group, its irreducible representations are either 1-dimensional or infinite dimensional. It follows that $C^*(G)$ has the following exact sequence:

$$0 \to \mathfrak{I} \to C^*(G) \to C_0(G_1^\wedge) \to 0,$$

where the closed ideal $\mathfrak{I}$ has only infinite dimensional representations. Since $C^*(G)$ is of type I, so is $\mathfrak{I}$. Therefore, $\mathfrak{I}$ has a composition series of closed ideals such that its subquotients have continuous trace. Furthermore, note that the spectrum of G (or $C^*(G)$) can be determined by a generalized orbit method which involves the coadjoint action of G on the real dual space of the Lie algebra of G, so that we may assume that the spectrums of those subquotients are paracompact and finite dimensional. By induction using local triviallity, it follows that each subquotient has a composition series of closed ideals such that its subquotients are trivial as a continuous field on their spectrums with fibers $\mathbb{K}$ only.

Since $2n \geq n+1$ for any integer $n \geq 1$, we have

$$n \geq (n+1)/2 \geq \lfloor (n+1)/2 \rfloor.$$

On the other hand, $0 = \lfloor (0+1)/2 \rfloor$. $\square$

Remark. Note that as a space, $G_1^\wedge$ is homeomorphic to $\mathbb{R}^k \times \mathbb{Z}^l$ for some $k, l \geq 0$ since the quotient group $G/[G,G]$ by the commutator $[G,G]$ of G is a connected commutative Lie group and its spectrum is identified with $G_1^\wedge$.

Corollary 2.2. *Let G be a simply connected solvable Lie group of type I. Then*

$$\mathrm{RR}(C^*(G)) = \dim G_1^\wedge.$$

Proof. In this case, $G_1^\wedge$ is homeomorphic to $\mathbb{R}^k$ for some $k \geq 1$ (see [21] and also [20]). $\square$

Example 2.3. Let $\mathbb{T}^n$ be the n-torus, which is also a connected Lie group. Then $C^*(\mathbb{T}^n) \cong C_0(\mathbb{Z}^n)$ by the Fourier transform. Therefore, $\mathrm{RR}(C^*(\mathbb{T}^n)) = 0 = \dim \mathbb{Z}^n$.

Moreover, let G be a compact group. Then $C^*(G)$ is isomorphic to the c_0-direct sum of matrix algebras over $\mathbb{C}$. Thus, $\mathrm{RR}(C^*(G)) = 0$, while $G_1^\wedge$ may be empty. For instance, let $G = \mathbb{Z}_n \rtimes \mathbb{Z}_n$ a semi-direct product of the n-cyclic group $\mathbb{Z}_n$, for which $C^*(G) \cong M_n(\mathbb{C})$.

More generally,

Proposition 2.4. *Let $\mathfrak{A}$ be a C^*-algebra. Then we have*

$$0 \to \mathfrak{I}_{\mathfrak{A}} \to \mathfrak{A} \to C_0(\mathfrak{A}_1^\wedge) \to 0,$$

where $\mathfrak{A}_1^\wedge$ is the space of all 1-dmensional representations of $\mathfrak{A}$. Assume that $\mathfrak{I}_{\mathfrak{A}}$ has a composition series of closed ideals such that its subquotients are stable. Then

$$\dim(\mathfrak{A}_1^\wedge)^+ \leq \mathrm{RR}(\mathfrak{A}) \leq \max\{1, \dim(\mathfrak{A}_1^\wedge)^+\}.$$

Namely, $$\mathrm{RR}(\mathfrak{A}) = \begin{cases} \dim(\mathfrak{A}_1^\wedge)^+ & \textit{if } \dim(\mathfrak{A}_1^\wedge)^+ \geq 1, \\ 0 \textit{ or } 1 & \textit{if } \dim(\mathfrak{A}_1^\wedge)^+ = 0 \textit{ or } -1, \end{cases}$$
where $\dim \emptyset = -1$ *and* $(\mathfrak{A}_1^\wedge)^+$ *means the one-point compactification of* $\mathfrak{A}_1^\wedge$ *if* $\mathfrak{A}_1^\wedge$ *non-compact, otherwise, just the same as* $\mathfrak{A}_1^\wedge$.

Using the above proposition we obtain

Theorem 2.5. *Let G be a non-compact, CCR locally compact group. Then*
$$\dim G_1^\wedge \leq \mathrm{RR}(C^*(G)) \leq \max\{1, \dim G_1^\wedge\}.$$
Let $C_r^*(G)$ be the reduced group C^*-algebra of G. Then $\mathrm{RR}(C_r^*(G)) \leq 1$.

Proof. Note that $C_r^*(G)$ has no finite dimensional irreducible representations. □

Corollary 2.6. *Let G be a non-comppact, reductive Lie group. Then we have* $\mathrm{RR}(C_r^*(G)) \leq 1$.

If the center of G has the spectrum with dimension non-zero, then we have $\mathrm{RR}(C_r^*(G)) = 1$.

Theorem 2.7. *Let G be a non-compact, connected semi-simple Lie group. Then* $\mathrm{RR}(C_r^*(G)) = 1$.

Proof. Note that G is CCR. Thus, $\mathrm{RR}(C_r^*(G)) \leq 1$. On the other hand, $C_r^*(G)$ has a closed ideal with no non-trivial projections (see [14]), so that it has real rank non-zero. Therefore, $\mathrm{RR}(C_r^*(G)) \geq 1$. □

3. The Case of Solvable Lie Groups of Non-Type I

Proposition 3.1. *Let $\mathfrak{A}$ be a C^*-algebra. Assume that the closed ideal $\mathfrak{I}_{\mathfrak{A}}$ defined above has a composition series of closed ideals $\mathfrak{I}_j$ such that its each subquotient $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ is given by one of the following*:
$$\begin{cases} (1) \quad C_0(X_j) \otimes \mathbb{K}, \quad (2) \quad C_0(X_j) \otimes \mathfrak{A}_{\Theta_j}, \\ (3) \quad C_0(X_j) \otimes \mathfrak{A}_{\Theta_j} \otimes \mathbb{K}, \\ (4) \quad C_0(X_j) \otimes \Gamma_0(Y_j, \{\mathfrak{A}_{\Theta_t}\}_{t \in Y_j}) \otimes \mathbb{K}, \end{cases}$$
where X_j is a locally compact Hausdorff space, $\mathfrak{A}_{\Theta_j}$ is a noncommutative torus, and $\Gamma_0(Y_j, \{\mathfrak{A}_{\Theta_t}\}_{t \in Y_j})$ is the C^*-algebra associated with a continuous field of

noncommutative tori $\mathfrak{A}_{\Theta_t}$ *over a locally compact Hausdorff space* Y_j. *Then*

$$\mathrm{RR}(\mathfrak{A}) \le \sup_j \max\{1, \dim(\mathfrak{A}_1^\wedge)^+, \mathrm{RR}(C_0(X_j) \otimes \mathfrak{A}_{\Theta_j}), \mathrm{csr}(C_0(X_j) \otimes \mathfrak{A}_{\Theta_j}) - 1\}.$$

If each $\mathfrak{A}_{\Theta_j}$ *is a simple AT algebra, then*

$$\dim(\mathfrak{A}_1^\wedge)^+ \le \mathrm{RR}(\mathfrak{A}) \le \max\{1, \dim(\mathfrak{A}_1^\wedge)^+\}.$$

Proof. The first estimate follows from applying the formulae in the list above and the estimate

$$\dim(\mathfrak{A}_1^\wedge)^+ \ge \lfloor(\dim(\mathfrak{A}_1^\wedge)^+ + 1)/2\rfloor$$

as shown in the proof of Theorem 2.1.

To see the second estimate, if $\mathfrak{A}_{\Theta_j}$ is a simple AT algebra, then the tensor product $C_0(X_j)\otimes\mathfrak{A}_{\Theta_j}$ is an inductive limit of matrix algebras $M_{n_k}(C_0(X_j\times\mathbb{T}))$ over $C_0(X_j\times\mathbb{T})$ for some $n_k \ge 1$ going to ∞ as $k\to\infty$. Since

$\mathrm{RR}(M_{n_k}(C_0(X_j\times\mathbb{T}))) \le \mathrm{RR}(M_{n_k}(C((X_j\times\mathbb{T})^+)) = \lceil \dim(X_j\times\mathbb{T})^+/(2n_k-1)\rceil$,

which is equal to 1 if n_k is large enough, we have $\mathrm{RR}(C_0(X_j)\otimes\mathfrak{A}_{\Theta_j}) \le 1$. Also,

$$\begin{aligned}\mathrm{csr}(M_{n_k}(C_0(X_j\times\mathbb{T}))) &\le \lceil(\mathrm{csr}(C_0(X_j\times\mathbb{T}))-1)/n_k\rceil + 1\\ &\le \lceil\lfloor(\dim(X_j\times\mathbb{T})^+ + 1)/2\rfloor/n_k\rceil + 1,\end{aligned}$$

which is equal to 2 if n_k is large enough. Hence $\mathrm{csr}(C_0(X_j)\otimes\mathfrak{A}_{\Theta_j}) \le 2$. □

Theorem 3.2. *Let G be a simply connected solvable Lie group that is a semi-direct product* $\mathbb{C}^n \rtimes \mathbb{R}$. *Then*

$$\mathrm{RR}(C^*(G)) = \dim(G_1^\wedge)^+.$$

Proof. By [15], $C^*(G)$ has a finite composition series of closed ideals such that its subquotients are given among $C_0(G_1^\wedge)$, the cases (1) or (3) in the proposition above, where $\mathfrak{A}_{\Theta_j}$ are the crossed products $C(\mathbb{T}^{n_j})\rtimes\mathbb{Z}$ for some $n_j\ge 1$, which are simple AT algebras by [6]. □

Example 3.3. Let $M^5 = \mathbb{C}^2 \rtimes_\alpha \mathbb{R}$ be the Mautner group, where the action α of $\mathbb{R}$ on $\mathbb{C}^2$ is defined by $\alpha_t(z,w) = (e^{2\pi i\theta t}z, e^{2\pi i t}w) \in \mathbb{C}^2$ for $t\in\mathbb{R}$ and θ a fixed irrational number. The group C^*-algebra $C^*(M^5)$ of M^5 has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^4$ of closed ideals with $\mathfrak{I}_0 = \{0\}$ and $\mathfrak{I}_4 = C^*(M^5)$ such that

$$\begin{aligned}\mathfrak{I}_1 &\cong C_0(\mathbb{R}^2)\otimes\mathbb{K}\otimes(C(\mathbb{T})\rtimes_\theta\mathbb{Z}),\\ \mathfrak{I}_j/\mathfrak{I}_{j-1} &\cong C_0(\mathbb{R}\times\mathbb{T})\otimes\mathbb{K} \quad (j=2,3),\\ \mathfrak{I}_4/\mathfrak{I}_3 &\cong C_0(\mathbb{R}),\end{aligned}$$

where $C(\mathbb{T})\rtimes_\theta\mathbb{Z}$ is a simple noncommutative 2-torus corresponding to θ (see [15]), so that it is an AT algebra by [5]. Hence we obtain $\mathrm{RR}(C^*(M^5)) = 1 =$

$\dim \mathbb{R}$, where $(M^5)_1^\wedge$ is homeomorphic to $\mathbb{R}$.

Example 3.4. Let $D^7 = \mathbb{C}^2 \rtimes_\alpha H^3$ be the Dixmier group, where the action α of the Heisenberg Lie group H^3 on $\mathbb{C}^2$ is defined by $\alpha_g(z,w) = (e^{ia}z, e^{ib}w) \in \mathbb{C}^2$ for $g = (c,b,a) \in H^3$ consisting of the following matrices:

$$g = (c,b,a) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

for $a,b,c \in \mathbb{R}$. The group C^*-algebra $C^*(D^7)$ of D^7 has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^4$ of closed ideals with $\mathfrak{I}_0 = \{0\}$ and $\mathfrak{I}_4 = C^*(D^7)$ such that

$$\begin{aligned} \mathfrak{I}_1 &\cong C_0(\mathbb{R}^2) \otimes C^*(H^3_{(1,1)}) \otimes \mathbb{K}, \\ \mathfrak{I}_2/\mathfrak{I}_1 &\cong C_0(\mathbb{R}) \otimes C^*(H^3_{(1,0)}) \otimes \mathbb{K}, \\ \mathfrak{I}_3/\mathfrak{I}_2 &\cong C_0(\mathbb{R}) \otimes C^*(H^3_{(0,1)}) \otimes \mathbb{K}, \\ \mathfrak{I}_4/\mathfrak{I}_3 &\cong C^*(H_3), \end{aligned}$$

where $H^3_{(1,1)}$, $H^3_{(1,0)}$, and $H^3_{(0,1)}$ are the stabilizers corresponding to the points $(1,1)$, $(1,0)$, and $(0,1)$ in $\mathbb{C}^2$ respectively, and their group C^*-algebras are decomposed into extensions of certain commutative C^*-algebras by closed ideals given among the cases (1) or (4) (see [16]). Furthermore, we have

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \to C^*(H^3) \to C_0(\mathbb{R}^2) \to 0.$$

Therefore, we obtain $\mathrm{RR}(C^*(D^7)) = 2 = \dim \mathbb{R}^2$, where $(D^7)_1^\wedge$ is homeomorphic to $\mathbb{R}^2$.

Theorem 3.5. *Let G be a disconnected solvable Lie group that is a semi-direct product $\mathbb{C}^n \rtimes \mathbb{Z}$. Then*

$$\begin{aligned} &\dim(G_1^\wedge)^+ \le \mathrm{RR}(C^*(G)) \\ &\le \max\{1, \dim(G_1^\wedge)^+, \sup_j \mathrm{RR}(C_0(\mathbb{R}^{n_j}) \otimes \mathfrak{A}_{\Theta_j}), \sup_j \mathrm{csr}(C_0(\mathbb{R}^{n_j}) \otimes \mathfrak{A}_{\Theta_j}) - 1\} \\ &\le \max\{\dim(G_1^\wedge)^+, \sup_j \lceil (n_j + 1 + s_j)/(2p_j - 1) \rceil, \sup_j \lceil \lfloor (n_j + 2 + s_j)/2 \rfloor / p_j \rceil\}, \end{aligned}$$

where $G_1^\wedge$ is homeomorphic to $\mathbb{C}^g \times \mathbb{T}$ for some $g \ge 0$, and some $n_j \ge 1$, and each $\mathfrak{A}_{\Theta_j} = C(\mathbb{T}^{s_j}) \rtimes_{\alpha_j} \mathbb{Z}$ for some $s_j \ge 1$ is a simple or non-simple noncommutative torus, where the action α_j is a multi-rotation, and each $p_j \ge 2$ for some $j \ge 1$ is a period corresponding to a non-simple noncommutative torus $\mathfrak{A}_{\Theta_j} = C(\mathbb{T}^{s_j}) \rtimes_{\alpha_j} \mathbb{Z}$.

Furthermore, if $n_j + s_j$ is odd or if $n_j + s_j$ is even and $n_j + s_j + 2 \ge 2p$, then

$$\lceil (n_j + 1 + s_j)/(2p_j - 1) \rceil \ge \lceil \lfloor (n_j + 2 + s_j)/2 \rfloor / p_j \rceil,$$

and its reverse inequality holds if $n_j + s_j$ is even and $n_j + s_j + 2 \leq 2p$.

Proof. By [17], $C^*(G)$ has a finite composition series of closed ideals such that its subquotients are given among $C_0(G_1^\wedge)$, the cases (1) or (2) in the proposition above, where X_j and $\mathfrak{A}_{\Theta_j}$ are given as in the statement above.

To check the last estimate in the statement, if $\mathfrak{A}_{\Theta_j}$ is simple, then it is an AT algebra by [6]. As in the proof of the proposition above, we obtain

$$\mathrm{RR}(C_0(\mathbb{R}^{n_j}) \otimes \mathfrak{A}_{\Theta_j}) \leq 1, \quad \mathrm{csr}(C_0(\mathbb{R}^{n_j}) \otimes \mathfrak{A}_{\Theta_j}) \leq 2.$$

If $\mathfrak{A}_{\Theta_j} = C(\mathbb{T}^{s_j}) \rtimes_{\alpha_j} \mathbb{Z}$ is not simple, then there exists a period $p_j \geq 2$ so that

$$0 \to C_0(\mathbb{R}) \otimes (C(\mathbb{T}^{s_j}) \rtimes_{\alpha_j} \mathbb{Z}_{p_j}) \to \mathfrak{A}_{\Theta_j} \to C(\mathbb{T}^{s_j}) \rtimes_{\alpha_j} \mathbb{Z}_{p_j} \to 0$$

and $C(\mathbb{T}^{s_j}) \rtimes_{\alpha_j} \mathbb{Z}_{p_j} \cong M_{p_j}(C(\mathbb{T}^{s_j}))$. Since

$$\begin{aligned} &\mathrm{RR}(C_0(\mathbb{R}^{n_j+1}) \otimes M_{p_j}(C(\mathbb{T}^{s_j})) = \mathrm{RR}(C_0(\mathbb{R}^{n_j+1} \times \mathbb{T}^{s_j}) \otimes M_{p_j}(\mathbb{C})) \\ &\leq \mathrm{RR}(C((\mathbb{R}^{n_j+1} \times \mathbb{T}^{s_j})^+) \otimes M_{p_j}(\mathbb{C})) = \lceil (n_j + 1 + s_j)/(2p_j - 1) \rceil \end{aligned}$$

and

$$\begin{aligned} &\mathrm{csr}(C_0(\mathbb{R}^{n_j+1}) \otimes M_{p_j}(C(\mathbb{T}^{s_j})) = \mathrm{csr}(C_0(\mathbb{R}^{n_j+1} \times \mathbb{T}^{s_j}) \otimes M_{p_j}(\mathbb{C})) \\ &\leq \lceil (\mathrm{csr}(C_0(\mathbb{R}^{n_j+1} \times \mathbb{T}^{s_j}) - 1)/p_j \rceil + 1 \leq \lceil \lfloor (n_j + 2 + s_j)/2 \rfloor / p_j \rceil + 1, \end{aligned}$$

we obtain the second estimate in the statement above.

Furthermore, we compare the following numbers ≥ 1:

$$\lceil (a+1)/(2p-1) \rceil, \quad \lceil \lfloor (a+2)/2 \rfloor / p \rceil$$

for integers $a \geq 2$ and $p \geq 2$. Assume first that $a = 2a' - 1$ for an integer $a' \geq 2$. Then

$$\lceil \lfloor ((2a'-1)+2)/2 \rfloor / p \rceil = \lceil a'/p \rceil = \lceil 2a'/2p \rceil$$

so that $\lceil ((2a'-1)+1)/(2p-1) \rceil = \lceil 2a'/(2p-1) \rceil \geq \lceil 2a'/2p \rceil$. Next assume that $a = 2a'$ for an integer $a' \geq 1$. Then, by direct computation,

$$(2a'+1)/(2p-1) \leq (2a'+2)/2p = (a'+1)/p = \lfloor (2a'+2)/2 \rfloor / p$$

if and only if $a' + 1 \leq p$. In this case, we have

$$\lceil (2a'+1)/(2p-1) \rceil \leq \lceil \lfloor (2a'+2)/2 \rfloor / p \rceil,$$

and its reverse inequality holds if $a' + 1 \geq p$. □

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