International Journal of Pure and Applied Mathematics
Volume 62 No. 4 2010, 369-379

THE BASIS NUMBER AND MINIMAL CYCLE BASES OF THE STRONG PRODUCT OF PATHS AND CYCLES WITH TENETS

Maref Y.M. Alzoubi
Department of Mathematics
College of Science
Al-Qassim University
P.O. Box 6655, Buraidah, Al-Qassim, KINGDOM OF SAUDI ARABIA
e-mail: maref@yu.edu.jo

Abstract: The basis number of the graph G, denoted by $\mathrm{b}(G)$, is the smallest integer k such that the cycle space, $\mathcal{C}(G)$, has a k-fold basis. A basis is called k-fold basis if each edge of G occurs in at most k of the cycles in the basis. In this paper we prove that the basis number of the strong product of paths with tenets is at most 4, the basis number of the strong product of cycles with tenets is at most 5. Also, we give explicit minimal cycle bases for these graphs.

AMS Subject Classification: 05C38, 15A03
Key Words: cycle basis, cycle space, minimal cycle basis, fold and basis number

1. Introduction

Throughout this paper we consider only finite, connected, simple, unweighted, undirected graphs. For the undefined terms see [9]. Let G be a graph, the set W_E of all subsets of the edge set of G, $E(G)$, form an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X \backslash Y) \cup (Y \backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in W_E$ ($\oplus$ is the ring sum). A cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(W_E, \oplus, \cdot)$ spanned by the cycles of G. It follows that the non-zero elements of $\mathcal{C}(G)$ are cycles or union of edge disjoint cycles. A *cycle basis* of G is a basis of $\mathcal{C}(G)$. A basis of a

Received: April 25, 2010

cycle space, $\mathcal{C}(G)$, is called a *k-fold basis* if each edge of G occurs in at most k of the cycles in the basis. *The basis number* of the graph G, denoted by $\mathrm{b}(G)$, is the smallest integer k such that the cycle space, $\mathcal{C}(G)$, has a k-fold basis. *The dimension of the cycle space* denoted $\dim\mathcal{C}(G)$ is $|E(G)| - |V(G)| + 1$, where denotes the vertex sew of G. The length of a cycle C is the number of its edges and denoted $|C|$. The length of a cycle basis $\mathcal{B}$, denoted $\rho(\mathcal{B})$, is the the sum of the lengths of all cycles in $\mathcal{B}$. A *minimal cycle basis* is a cycle basis of minimum length. In general, the basis number and the minimal cycle bases are not very well behaved under graph operations in the sense that there is no way to extend these properties from subgraphs to the whole graph.

The cycle space and its analysis has various applications in different fields of sciences like electrical engineering (see [12]), structural analysis (see [7]), biology and chemistry (see [8]) and periodic timetabling (see [13]). Some of these applications require cycle bases with special properties (see [14]).

In 1937 [15] MacLane proved that the basis number of a graph G is at least 3 iff the graph is nonplanar. In 1981, Schmeichel [16] introduced the definition of the basis number of a graph and found the basis number of the most known nonplanar graphs like the complete graph, the complete bipartite graphs and he proved the existence of graphs of arbitrary basis number. Then, Banks and Schmeichel proved that the basis number of the n-cube is 4, see [6]. In light of MacLane's ideas and Schmeichel's formal definition we notice that studying the basis number of nonplanar graphs is very interesting.

Graph products are the best natural way to enlarge the space of graphs. In the literature, one can find a lot of graph products such as; Cartesian product, direct product, strong product, semi strong product and lexicographic product. Many authors have investigated the basis number of different graph products; see [1-5,10] and their references.

Definition 1.1. The strong product of two graphs G_1 and G_2, denoted by $G_1 \otimes G_2$, is a graph whose vertex set is $V_1 \times V_2$ and edge set is

$$E = \left\{ \begin{array}{c} (u_1,v_1)(u_2,v_2): \text{ either } [u_1 = u_2 \text{ and } v_1v_2 \in E_2] \\ \text{or } [u_1u_2 \in E_1 \text{ and } v_1 = v_2] \\ \text{or } [u_1u_2 \in E_1 \text{ and } v_1v_2 \in E_2] \end{array} \right\}.$$

Imrich and Stadler [11] constructed minimal cycle bases for the direct, the Cartesian and strong products of graphs. For the strong product of graphs they proved that the minimal cycle bases must contain cycles of length 3 and 4. In this paper, we give explicit examples of such bases.

The purpose of this paper is to investigate the basis number of the strong product of paths and cycles with tenets. Also, we give explicit minimal cycle bases for the strong product of paths and cycles with tenets.

2. The Main Results

In this section, we use the notations P_n, C_n, T_{2m+1} to denote a path, a cycle and a tenet respectively. We will investigate the basis number of the graphs $P_n \otimes T_{2m+1}$ and $C_n \otimes T_{2m+1}$.

Throughout this work, we use P_n to denote the path 1, 2...n and C_n to denote the cycle 1, 2 ...n1, such that $E(P_n) = \{i(i+1)\colon\ 1 \leq i \leq n-1\}$ and $E(C_n) = E(P_n) \cup \{n1\}$. We consider the tenet graph, T_{2m+1}; $m \geq 3$, as a graph consisting of a center vertex a and two concentric m-cycles C_u and C_v in addition to m paths of the form $\alpha u_i v_i$ for each $i = 1, 2, \ldots, m$; where $C_u = u_1 u_2 \ldots u_m u_1$ is the inner cycle and $C_v = v_1 v_2 \ldots v_m v_1$ is the outer cycle.

We consider the graph of $P_n \otimes T_{2m+1}$, where P_n and T_{2m+1} are as defined above. It is clear that T_{2m+1} has the following path as a spanning tree:

$$P_\alpha = \alpha u_1 u_2 \ldots u_m v_m v_{m-1} ... v_2 v_1 = a u_1 u_2 \ldots u_m u_{m+1} ... u_{2m},$$

where $u_{2m-j+1} = v_j$ for all $j = 1, 2, \ldots, m$. Note that $P_n \otimes P_\alpha$ is a subgraph of $P_n \otimes T_{2m+1}$, so we will start by finding a basis for $\mathcal{C}\left(P_n \otimes P_\alpha\right)$ and then enlarge it to get a basis for $\mathcal{C}\left(P_n \otimes T_{2m+1}\right)$. In fact, it is proved by Ali and Marougi [2] that b$(H \otimes G) = 3$, where H is a path of order at least 2 and G is a path of order at least 3, and here we construct a basis similar to their basis. It is easy to see that if we remove the set of edges;

$$E_* = \bigcup_{i=1}^{n-1} \left(\{(i, u_j)\,(i+1, u_{j+1}) : j = 1, 2, \ldots, 2m-1\} \cup \{(i, \alpha)\,(i+1, u_1)\}\right),$$

from the graph $P_n \otimes P_\alpha$ we get a planar subgraph call it F. Let $\mathcal{B}_F$ be the set of all cycles that represent the boundaries of all the finite faces of F then $\mathcal{B}_F$ is a basis of $\mathcal{C}\left(F\right)$ which is a subspace of $\mathcal{C}\left(P_n \otimes P_\alpha\right)$. Note that $|\mathcal{B}_F| = 4mn - 4m$.

Define the set of cycles $\mathcal{B}_* = \mathcal{B}_\alpha \cup \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{i,i+1}\right)$, $\mathcal{B}_\alpha$ and $\mathcal{B}_{i,i+1}$ are defined as follows:

$$\mathcal{B}_\alpha = \left\{(i, \alpha)\,(i+1, u_1)\,(i+1, \alpha)\,(i, \alpha) : i = 1, 2, ..., n-1\right\},$$

$$\mathcal{B}_{i,i+1} = \left\{(i, u_j)\,(i+1, u_{j+1})\,(i+1, u_j)\,(i, u_j) : j = 1, 2, ..., 2m-1\right\},$$

where $|\mathcal{B}_*| = 2mn - 2m$. If we define $\mathcal{B}\left(P_n \otimes P_\alpha\right) = \mathcal{B}_F \cup \mathcal{B}_*$, then $|\mathcal{B}\left(P_n \otimes P_\alpha\right)| = 6mn - 6m$.

Lemma 2.1. (see [2]) *The set* $\mathcal{B}(P_n \otimes P_\alpha) = \mathcal{B}_F \cup \mathcal{B}_*$ *is a 3-fold basis of* $\mathcal{C}(P_n \otimes P_\alpha)$.

Now, let $T_i = i \times T_{2m+1}$ for each $i = 1, 2, ..., n$. Then, T_i is homeomorphic to the planar graph T_{2m+1}, and so $\bigcup_{i=1}^{n} T_i$ is a planar subgraph of $P_n \otimes T_{2m+1}$. We define $\mathcal{B}(T_i)$ to be the set of all cycles that obtained from the boundaries of the finite faces of T_i.

Note. $E(\mathcal{B})$ denotes the set of all edges that used to form the cycles in $\mathcal{B}$, where $\mathcal{B}$ is a set of cycles.

Lemma 2.2. $\mathcal{B}_T = \bigcup_{i=1}^{n} \mathcal{B}(T_i)$ *is linearly independent set of cycles in* $\mathcal{C}(P_n \otimes T_{2m+1})$.

Proof. It is clear that $\mathcal{B}(T_i)$ is a basis of the cycle subspace $\mathcal{C}(T_i)$, hence $\mathcal{B}(T_i)$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$, moreover

$$E(\mathcal{B}(T_i)) \cap E(\mathcal{B}(T_j)) = \phi$$

for all $i \neq j$ for which $1 \leq i, j \leq n$. Therefore, $\bigcup_{i=1}^{n} \mathcal{B}(T_i)$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. □

It is easy to find that $|\mathcal{B}_T| = 2mn$, and $|\mathcal{B}_T \cup \mathcal{B}(P_n \otimes P_\alpha)| = 8mn - 6m$.

Lemma 2.3. $\mathcal{B}_T \cup \mathcal{B}(P_n \otimes P_\alpha)$ *is linearly independent set of cycles in* $\mathcal{C}(P_n \otimes T_{2m+1})$.

Proof. Every linear combination of cycles from $\mathcal{B}_T$ contains an edge of the form $(i, \alpha)(i, u_j)$; $j = 2, ...m$, $(i, u_j)(i, v_j)$; $j = 1, ..., m$, $(i, u_1)(i, u_m)$ or $(i, v_1)(i, v_m)$, where $i = 1, 2, ..., n$, and such an edge does not occur in any linear combination of cycles from $\mathcal{B}(P_n \otimes P_\alpha)$, and so all cycles in $\mathcal{B}_T$ are linearly independent with the cycles of $\mathcal{B}(P_n \otimes P_\alpha)$, hence $\mathcal{B}_T \cup \mathcal{B}(P_n \otimes P_\alpha)$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. □

Define $\mathcal{B}_{uv} = \bigcup_{i=1}^{n-1} \mathcal{B}_{uv}^{i,i+1}$, where

$$\begin{aligned} \mathcal{B}_{uv}^{i,i+1} &= \{\beta_j = (i, u_j)(i+1, v_j)(i, v_j)(i, u_j) : 1 \leq j \leq m-1\} \cup \\ &\quad \{\gamma_j = (i+1, u_j)(i+1, v_j)(i, v_j)(i+1, u_j) : 1 \leq j \leq m-1\}. \end{aligned}$$

It is clear that $|\mathcal{B}_{uv}| = \sum_{i=1}^{n-1} \left|\mathcal{B}_{uv}^{i,i+1}\right| = 2mn - 2m - 2(n-1)$.

Lemma 2.4. $\mathcal{B}_{uv}$ *is linearly independent set of cycles in* $\mathcal{C}(P_n \otimes T_{2m+1})$.

Proof. For each $j = 1, 2, ...m - 1$ the cycles β_j and γ_j share the edge $(i+1, v_j)(i, v_j)$ and their sum is a 4-cycle, and so they are linearly independent and $\{\beta_j, \gamma_j\} \cap \{\beta_k, \gamma_k\} = \phi$ for all $j \neq k$; $1 \leq j, k \leq m-1$. Thus, for fixed i, $\mathcal{B}_{uv}^{i,i+1}$ is linearly independent. To prove that $\mathcal{B}_{uv} = \bigcup_{i=1}^{n-1} \mathcal{B}_{uv}^{i,i+1}$ is linearly independent we use induction on n. For $n = 3$; consider $\mathcal{B}_{uv}^{1,2} \cup \mathcal{B}_{uv}^{2,3}$, then every linear combination of cycles from $\mathcal{B}_{uv}^{2,3}$ contains an edge of the form $(2, u_j)(3, v_j)$, $(2, v_j)(3, v_j)$ or $(2, v_j)(3, u_j)$ and such an edge does not occur in any linear combination of cycles from $\mathcal{B}_{uv}^{1,2}$, thus every linear combination of cycles from $\mathcal{B}_{uv}^{2,3}$ is linearly independent with all cycles in $\mathcal{B}_{uv}^{1,2}$, therefore $\mathcal{B}_{uv}^{1,2} \cup \mathcal{B}_{uv}^{2,3}$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. Suppose that $\bigcup_{i=1}^{k-1} \mathcal{B}_{uv}^{i,i+1}$ is linearly independent, we will prove that $\bigcup_{i=1}^{k} \mathcal{B}_{uv}^{i,i+1}$ is linearly independent. Since $\bigcup_{i=1}^{k} \mathcal{B}_{uv}^{i,i+1} = \left(\bigcup_{i=1}^{k-1} \mathcal{B}_{uv}^{i,i+1} \right) \cup \mathcal{B}_{uv}^{k,k+1}$, every linear combination of cycles from $\mathcal{B}_{uv}^{k,k+1}$ contains an edge of the form $(k, u_j)(k+1, v_j)$, $(k, v_j)(k+1, v_j)$ or $(k, v_j)(k+1, u_j)$ and each of which does not occur in any linear combination of cycles from $\bigcup_{i=1}^{k-1} \mathcal{B}_{uv}^{i,i+1}$, thus all linear combinations of cycles from $\mathcal{B}_{uv}^{k,k+1}$ cannot be obtained from linear combinations of cycles from $\bigcup_{i=1}^{k-1} \mathcal{B}_{uv}^{i,i+1}$, hence $\bigcup_{i=1}^{k} \mathcal{B}_{uv}^{i,i+1}$ is linearly independent set of cycles. Therefore, $\mathcal{B}_{uv} = \bigcup_{i=1}^{n-1} \mathcal{B}_{uv}^{i,i+1}$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. □

Let $\mathcal{B}_{\alpha u} = \bigcup_{i=1}^{n-1} \mathcal{B}_{\alpha u}^{i,i+1}$, where for each $i = 1, 2, ...n-1$, we define $\mathcal{B}_{\alpha u}^{i,i+1}$ as follows;

$$\mathcal{B}_{\alpha u}^{i,i+1} = \left\{ \beta_{\alpha_j} = (i, \alpha)(i+1, u_j)(i, u_j)(i, \alpha) : 2 \leq j \leq m \right\} \cup \left\{ \gamma_{\alpha_j} = (i+1, \alpha)(i+1, u_j)(i, u_j)(i+1, \alpha) : 2 \leq j \leq m \right\}.$$

Lemma 2.5. $\mathcal{B}_{\alpha u} = \bigcup_{i=1}^{n-1} \mathcal{B}_{\alpha u}^{i,i+1}$ *is linearly independent set of cycles in* $\mathcal{C}(P_n \otimes T_{2m+1})$.

Proof. For each $j = 2, ..., m$, β_{α_j} and γ_{α_j} are linearly independent with common edge $(i+1, u_j)(i, u_j)$ and $\beta_{\alpha_j} \oplus \gamma_{\alpha_j}$ is a 4-cycle. So, $\left\{\beta_{\alpha_j}, \gamma_{\alpha_j}\right\}$ is linearly independent set of cycles and $\left\{\beta_{\alpha_j}, \gamma_{\alpha_j}\right\} \cap \{\beta_{\alpha_k}, \gamma_{\alpha_k}\} = \phi$, for all $j \neq k$; $2 \leq j, k \leq m$, thus $\mathcal{B}_{\alpha u}^{i,i+1}$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$.

To prove that $\bigcup_{i=1}^{n-1} \mathcal{B}_{\alpha u}^{i,i+1}$ is linearly independent, it is easy to notice that $\mathcal{B}_{\alpha u}^{i,i+1} \cap \mathcal{B}_{\alpha u}^{k,k+1} = \phi$ if $k \neq i+1$, also if $k = i+1$, then $E\left(\mathcal{B}_{\alpha u}^{i,i+1}\right) \cap E\left(\mathcal{B}_{\alpha u}^{k,k+1}\right)$ generates the zero subspace of $\mathcal{C}(P_n \otimes T_{2m+1})$; i.e., it contains no cycles, and so $\left\langle \mathcal{B}_{\alpha u}^{i,i+1} \right\rangle \cap \left\langle \mathcal{B}_{\alpha u}^{k,k+1} \right\rangle = \langle \mathbf{0} \rangle$. Thus, $\mathcal{B}_{\alpha u}^{i,i+1} \cup \mathcal{B}_{\alpha u}^{k,k+1}$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. Therefore, $\bigcup_{i=1}^{n-1} \mathcal{B}_{\alpha u}^{i,i+1} = \mathcal{B}_{\alpha u}$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. □

Note that we can use induction with similar arguments to those used in the proof of Lemma 2.5 to prove that $\bigcup_{i=1}^{n-1} \mathcal{B}_{\alpha u}^{i,i+1}$ is linearly independent.

Define $\mathcal{B}_{u_1,u_m} = \bigcup_{i=1}^{n-1} \mathcal{B}_{u_1,u_m}^{i,i+1}$, where

$$\mathcal{B}_{u_1,u_m}^{i,i+1} = \{(i,u_1)(i+1,u_1)(i,u_m)(i,u_1),\ (i+1,u_1)(i+1,u_m)(i,u_1)(i+1,u_1)\},$$

and $\mathcal{B}_{v_1,v_m} = \bigcup_{i=1}^{n-1} \mathcal{B}_{v_1,v_m}^{i,i+1}$, where

$$\mathcal{B}_{u_1,u_m}^{i,i+1} = \{(i,v_1)(i+1,v_1)(i,v_m)(i,v_1),\ (i+1,v_1)(i,v_1)(i+1,v_m)(i+1,v_1)\}.$$

Lemma 2.6. *$\mathcal{B}^* = \mathcal{B}_{u_1,u_m} \cup \mathcal{B}_{v_1,v_m}$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$.*

It is easy to notice that the cycles of $\mathcal{B}_{u_1,u_m}$ when we put them together they form boundaries of the finite faces of a planar graph, so they constitute a basis for this planar subgraph, thus $\mathcal{B}_{u_1,u_m}$ is linearly independent set of cycles. The same words work to prove that $\mathcal{B}_{v_1,v_m}$ is linearly independent. Moreover $E(\mathcal{B}_{u_1,u_m}) \cap E(\mathcal{B}_{v_1,v_m}) = \phi$, therefore $\mathcal{B}_{u_1,u_m} \cup \mathcal{B}_{v_1,v_m}$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. □

We define the following set of cycles:

$$\mathcal{B}(P_n \otimes T_{2m+1}) = \mathcal{B}_T \cup \mathcal{B}(P_n \otimes P_\alpha) \cup \mathcal{B}_{uv} \cup \mathcal{B}_{\alpha u} \cup \mathcal{B}^*.$$

Theorem 2.1. *For $m, n \geq 3$, $\mathcal{B}(P_n \otimes T_{2m+1})$ is a minimal cycle basis of $\mathcal{C}(P_n \otimes T_{2m+1})$.*

Proof. It is clear from the way we have defined the cycles in each of $\mathcal{B}_{uv}$ and $\mathcal{B}_{\alpha u}$ that $E(\mathcal{B}_{uv}) \cap E(\mathcal{B}_{\alpha u}) = \phi$, then $\mathcal{B}_{uv} \cup \mathcal{B}_{\alpha u}$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. Moreover, every linear combination of cycles from

$\mathcal{B}^*$ contains an edge of the form

$$(i,u_m)(i+1,u_1)\,,\quad (i,v_m)(i+1,v_1)\,,\quad (i,u_1)(i+1,u_m)\,,\quad (i,v_1)(i+1,v_m)$$

which does not occur in any linear combination of cycles of $\mathcal{B}_{uv} \cup \mathcal{B}_{\alpha u}$, so $\mathcal{B}_{uv} \cup \mathcal{B}_{\alpha u} \cup \mathcal{B}^*$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. Also, we have proved in Lemma 2.3 that $\mathcal{B}_T \cup \mathcal{B}(P_n \otimes P_\alpha)$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. Now, every linear combination of cycles from $\mathcal{B}_{uv} \cup \mathcal{B}_{\alpha u} \cup \mathcal{B}^*$ contains at least an edge from the set $E(P_n \otimes T_{2m+1}) \backslash (E(P_n \otimes P_\alpha) \cup E(\mathcal{B}_T))$ and such an edge does not occur in any linear combination of cycles from $\mathcal{B}_T \cup \mathcal{B}(P_n \otimes P_\alpha)$, thus it is clear that $\mathcal{B}_T \cup \mathcal{B}(P_n \otimes P_\alpha) \cup \mathcal{B}_{uv} \cup \mathcal{B}_{\alpha u} \cup \mathcal{B}^* = \mathcal{B}(P_n \otimes T_{2m+1})$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. Moreover,

$$\begin{aligned}|\mathcal{B}(P_n \otimes T_{2m+1})| &= |\mathcal{B}(P_n \otimes P_\alpha)| + |\mathcal{B}_T| + |\mathcal{B}_{uv}| + |\mathcal{B}_{\alpha u}| + |\mathcal{B}^*| \\ &= (6mn - 6m) + 2mn + (2mn - 2m - 2(n-1)) + (2mn - 2m - 2(n-1)) \\ &\quad + 4n - 1 = 12mn - 10m = dim\mathcal{C}(P_n \otimes T_{2m+1})\,.\end{aligned}$$

Therefore, $\mathcal{B}(P_n \otimes T_{2m+1})$ is a basis for $\mathcal{C}(P_n \otimes T_{2m+1})$.

Furthermore, from the way we have defined the cycles of $\mathcal{B}(P_n \otimes T_{2m+1})$ it is clear that we have found a maximal linearly independent set of triangles and we added a set of linearly independent rectangles to end up with a minimal cycle basis for $\mathcal{C}(P_n \otimes T_{2m+1})$. □

We consider the graph of $C_n \otimes T_{2m+1}$as a graph obtained from the graph of $P_n \otimes T_{2m+1}$ by adding the set of edges

$E_{n1} = E(C_n \otimes T_{2m+1}) \backslash E(P_n \otimes T_{2m+1})$, where $|E_{n1}| = 10m + 1$.

It is clear that

$$|E(C_n \otimes T_{2m+1})| = 14mn + n, \quad |V(C_n \otimes T_{2m+1})| = 2mn + n,$$

and

$$dim\mathcal{C}(C_n \otimes T_{2m+1}) = 12mn + 1.$$

We define the set $\mathcal{B}(n1 \otimes P_\alpha)$ in the same way we have defined the set $\mathcal{B}(P_n \otimes P_\alpha)$ just for the case $P_n = n1$ (a path of order 2). We define the sets of cycles $\mathcal{B}_{uv}^{n,1}$, $\mathcal{B}_{\alpha u}^{n,1}$ and $\mathcal{B}_{n1}^* = \mathcal{B}_{u_1,u_m}^{n,1} \cup \mathcal{B}_{v_1,v_m}^{n,1}$ where these sets are defined in the same way we have defined the sets in Lemmas 2.4, 2.5 and 2.6, respectively, with replacing i by n and $i+1$ by 1. Using similar arguments to those used in the proofs of the above lemmas one can easily prove that $\mathcal{B}(n1 \otimes P_\alpha) \cup \mathcal{B}_{uv}^{n,1} \cup \mathcal{B}_{\alpha u}^{n,1} \cup \mathcal{B}_{n1}^*$ is linearly independent set of cycles. Define the set of cycles

$$\mathcal{B}_{n1} = \mathcal{B}(n1 \otimes P_\alpha) \cup \mathcal{B}_{uv}^{n,1} \cup \mathcal{B}_{\alpha u}^{n,1} \cup \mathcal{B}_{n1}^* \cup \{Q\}\,,$$

where $Q = (n,v_1)(1,v_1)(2,v_1)(1,v_2)(n,v_1)$. It is clear that Q is linearly independent with all other cycles of $\mathcal{B}_{n1}$ being it contains 2 edges that do not join in forming any of the cycles of $\mathcal{B}_{n1}$, and so $\mathcal{B}_{n1}$ is a linearly independent set of cycles, moreover any linear combination of cycles from $\mathcal{B}_{n1}$ contains at least one edge from E_{n1} which does not occur in any linear combination of cycles from $\mathcal{B}(P_n \otimes T_{2m+1})$. Thus, the set $\mathcal{B}(C_n \otimes T_{2m+1}) = \mathcal{B}_{n1} \cup \mathcal{B}(P_n \otimes T_{2m+1})$ is linearly independent set of cycles and we have

$$\begin{aligned} |\mathcal{B}(C_n \otimes T_{2m+1})| &= |\mathcal{B}(P_n \otimes T_{2m+1})| + |\mathcal{B}_{n1}| \\ &= (12mn - 10m) + (10m + 1) = \dim \mathcal{C}(C_n \otimes T_{2m+1}). \end{aligned}$$

Remark. $|\mathcal{B}(n1 \otimes P_\alpha)| = 6m$, $\left|\mathcal{B}^{n,1}_{uv}\right| = 2m - 2 = \left|\mathcal{B}^{n,1}_{\alpha u}\right|$ and $|\mathcal{B}^*_{n1}| = \left|\mathcal{B}^{n,1}_{u_1,u_m} \cup \mathcal{B}^{n,1}_{v_1,v_m}\right| = 2 + 2 = 4$, and so $|\mathcal{B}_{n1}| = 10m + 1$.

Therefore, $\mathcal{B}(C_n \otimes T_{2m+1})$ is a basis of $\mathcal{C}(C_n \otimes T_{2m+1})$. From the way we have defined $\mathcal{B}(C_n \otimes T_{2m+1})$ it is clear that it consists of a maximal linearly independent set of triangles in addition to a linearly independent set of rectangles to end up with a minimal cycle basis for $\mathcal{C}(P_n \otimes T_{2m+1})$. Hence, we have proved the following theorem.

Theorem 2.2. *For each* $m, n \geq 3$, $\mathcal{B}(C_n \otimes T_{2m+1})$ *is a minimal cycle basis of* $\mathcal{C}(C_n \otimes T_{2m+1})$.

Theorem 2.3. *For each* $m, n \geq 3$, *we have* $3 \leq \mathrm{b}(C_n \otimes T_{2m+1}) \leq 5$.

Proof. Since $C_n \otimes T_{2m+1}$ is nonplanar graph then by MacLane's Theorem (see [15]) we have $\mathrm{b}(C_n \otimes T_{2m+1}) \geq 3$. To prove that $\mathrm{b}(C_n \otimes T_{2m+1}) \leq 4$, we exhibit a 5-fold basis for $C_n \otimes T_{2m+1}$. In fact, one can easily see from counting folds of edges that the basis $\mathcal{B}(C_n \otimes T_{2m+1})$ which has been constructed in Theorem 2.2 is a 5-fold basis which completes the proof. □

We are going now to exhibit another basis for the cycle space $\mathcal{C}(P_n \otimes P_\alpha)$ which will be a 4-fold basis. Define $\mathcal{H}_* = \bigcup_{i=1}^{n-1} H_i$; where for $1 \leq i \leq n-2$ we define

$$\begin{aligned} H_i = &\{(i, u_j)(i+1, u_{j-1})(i+2, u_j)(i+1, u_{j+1})(i, u_j) : 1 \leq j \leq 2m-1\} \cup \\ &\{(i, \alpha)(i+1, u_1)(i+2, \alpha)(i+1, \alpha)(i, \alpha)\}, \end{aligned}$$

and for $i = n-1$ we define

$$\begin{aligned} H_{n-1} = &\{(n-1, u_j)(n, u_{j+1})(n, u_j)(n-1, u_j) : 1 \leq j \leq 2m-1\} \cup \\ &\{(n-1, \alpha)(n, u_1)(n, \alpha)(n-1, \alpha)\}. \end{aligned}$$

It is clear that $|H_i| = 2m$ for all $i = 1, 2, \ldots, n-1$, and so $|H| = 2m(n-1)$.

Lemma 2.7. *The set* $\mathcal{B}_H(P_n \otimes P_\alpha) = \mathcal{B}_F \cup \mathcal{H}_*$ *is a 4-fold basis of* $\mathcal{C}(P_n \otimes$

P_α).

Proof. If we fix i with $i = 1, 2, ..., n-1$, then the cycles of H_i are edge disjoint and so the cycles of H_i are linearly independent. Alo, if we fix j for $j = 1, 2, ..., 2m-1$ and let i changes from $i = 1$ to $n-1$, then we get a linearly independent set of cycles in $\mathcal{C}(P_n \otimes P_\alpha)$. Thus, it easy to notice that $\mathcal{H}_*$ is a linearly independent set of cycles in $\mathcal{C}(P_n \otimes P_\alpha)$. Every linear combination of a set of cycles from $\mathcal{H}_*$ contains at least a n edge from the set E_* which does not occur in any linear combination of cycles from the set $\mathcal{B}_F$, so $\mathcal{B}_H(P_n \otimes P_\alpha) = \mathcal{B}_F \cup \mathcal{H}_*$ is a linearly independent set of cycles in $\mathcal{C}(P_n \otimes P_\alpha)$. It is clear that $|\mathcal{B}_H(P_n \otimes P_\alpha)| = \dim \mathcal{C}(P_n \otimes P_\alpha)$, thus $\mathcal{B}_H(P_n \otimes P_\alpha)$ is a basis of $\mathcal{C}(P_n \otimes P_\alpha)$. Moreover, it is easy to see that it is a 4-fold basis $\mathcal{C}(P_n \otimes P_\alpha)$. □

Define the following sets of cycles:

$$\begin{aligned}\mathcal{N}_{uv}^{n-1,n} &= \{(n-1, u_j)(n-1, v_j)(n, v_j)(n-1, u_j) : j = 1, ..., m-1\} \\ &\cup \{(n-1, u_j)(n-1, v_j)(n, u_j)(n, v_j)(n-1, u_j) : j = 1, ..., m-1\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}_{\alpha u}^{n-1} &= \{(n-2, \alpha)(n-1, u_j)(n, \alpha)(n-1, u_{j+1})(n-2, \alpha) : j = 1, ..., m-1\} \\ &\cup \{(n-1, \alpha)(n, u_j)(n-1, u_{j-1})(n-1, \alpha) : j = 3, ..., m\} \\ &\cup \{(n-1, \alpha)(n, u_2)(n-1, u_1)(n, u_1)(n-1, \alpha)\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_{u_1,u_m}^{n-1} &= \{(n-1, u_1)(n, u_m)(n-1, u_m)(n-1, u_1)\} \\ &\cup \{(n-1, u_1)(n-1, u_m)(n, u_1)(n, u_m)(n-1, u_1)\}.\end{aligned}$$

Also, define $\mathcal{N}_{uv} = \left(\bigcup_{i=1}^{n-2} \mathcal{B}_{uv}^{i,i+1}\right) \cup \mathcal{N}_{uv}^{n-1,n}$, $\mathcal{N}_{\alpha u} = \left(\bigcup_{i=1}^{n-2} \mathcal{B}_{\alpha u}^{i,i+1}\right) \cup \mathcal{N}_{\alpha u}^{n-1}$ and $\mathcal{N}^* = \mathcal{N}_{u_1,u_m} \cup \mathcal{B}_{v_1,v_m}$ where $\mathcal{N}_{u_1,u_m} = \left(\bigcup_{i=1}^{n-2} \mathcal{B}_{u_1,u_m}^{i,i+1}\right) \cup N_{u_1,u_m}^{n-1}$.

Theorem 2.3. *For each $m, n \geq 3$, we have $3 \leq \mathrm{b}(P_n \otimes T_{2m+1}) \leq 4$.*

Proof. For each $n, m \geq 3$, the graph $P_n \otimes T_{2m+1}$ is nonplanar, so by MacLane's Theorem (see [15]) we have $\mathrm{b}(P_n \otimes T_{2m+1}) \geq 3$. On the other hand, we define the following set of cycles

$$\mathcal{B}_H(P_n \otimes T_{2m+1}) = \mathcal{B}_T \cup \mathcal{B}_H(P_n \otimes P_\alpha) \cup \mathcal{N}_{uv} \cup \mathcal{N}_{\alpha u} \cup \mathcal{N}^*.$$

Now, in the proof of Lemma 2.3 if we replace $\mathcal{B}(P_n \otimes P_\alpha)$ by $\mathcal{B}_H(P_n \otimes P_\alpha)$, we conclude that $\mathcal{B}_T \cup \mathcal{B}_H(P_n \otimes P_\alpha)$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes P_\alpha)$. Also, we notice that $\mathcal{N}_{uv}$, $\mathcal{N}_{\alpha u}$ and $\mathcal{N}^*$ differ, respectively, from the sets $\mathcal{B}_{uv}$, $\mathcal{B}_{\alpha u}$ and $\mathcal{B}^*$ in the last layers $(n-1)\, n \otimes T_{2m+1}$. So if we mimic the proofs of Lemmas 2.4, 2.5 and 2.6, we can easily prove that $\mathcal{B}_H(P_n \otimes P_\alpha)$ is linearly independent set of cycles in $\mathcal{C}(P_n \otimes T_{2m+1})$. Thus, $\mathcal{B}_H(P_n \otimes T_{2m+1})$ is a basis of $\mathcal{C}(P_n \otimes T_{2m+1})$, being $|\mathcal{B}_H(P_n \otimes T_{2m+1})| = \dim\mathcal{C}(P_n \otimes T_{2m+1})$. Moreover, from the construction of $\mathcal{B}_H(P_n \otimes T_{2m+1})$ one can easily follow the folds of the edges and see that $\mathcal{B}_H(P_n \otimes T_{2m+1})$ is a 4-fold basis of $\mathcal{C}(P_n \otimes T_{2m+1})$ which completes the proof. □

References

[1] A.A. Ali, The basis number of complete multipartite graphs, *Ars. Combinatorial*, **28** (1989), 41-49.

[2] A.A. Ali, T. Marougi, The basis number of the strong product of graphs, *Mutah. Lil-Buhoth wa Al-Dirasat*, **7**, No. 1 (1992), 211-222.

[3] A.S. Alsardary, J. Wojciechowski, The basis number of the powers of the complete graph, *Discrete Math.*, **188**, No-s: 1-3 (1998), 13-25.

[4] M.Y.M. Alzoubi, The basis number of Cartesian product of stars and wheels with different ladders, *Act. Math. Hungar.*, **117**, No. 4 (2007), 373-381.

[5] M.Y.M. Alzoubi, R.A. Al-Ta'ani, The basis number of the lexicographic product of different ladders with paths and cycles, *Kyungpook Mathematical Journal*, **48**, No. 2 (2008), 303-315.

[6] J.A. Banks, E.F. Schmeichel, The basis number of n-cube, *J. Combin. Theory*, Ser. B, **33**, No. 2 (1982), 95-100.

[7] A.C. Cassell, J.C. Henderson, K. Ramachandran, Cycle bases of minimal measure for the structural analysis of skeletal structures by the flexibility method, In: *Proc. Royal Society of London*, Series A, **350** (1976), 61-70.

[8] P.M. Gleiss, *Short Cycles, Minimum Cycle Bases of Graphs from Chemisrty and Biochemistry*, Ph.D. Thesis, Fakultät für Naturwissenschaften und Mathematik der Universtät Wien (2001).

[9] J. Gross, J. Yelle, *Graph Theory and its Applications*, CRC Press, New York (1999).

[10] M.Q. Hailat, M.Y. Alzoubi, The basis number of the composition of graphs, *Istanbul Univ. Fen Fak. Mat. Der.*, **53** (1994), 43-60.

[11] W. Imrich, P.F. Stadler, Minimum cycle basesof product graphs, *Australasian J. Comb.*, **26** (2002), 233-244.

[12] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchungen der linearen Verteilung galvanischer Ströme geführt wird, *Poggendorf Ann. Phy. Chem.*, **72** (1847), 497-508.

[13] Ch. Liebchen, *Periodic Timetable Optimization in Public Transport*, Ph.D. Thesis, TU Berlin (2006).

[14] Ch. Liebchen, R. Rizzi, Classes of cycle bases, *Discrete Applied Mathematics*, **155**, No. 5 (2007), 337-355.

[15] S.F. Maclane, A combinatorial condition for planar graphs, *Fundamenta Math.*, **28** (1937), 22-32.

[16] E.F. Schmeichel, The basis number of graphs, *J. Combin. Theory*, Ser. B, **30**, No. 2 (1981), 123-129.