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ASYMPTOTIC EXPRESSIONS FOR THE EIGENVALUES AND EIGENVECTORS OF A SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH A TURNING POINT

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Abstract: The present paper deals with a system of second order differential equations with turning points.

In particular we consider the second order differential system

$$y''(x) + (\lambda^2 R(x) + Q(x))y(x) = 0, \quad 0 \leq x \leq \pi,$$

where $y(x) = (y_1(x), y_2(x))^T$,

$$Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}, \qquad R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix},$$

$s(x) = x^m s_1(x)$, $t(x) = x^m t_1(x)$, $s_1(x) > 0$, $t_1(x) > 0$ and $p(x)$, $q(x)$, $r(x)$, $s_1(x)$, $t_1(x)$ are real-valued functions having continuous second order derivatives at x, $0 \leq x \leq \pi$, m being a positive constant and λ, a real parameter.

We determine the asymptotic solutions for such a system for large values of the parameter λ and apply these to determine the asymptotic distributions of the eigenvalues and the asymptotic values of the normalizing constants in two cases when the boundary conditions are: (i) the Dirichlet and (ii) the Neumann.

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1. Introduction

J. Horn (see [5]) was the first who obtained the aysmptotic representation of solutions for large parameter λ and the aymptotic representation for the n-th eigenvalue for the differential equation

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) + \left(\lambda^2 r(x) + q(x)\right) y = 0, \tag{1}$$

where $k(x), r(x) > 0$, $a \le x \le b$.

The method adopted by Horn for the solution of the problem was a modification of the W.B.K. method.

The problem with $k(x) = 1$ and $r(x) = x^m r_1(x)$, $r_1(x) > 0$, m positive or negative so that $r(x)$ has either a zero or pole of a given order m at $x = 0$, $0 \le x \le \pi$ was considered by Petrasen [8] and Langer [6]. Langer however considered a general equation in which the singularities of the coefficients of the parameter λ were of more complicated nature. The point at which singularities appear was termed by Langer, a “turning point”. Dorodnicyn [4] considered the equations

$$\left.\begin{aligned} &y''(x) + [\lambda^2 r(x) + q(x)]y(x) = 0,\\ &\text{and}\;\; xy''(x) + p(x)y'(x) + [\lambda^2 r(x) + q(x)]y(x) = 0, \end{aligned}\right\} \tag{2}$$

where $r(x) = x^{\alpha} r_{\alpha}(x)$, $r_{\alpha}(x) > 0$, $0 \le x \le \pi$.

Dorodnicyn obtained the asymptotic representation for the solutions and their derivatives and discussed the problem of the distribution of the eigenvalues and the asymptotic representation of the norms of the eigenfunctions for various values of α and other associated problems.

In the present paper, we consider the second order differential system

$$y''(x) + (\lambda^2 R(x) + Q(x))y(x) = 0, \tag{3}$$

where

$$y(x) = (y_1(x), y_2(x))^T, \qquad R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix},$$

$$Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}, \qquad 0 \le x \le \pi;$$

$p(x)$, $q(x)$, $r(x)$ are real-valued functions having continuous second order derivatives at x, $0 \le x \le \pi$ and $s(x) = x^m s_1(x)$, $t(x) = x^m t_1(x)$, $s_1(x)$, $t_1(x) > 0$, have continuous second order derivatives at x, $0 \le x \le \pi$, m being a positive constant.

The problem considered may be called after Langer, “the singular second order system with a turning point." The reason for choosing $R(x)$ as a diagonal matrix as considered here may be explained in the way as discussed in Neumark [7].

The existence of non-trivial solutions of such problems follows in the usual manner. The problem for which $s(x)$, $t(x) = 1$ is well-known (see Chakraborty [2]). The problem for which $s(x), t(x) > 0$ was touched upon by Chakravarty and Acharyya [3] but the general problem formulated above in (3) does not appear to have been taken as yet. For simplicity of our discussions we consider in the present paper, the system (3) where

$$s(x) = xs_1(x), \quad t(x) = xt_1(x), \quad s_1(x), \quad t_1(x) > \rho^2 > 0 \tag{4}$$

are continuous for x, $0 \leq x \leq \pi$.

The boundary conditions considered in the problem are

$$y_1(0) = y_2(0) = y_1(\pi) = y_2(\pi) = 0 \text{ (the Dirichlet boundary conditions)} \tag{5}$$

or

$$y_1'(0) = y_2'(0) = y_1'(\pi) = y_2'(\pi) = 0 \text{ (the Neumann boundary conditions),} \tag{6}$$

satisfied by the solution $y(x) = (y_1(x),\ y_2(x))^T$ of the system (3) at $x = 0$, $x = \pi$.

As pointed out by Dorodnicyn [4], the choice of the comparing equation is essential for the study of the asymptotic relations for the eigenvalues and eigenvectors of the problem under consideration when $s(x), t(x) > 0$, so that the system (3) is regular, the corresponding comparing equation being the usual Fourier system:

$$y''(x) + \lambda^2 y(x) = 0, \tag{7}$$

where $y(x) = (y_1(x), y_2(x))^T$ (see Chakravarty [2]). But this is not the case for the singular problem with the turning point at $x = 0$. For, the behaviour of the coefficients of the original equation have to be reflected in the coefficients of the comparing equation. Thus to construct the comparing equation we replace by constants those coefficients of the original equation which throughout the interval remain well–behaved and then observe that all the singularities of the coefficients of the original equation are retained in the coefficient of the comparing equation (see Dorodnicyn [4] for a thorough discussions on the construction of the comparing equation).

In the present problem (3) with $s(x), t(x)$ satisfying (4), comparing equation considered is the system of Airy equation

$$u''(x) + xu(x) = 0, \tag{8}$$

where $u(x) = (u_1(x), u_2(x))^T$, satisfying at $x = 0$, the initial conditions

$$\left.\begin{array}{ll} u_1(0) = 1, & u_2(0) = 0, \\ u_1'(0) = 0, & u_2'(0) = 1. \end{array}\right\} \tag{9}$$

In this paper, for the system (3) with $s(x)$, $t(x)$ satisfying conditions (4) we first determine the asymptotic representations for the solutions and their derivatives for large λ and then apply these to determine the asymptotic expressions for the n-th eigenvalue λ_n and the corresponding normalized eigenvector $\psi_n(x)$ under the Dirichlet and Neumann boundary conditions (5) and (6), respectively.

2. A Particular Solution

Let us put

$$w_1(x) = \left(\frac{3}{2}\int_0^x \sqrt{s(z)}dz\right)^{2/3}, \qquad w_2(x) = \left(\frac{3}{2}\int_0^x \sqrt{t(z)}dz\right)^{2/3}, \tag{10}$$

so that

$$\sqrt{s(x)} = w_1'(x)\sqrt{w_1(x)}, \quad \sqrt{t(x)} = w_2'(x)\sqrt{w_2(x)}, \tag{11}$$

$(w_1'(0))^3 = s_1(0) \neq 0$. Then by L'Hospitals rule $(w_2'(0))^3 = t_1(0) \neq 0$, i.e., $w_i'(0) \neq 0$, $i = 1, 2$, and

$$Q_0(x) = \begin{pmatrix} p_0(x) & 0 \\ 0 & q_0^{(}x) \end{pmatrix}. \tag{12}$$

Let

$$L(y) \equiv y''(x) + (\lambda^2 R(x) + Q_0(x))y(x) = 0,$$

where

$$\begin{array}{l} p_0(x) = -\left(w_1'(x)\right)^{1/2} D^2 \left(w_1'(x)\right)^{-1/2}, \\ q_0(x) = -\left(w_2'(x)\right)^{1/2} D^2 \left(w_2'(x)\right)^{-1/2}, \end{array} \tag{13}$$

$D \equiv \frac{d}{dx}$, and moreover let $Q_0(x)$ be related to $Q(x)$ of (3) by means of the relations

$$|p(x) - p_0(x)|, |q(x) - q_0(x)|, |r(x)| \leq c.x^m, \tag{14}$$

where $m > 0$ and c is a non-zero constant.

This ensures that

$$\lim_{x\to 0} |p(x) - p_0(x)|\, x^{-\alpha}, \lim_{x\to 0} |q(x) - q_0(x)|\, x^{-\alpha}, \lim_{x\to 0} |r(x)|\, x^{-\alpha}$$

exist finitely for all $\alpha > 0$.

We establish the following theorem.

Theorem 1. *Let*

$$y_i(x) = c_{i1}\left(w_i'(x)\right)^{-1/2} u_1\left(\lambda^{2/3} w_i(x)\right) + c_{i2}\left(w_i'(x)\right)^{-1/2} u_2\left(\lambda^{2/3} w_i(x)\right),$$

where $i = 1, 2$, c_{ij}, $i, j = 1, 2$ are constants, $(u_1, u_2)^T$ is the solution of the Airy system (8) satisfying at $x = 0$, the conditions (9) and $w_i(x)$, $i = 1, 2$ are those defined in (10). Then $y(x) = (y_1(x),\, y_2(x))^T$ satisfies the system (12).

Proof. Let $A_{ij}(x) = c_{ij},\, (w_i'(x))^{-1/2}$, and

$$\phi_i(x) = \lambda^{2/3} w_i(x), \quad i, j = 1, 2, \tag{16}$$

where c_{ij} are constants and $A_{ij}(x)$ are continuously twice differentiable, since $R(x)$ is so. It therefore follows that

$$\begin{aligned} A_{11}''(x)/A_{11}(x) = A_{12}''(x)/A_{12}(x) = -p_0(x), \\ A_{21}''(x)/A_{21}(x) = A_{22}^{11}(x)/A_{22}(x) = -q_0(x), \end{aligned} \tag{17}$$

where the relations (13) are used.

Also from (16)

$$A_{j1}'(x)/A_{j1}(x) = A_{j2}'(x)/A_{j2}(x) = -\phi_j''(x)/2\phi_j'(x) = -w_j''(x)/2w_j'(x), \quad j = 1, 2.$$

Thus

$$\begin{aligned} 2A_{j1}'(x)\phi_j'(x) + A_{j1}(x)\phi_j''(x) = 0, \\ 2A_{j2}'(x)\phi_j'(x) + A_{j2}(x)\phi_j''(x) = 0, \quad j = 1, 2. \end{aligned} \tag{18}$$

From (11), (14) and (16) it follows that

$$\begin{aligned} & -\phi_1(x)\left(\phi_1'(x)\right)^2 + \lambda^2 s(x) = 0 \\ \text{and} \quad & -\phi_2(x)\left(\phi_2'(x)\right)^2 + \lambda^2 t(x) = 0. \end{aligned} \tag{19}$$

Let us put

$$y_i(x) = A_{i1}(x) u_1(\phi_i(x)) + A_{i2} u_2(\phi_i(x)), \quad i = 1, 2. \tag{20}$$

Using the relations (17), (18) and (19), it follows after some reductions that

$$y_1''(x) + p_0(x) y_1(x) + \lambda^2 s(x) y_1(x) = 0.$$

Similarly

$$y_2''(x) + q_0(x) y_2(x) + \lambda^2 t(x) y_2(x) = 0.$$

Thus $y(x) = (y_1(x), y_2(x))^T$ defined by (15) satisfy the system (12). Hence the theorem is proved. □

Since $(u_1(x), u_2(x))^T$ are the solutions of (8) satisfying at $x = 0$, the relations (9), therefore, putting $\alpha_i(x) = (\alpha_{i1}(x),\, \alpha_{i2}(x))^T$, $i = 1, 2$, where

$$\alpha_{ij}(x) = \left(w_j'(x)\right)^{-1/2} u_i\left(\lambda^{2/3} w_j(x)\right), \quad i, j = 1, 2, \tag{21}$$

it follows from Theorem 1, that $\alpha_1(x)$, $\alpha_2(x)$ are the two linearly independent solutions of (12), where $\alpha_{1j}(0) = (w'_j(0))^{-1/2}$, $\alpha_{2j}(0) = 0$,

$$\alpha'_{1j}(0) = -w''_j(0)/2(w'_j(0))^{3/2}, \quad \alpha'_{2j}(0) = \lambda^{2/3}(w'_j(0))^{1/2}, \quad j = 1, 2. \tag{22}$$

The bilinear concomitant

$$[\alpha_1(x), \alpha_2(x)] = \begin{vmatrix} \alpha_{11}(x) & \alpha_{21}(x) \\ \alpha'_{11}(x) & \alpha'_{21}(x) \end{vmatrix} + \begin{vmatrix} \alpha_{12}(x) & \alpha_{22}(x) \\ \alpha'_{12}(x) & \alpha'_{22}(x) \end{vmatrix},$$

as can be easily verified, is independent of x and is a function of λ alone.

In fact,

$$[\alpha_1(x), \alpha_2(x)] = 2\lambda^{2/3} = \frac{1}{\delta} \quad \text{(say)}. \tag{23}$$

3. Asymptotic Estimates of Certain Integrals

Let

$$f(x) = Q(x) - Q_0(x) = (f_{ij}(x)), \quad i, j = 1, 2, \tag{24}$$

where $Q_0(x)$ is given by (13). Then using conditions (14), it easily follows that

$$\int_0^x f_{11}(z)/\sqrt{s(z)}dz, \quad \int_0^x f_{12}(z)/\sqrt[4]{s(z)t(z)}dz, \quad \int_0^x f_{22}(z)/\sqrt{t(z)}dz, \tag{25}$$

are all finite.

Also by making use of the asymptotic relations for the Airy functions $u_1(x)$, $u_2(x)$, i.e.

$$u_i(x) \cong c_i x^{-\frac{1}{4}} + 0(x^{-\frac{13}{4}}), \quad i = 1, 2, \tag{26}$$

where c_i are different numerical constants for the two functions $u_1(x)$, $u_2(x)$ (see Dorodnicyn [4]), it easily follows that

$$\int_0^x z u_i(z) u_j(z) dz = c x^{3/2} + 0(1), \quad i, j = 1, 2, \tag{27}$$

where c takes different numerical values in the three cases $i = j = 1$, $i = j = 2$, $i = 1$, $j = 2$. Evidently the asymptotic values of $\alpha_{ij}(x) \equiv \alpha_{ij}(x, \lambda)$, $i, j = 1, 2$ of (21) can be determined from the asymptotic values of $u_j(x)$ as given by (26) by replacing x by $\lambda^{2/3} w_j(x)$, $j = 1, 2$, allowing λ to assume large values and keeping x fixed.

In what follows, since exact numerical values are not required in our analysis we allow c to represent different numerical constants arising from that utilization of those associated with the asymptotic relations (26).

Then, as $\int_0^x f_{11}(z)/w_1^3(z)\sqrt{s(z)}dz$ is finite (proved by using (25)) and the inequality $w_1(x) > x\rho^{2/3}$, which follows from (4) and (11), it follows that for large λ,

$$\int_0^x \alpha_{11}^2(z)f_{11}(z)dz = c\lambda^{-1/3}\int_0^x f_{11}(z)/\sqrt{s(z)}dz + 0(\lambda^{-7/3}). \tag{28}$$

Similarly for large λ:

$$\int_0^x \alpha_{21}^2(z)f_{11}(z)dz,\ \int_0^x \alpha_{21}(z)\alpha_{11}(z)f_{11}(z)dz$$
$$= c\lambda^{-1/3}\int_0^x f_{11}(z)/\sqrt{s(z)}dz + 0\left(\lambda^{-7/3}\right), \tag{29}$$

$$\int_0^x \alpha_{22}^2(z)f_{22}(z)dz,\ \int_0^x \alpha_{12}^2(z)f_{22}(z)dz,\ \int_0^x \alpha_{12}(z)\alpha_{22}(z)f_{22}(z)dz$$
$$= c\lambda^{-1/3}\int_0^x f_{22}(z)/\sqrt{t(z)}dz + 0\left(\lambda^{-7/3}\right), \tag{30}$$

$$\int_0^x \alpha_{12}(z)\alpha_{21}(z)f_{12}(z)dz,\ \int_0^x \alpha_{21}(z)\alpha_{22}(z)f_{12}(z)dz,$$

$$\int_0^x \alpha_{11}(z)\alpha_{22}(z)f_{12}(z)dz,\ \int_0^x \alpha_{11}(z)\alpha_{12}(z)f_{12}(z)dz$$
$$= c.\lambda^{-1/3}\int_0^x f_{12}(z)/\sqrt[4]{s(z)t(z)}dz + 0(\lambda^{-7/3}). \tag{31}$$

4. Asymptotic Relations for the Solutions

Let

$$\alpha(z) = \begin{pmatrix} \alpha_{11}(z) & \alpha_{21}(z) \\ \alpha_{12}(z) & \alpha_{22}(z) \end{pmatrix}, \quad \alpha_i(z) = (\alpha_{i1}(z), \alpha_{i2}(z))^T, \quad i = 1, 2, \tag{32}$$

$$y(x,z) = \begin{pmatrix} y_{11}(x,z) & y_{21}(x,z) \\ y_{12}(x,z) & y_{22}(x,z) \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_{21}(z)\alpha_{11}(x) - \alpha_{11}(z)\alpha_{21}(x) & \alpha_{22}(z)\alpha_{11}(x) - \alpha_{12}(z)\alpha_{21}(x) \\ \alpha_{21}(z)\alpha_{12}(x) - \alpha_{11}(z)\alpha_{22}(x) & \alpha_{22}(z)\alpha_{12}(x) - \alpha_{12}(z)\alpha_{22}(x) \end{pmatrix}, \tag{33}$$
$$y_j(x,z) = (y_{1j}(x,z), y_{2j}(x,z))^T, \quad j = 1, 2,$$

where $\alpha_{ij}(x)$ are given by (21).

Further, let

$$\beta(z) = \begin{pmatrix} \beta_{11}(z) & \beta_{21}(z) \\ \beta_{12}(z) & \beta_{22}(z) \end{pmatrix}, \beta_j(z) = (\beta_{j1}(z), \beta_{j2}(z))^T, \quad j = 1, 2, \tag{34}$$

We establish the following theorem.

Theorem 2. *Let the elements of the matrices $Q(x)$, $Q_0(x)$ satisfy the additional conditions (14). Then the two linearly independent solutions of (3) are given by*

$$y_j(x) = (y_{1j}(x), y_{2j}(x))^T = \alpha_j(x) + \beta_j(x), \quad j = 1, 2, \tag{35}$$

where $\alpha_j(x)$, $j = 1, 2$ as defined by (21) are the linearly independent solutions of (12) i.e., $L(y) = 0$ and $\beta_j(x)$, $j = 1, 2$ defined by (34) are the column vectors of

$$\beta(x) = \delta \int_0^x y(x, z) f(z)(\alpha(z) + \beta(z)) dz, \tag{36}$$

$f(z)$, $y(x, z)$ and δ are those defined by (24), (33) and (23), respectively.

Proof. The system (3) can be expressed in the form

$$L(y) = -f(x)y(x), \tag{37}$$

where $L(y)$ and $f(x)$ are those defined by (12) and (24), respectively.

Since $\alpha_j(x)$, $j = 1, 2$ are the solutions of (12), therefore putting

$$y_j(x) = \alpha_j(x) + \beta_j(x), \quad j = 1, 2,$$

we have

$$L(y_j(x)) = L(\alpha_j(x) + \beta_j(x)) = L(\alpha_j(x)) + L(\beta_j(x)) = L(\beta_j(x)), \quad j = 1, 2.$$

Hence from(37),

$$L(\beta_j(x)) = -f(x)y_j(x) = -f(x)(\alpha_j(x) + \beta_j(x)), \quad j = 1, 2. \tag{38}$$

Put

$$\beta_j(x) = k_{j2}(x)\alpha_1(x) + k_{j1}(x)\alpha_2(x), \quad j = 1, 2, \tag{39}$$

where we have to determine $k_{ij}(x)$, $i, j = 1, 2$, suitably.

For this, let

$$K'_{j2}(x)\alpha_1(x) + K'_{j1}(x)\alpha_2(x) = 0, \quad j = 1, 2, \tag{40}$$

so that

$$\beta'_j(x) = K_{j2}(x)\alpha'_1(x) + K_{j1}(x)\alpha'_2(x), \quad j = 1, 2. \tag{41}$$

Also from (38), by a simple use of the expression $L(y)$, the relations (40), (41) we have

$$K'_{j2}(x)\alpha'_1(x) + K'_{j1}(x)\alpha'_2(x) = L(\beta_j) = -f(x)(\alpha_j(x) + \beta_j(x)), \quad j = 1, 2. \tag{42}$$

Solving for $K'_{ij}(x)$, it follows from (40) and (42) that

$$K_{ij}(x) = (-1)^j \delta \int_0^x \left(f(z)\left(\alpha_i(z) + \beta_i(z)\right), \alpha_j(z)\right) dz, \quad i,j = 1,2, \tag{43}$$

where $(f(z)(\alpha_i(z) + \beta_i(z)),\ \alpha_j(z))$ is the inner product of the corresponding vectors and δ is given by (23).

Substituting for $K_{ij}(x)$, given by (43) in (39), it follows that

$$\beta_j(x) = \delta \int_0^x y(x,z) f(z)\left(\alpha_j(z) + \beta_j(z)\right) dz, \quad j = 1,2. \tag{44}$$

Since $\alpha_i(x)$ are bounded in $[0,\pi]$, we apply the usual method of successive approximation to show that (44) has a unique solution in $[0,\pi]$. For this, let

$$\beta(x) = \sum_{n=0}^{\infty} D_n(x), \tag{45}$$

where

$$\begin{aligned} D_0(x) &= \delta \int_0^x y(x,z) f(z)\alpha(z) dz, \\ \text{and} \quad D_{n+1}(x) &= \delta \int_0^x y(x,z) f(z) D_n(z) dz, \quad n \geq 1. \end{aligned} \tag{46}$$

Then making use of the asymptotic relations of Section 3 it easily follows that

$$D_0(x) = \lambda^{-1}\alpha(x) M.P(x) + 0\left(\lambda^{-3}\left\|\alpha(x)\right\|\right), \tag{47}$$

where $||\alpha(x)||$ is the usual sum of the moduli of the elements of the matrix $\alpha(x)$, M is a numerical constant matrix whose elements are those derived from the various numerical constants involved in the asymptotic relations of Section 3. In fact

$$M = \begin{pmatrix} \frac{1}{\sqrt{3}} & K_2^2/2 \\ -K_1^2/2 & -1/\sqrt{3} \end{pmatrix},$$

where

$$K_1 = 3^{1/6}\pi^{-1/2}\Gamma\left(2/3\right), \quad K_2 = 3^{1/6}\pi^{-1/2}\Gamma(1/3),$$

and

$$P(x) = \int_0^x \left[f_{11}(z)/\sqrt{s(z)} + 2f_{12}(z)/\sqrt[4]{s(z)t(z)} + f_{22}(z)/\sqrt{t(z)}\right] dz. \quad (48)$$

Since $y(x,z)f(z)$ is bounded, $||y(x,z)f(z)|| \leq K$, K being a constant, it follows from (46), (47) that

$$\|D_1(x)\| \leq \eta\lambda^{-2}, \quad \text{where } \eta \text{ is a constant}, \quad (49)$$

and generally for $n > 1$,

$$\|D_{n+1}(x)\| \leq \eta K^n \lambda^{-(2n+6)/3} \left|x\right|^n /n!. \quad (50)$$

It therefore follows that the series $\sum_{n=0}^{\infty} D_n(x)$ converges absolutely and uniformly for $x \in [0,\pi]$ and by familiar arguments it converges to $\beta(x)$, the unique solution of the integral equation (36). The theorem therefore follows completely. □

For large λ, $\beta(x) \cong D_0(x)$ and the asymptotic representations of the solutions $y_i(x) = (y_{i1}(x),\, y_{i2}(x))^T$, $i = 1,2$ defined by (35) are given by

$$y(x) = \alpha(x) + D_0(x), \quad (51)$$

$D_0(x)$ being given by (47).

To find the asymptotic estimates of $y_i'(x)$, we have from (35)

$$y_i'(x) = \alpha_i'(x) + \beta_i'(x), \quad i = 1,2,$$

where from (36)

$$\beta'(x) = \delta y(x,x)f(x)\alpha(x) + \delta y(x,x)f(x)\beta(x) + \delta\int_0^x \frac{\partial}{\partial x}\left\{y(x,z)f(z)\alpha(z)\right\} dz + \delta\int_0^x \frac{\partial}{\partial x}\left\{y(x,z)f(z)\beta(z)\right\} dz. \quad (52)$$

Hence, separately estimating the terms of (52) by means of the asymptotic relations as given in Section 3, we have

$$y'(x) = \alpha'(x) + \frac{1}{2}\lambda^{-2/3}y(x,x)f(x)\alpha(x) + \lambda^{-1}\alpha'(x)MP(x) + 0(\lambda^{-3}||\alpha'(x)||). \quad (53)$$

From (22), (47), (51) and the asymptotic relations for the Airy functions $u_j(x)$, we obtain

$$Y_{1j}(0) = y_{1j}(0,\lambda) = (w_j'(0))^{-1/2} + 0(\lambda^{-3}), \quad (54)$$

$$y_{2j}(0) = y_{2j}(0,\lambda) = 0(\lambda^{-3}), \quad j = 1,2,$$

$$y_{2j}(\pi) = c\lambda^{-1/6}(w_j(\pi))^{-1/4}(w_j'(\pi))^{-1/2}\text{Sin}\left\{\frac{2}{3}\lambda(w_j(\pi))^{3/2}+\frac{\pi}{12}\right\}+0(\lambda^{-7/6}),\quad j=1,2, \quad (55)$$

and

$$y_{1j}(\pi) = c\lambda^{-1/6}(w_j(\pi))^{-1/4}(w_j'(\pi))^{-1/2}\text{Cos}\left\{\frac{2}{3}\lambda(w_j(\pi))^{3/2}+\frac{\pi}{12}\right\}+0(\lambda^{-7/6}),$$

where c are different numerical constants and $y(\pi) = y(\pi, \lambda)$.

Similarly, from (22), (53) and the asymptotic relations for the derivatives of the Airy functions $u_j(z)$, it follows that

$$y_{1i}'(0,\lambda) = -\frac{1}{2}w_i''(0)(w_i'(0))^{-3/2}+0(\lambda^{-3}),$$

$$y_{2i}'(0,\lambda) = \lambda^{2/3}\left(w_i'(0)\right)^{1/2}+0(\lambda^{-3}),$$

and

$$y_{1i}'(\pi,\lambda) = c\lambda^{5/6}(w_i'(\pi))^{1/2}(w_i(\pi))^{1/4}\left[-\text{Sin}\left\{\frac{2}{3}\lambda(w_i(\pi))^{3/2}-\frac{\pi}{12}\right\}\right]+0\left(\lambda^{-1/6}\right),$$

$$y_{2i}'(\pi,\lambda) = c\lambda^{5/6}(w_i'(\pi))^{1/2}(w_i(\pi))^{1/4}\left[\text{Cos}\left\{\frac{2}{3}\lambda(w_i(\pi))^{3/2}+\frac{\pi}{12}\right\}\right]+0\left(\lambda^{-1/6}\right),\quad i=1,2, \quad (56)$$

where as before c are different numerical constants.

5. Asymptotic Representations for the Eigenvalues

The Wronskian of the four vectors $\phi_i = (x_i, y_i)^T$, $i = 1, 2, 3, 4$ is defined by

$$W = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1' & x_2' & x_3' & x_4' \\ y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \end{vmatrix} = p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23},$$

where

$$p_{ij} = [\phi_i, \phi_j] = \begin{vmatrix} x_i & y_i \\ x_i' & y_i' \end{vmatrix} + \begin{vmatrix} x_j & y_j \\ x_j' & y_j' \end{vmatrix}$$

are the bilinear concomitants of the vectors ϕ_i, ϕ_j, $i, j = 1, 2, 3, 4$ (see Chakravarty and Acharyya [3]).

It follows from (21) that $\bar{u}_1(x) = (\alpha_{11}(x), 0)^T$, $\bar{u}_2(x) = (\alpha_{21}(x), 0)^T$, $\bar{u}_3(x) = (0, \alpha_{12}(x))^T$, $\bar{u}_4, (x) = (0, \alpha_{22}(x))^T$ are the four solutions of the system (12) and from considerations of the Wronskian of the four vectors $\bar{u}(x)$, $i = 1, 2, 3, 4$, these solutions are linearly independent. In particular, the pairs $\bar{u}_1(x), \bar{u}_2(x)$ and $\bar{u}_3(x), \bar{u}_4(x)$ are linearly independent. Put

$$A(z) = \begin{pmatrix} \alpha_{11}(z) & \alpha_{21}(z) \\ 0 & 0 \end{pmatrix},$$

$$A_i(z) = (\alpha_{i1}(z), 0)^T, \quad i = 1, 2,$$

$$a(z, x) = \alpha_{21}(z)\alpha_{11}(x) - \alpha_{11}(z)\alpha_{21}(x);$$

$$c(z, x) = a(z, x)I_0, \quad I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\beta_j(x)$ as given by (44) changes to

$$B_j(x) = \delta \int_0^x c(z, x) f(z)(A_j(z) + B_j(z))dz, \quad j = 1, 2, \tag{57}$$

where $\delta = \lambda^{-2/3}$.

Similarly, putting

$$\bar{A}(z) = \begin{pmatrix} 0 & 0 \\ \alpha_{12}(z) & \alpha_{22}(z) \end{pmatrix}, \quad \bar{A}_i(z) = (0, \alpha_{i2}(z))^T, \quad i = 1, 2,$$

$$\bar{a}(z, x) = \alpha_{22}(z)\alpha_{12}(x) - \alpha_{12}(z)\alpha_{22}(x),$$

$$\bar{c}(z, x) = \bar{a}(z, x)\bar{I}_0, \quad \bar{I}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

it follows that $\beta_j(x)$ as given by (44) changes to

$$\bar{B}_j(x) = \delta \int_0^x \bar{c}(z, x) f(z)(\bar{A}_j(z) + \bar{B}_j(z))dz, \quad j = 1, 2. \tag{58}$$

Then by Theorem 2:

$$U_j(x) = A_j(x) + B_j(x), \quad j = 1, 2,$$

i.e.,

$$U_1(x) = (U_{11}(x), U_{12}(x))^T = (\alpha_{11}(x) + B_{11}(x), B_{12}(x))^T,$$

$$U_2(x) = (U_{21}(x), U_{22}(x))^T = (\alpha_{21}(x) + B_{21}(x), B_{22}(x))^T \tag{59}$$

are solutions of the system (3).

Similarly

$$\bar{U}_i(x) = \bar{A}_j(x) + \bar{B}_j(x), \quad i = 3, j = 1; i = 4, j = 2,$$

i.e.

$$\bar{U}_3(x) = \left(\bar{U}_{11}(x), \bar{U}_{12}(x)\right)^T = \left(\bar{B}_{11}(x), \alpha_{12}(x) + \bar{B}_{12}(x)\right)^T,$$

$$\bar{U}_4(x) = \left(\bar{U}_{21}(x), \bar{U}_{22}(x)\right)^T = \left(\bar{B}_{21}(x), \alpha_{22}(x) + \bar{B}_{22}(x)\right)^T \tag{60}$$

are also solutions of the system (3).

By considering the wronskian of the four vectors $U_1(x)$, $U_2(x)$, $\bar{U}_3(x)$, $\bar{U}_4(x)$, it follows that these four solutions are linearly independent. Also $U_i(x)$, $\bar{U}_j(x)$, $B_i(x)$, $\bar{B}_i(x)$, $i = 1, 2$; $j = 3, 4$, behave asymptotically in the same way as $y_i(x)$, $\beta_i(x)$, $i = 1, 2$ of Theorem 2.

The asymptotic values of $U_i(x)$, $\bar{U}_j(x)$, $i = 1, 2$, $j = 3, 4$ and their derivatives at $x = 0$ and $x = \pi$ are therefore the same as those given by (54), (55) and (56).

The asymptotic formulae for $\bar{B}'_{ij}(\pi)$, $\bar{B}_{ij}(0)$, $B_{ij}(\pi)$, $\bar{B}_{ij}(\pi)$, $B'_{ij}(0)$, $\bar{B}'_{ij}(0)$, $B_{ij}(0)$, $B'_{ij}(\pi)$ are given by

$$B_{ij}(0),\ \bar{B}_{ij}(0) = 0(1),\ B_{ij}(\pi),\ \bar{B}_{ij}(\pi) = 0\left(1/\lambda\right), \quad \text{as} \quad \lambda \to \infty,$$

and

$$B'_{ij}(0),\ \bar{B}'_{ij}(0),\ B'_{ij}(\pi),\ \bar{B}'_{ij}(\pi) = 0(1), \quad \text{as} \quad \lambda \to \infty. \tag{61}$$

The general solution of the system (3) is given by

$$V(x) = AU_1(x) + BU_2(x) + C\bar{U}_3(x) + D\bar{U}_4(x), \tag{62}$$

A, B, C, D are constants.

When the boundary conditions are Dirichlet as given by (5) it follows from (62) that

$$A = C = 0, \quad BD \neq 0,$$

$$BU_{21}(\pi) + D\bar{B}_{21}(\pi) = 0,$$

$$BB_{12}(\pi) + D\bar{U}_{22}(\pi) = 0.$$

Hence eliminating B, D it follows that

$$U_{21}(\pi)\bar{U}_{22}(\pi) = B_{12}(\pi)\bar{B}_{21}(\pi). \tag{63}$$

To find the asymptotic expressions for the eigenvalues λ_n for large n, we obtain from (63) by using the asymptotic formulae for $U_{ij}(\pi)$, $\bar{U}_{ij}(\pi)$, $\bar{B}_{ij}(\pi)$, $B_{ij}(\pi)$, as given in (55) and (61), the relation

$$\text{Sin}\left[\frac{2}{3}\lambda_n\left(w_1(\pi)\right)^{3/2} + \frac{\pi}{12}\right].\text{Sin}\left[\frac{2}{3}\lambda_n\left(w_2(\pi)\right)^{3/2} + \frac{\pi}{12}\right] = 0\left(\lambda_n^{-4/3}\right). \tag{64}$$

Since eigenvalues form a monotonically increasing sequence with limit point at infinity and $\lambda_n \simeq n$, as $n \to \infty$ (see Chakravarty and Acharyya [1]), it follows from (64) that as $n \to \infty$, either

$$\lambda_n = \pi \left(n - \frac{1}{12}\right) \Big/ \int_0^\pi \sqrt{s(x)}dx + O\left(\frac{1}{n}\right) \tag{65}$$

or

$$\lambda_n = \pi \left(n - \frac{1}{12}\right) \Big/ \int_0^\pi \sqrt{t(x)}dx + O\left(\frac{1}{n}\right), \tag{66}$$

where (65) or (66) gives the asymptotic expressions for the eigenvalue λ_n according as $\int_0^\pi \sqrt{s(x)}dx >$ or $< \int_0^\pi \sqrt{t(x)}dx$.

When the boundary conditions are Neumann as given by (6) so that from (62), $V'(0) = 0$, $V'(\pi) = 0$. Hence from (62) it follows by elimination of A, B, C, D that:

$$\begin{vmatrix} U'_{11}(0) & U'_{21}(0) & 0 & 0 \\ 0 & 0 & \bar{U}'_{12}(0) & \bar{U}'_{22}(0) \\ U'_{11}(\pi) & U'_{21}(\pi) & \bar{B}'_{11}(\pi) & \bar{B}'_{21}(\pi) \\ B'_{12}(\pi) & B'_{22}(\pi) & \bar{U}'_{12}(\pi) & \bar{U}'_{22}(\pi) \end{vmatrix} = 0. \tag{67}$$

Expanding (67) by Laplace's expansion and using the relations $\bar{U}'_{ij}(\pi)$, $B'_{ij}(\pi)$, $\bar{B}'_{ij}(\pi)$, $\bar{U}'_{ij}(0)$ as given in (56) and (61), it follows that the eigenvalues λ_n are determined, as $n \to \infty$, from the following relation

$$\text{Sin}\left[\frac{2}{3}\lambda_n(w_1(\pi))^{3/2} - \pi/12\right] \text{Sin}\left[\frac{2}{3}\lambda_n(w_2(\pi))^{3/2} - \pi/12\right] = 0\left(\lambda_n^{-2/3}\right). \tag{68}$$

From this, as $\lambda_n \simeq n$, as $n \to \infty$, it follows that either

$$\lambda_n = \left(n\pi + \frac{\pi}{12}\right) \Big/ \int_0^\pi \sqrt{s(x)}dx + 0\left(\bar{n}^{2/3}\right), \tag{69}$$

or

$$\lambda_n = \left(n\pi + \frac{\pi}{12}\right) \Big/ \int_0^\pi \sqrt{t(x)}dx + 0\left(\bar{n}^{2/3}\right), \tag{70}$$

where (69) or (70) is the asymptotic expression for the eigenvalue λ_n, as $n \to \infty$ according as

$$\int_0^\pi \sqrt{s(x)}dx > \text{ or } < \int_0^\pi \sqrt{t(x)}dx.$$

6. Asymptotic Representation of the Normalized Eigenvector

It follows from the analysis of Section 5 and from Chakravarty and Acharyya [1], that when the boundary conditions are Dirichlet, the linear combination

$$\phi(x,\lambda_n) = (\phi_1(x,\lambda_n),\phi_2(x,\lambda_n))^T = a_{2n}U_2(x,\lambda_n) - a_{1n}\bar{U}_4(x,\lambda_n), \qquad (71)$$

represents an eigenvector corresponding to the eigenvalue λ_n, where $U_2(x,\lambda_n)$, $\bar{U}_4(x,\lambda_n)$ are those defined in (59) and (60), respectively, and

$$a_{1n} = \int_0^\pi U_2^T(x,\lambda_n)\, R(x) U_2(x,\lambda_n) dx,$$

$$a_{2n} = \int_0^\pi \bar{U}_4^T(x,\lambda_n)\, R(x) \bar{U}_4(x,\lambda_n) dx, \qquad (72)$$

Put

$$a_{3n} = \int_0^\pi U_2^T(x,\lambda_n) R(x) \bar{U}_4(x,\lambda_n) dx \qquad (73)$$

and

$$A_n = a_{1n}a_{2n}(a_{1n} + a_{2n} - 2a_{3n}). \qquad (74)$$

Then

$$\psi(x,\lambda_n) = (\psi_1(x,\lambda_n),\psi_2(x,\lambda_n))^T = A_n^{-\frac{1}{2}}\phi(x,\lambda_n) \qquad (75)$$

is a normalized eigenvector with normalizing constant A_n, normalized in the sense that

$$\int_0^\pi \psi^T(x,\lambda_n) R(x) \psi(x,\lambda_n) dx = 1. \qquad (76)$$

Replacing $\alpha(x)$ by

$$A(x) = \begin{pmatrix} \alpha_{11}(x) & \alpha_{21}(x) \\ 0 & 0 \end{pmatrix},$$

and $D_0(x)$ by $D_0^{(1)}(x)$ in (47), we obtain

$$D_0^{(1)}(x) = \lambda^{-1} \begin{pmatrix} \frac{1}{\sqrt{3}}\alpha_{11}(x) - \frac{K_1^2}{2}\alpha_{21}(x) & \frac{K_2^2}{2}\alpha_{11}(x) - \frac{1}{\sqrt{3}}\alpha_{21}(x) \\ 0 & 0 \end{pmatrix} .P(x) + 0\left(\lambda^{-3}\|A(x)\|\right), \qquad (77)$$

where K_1, K_2, $P(x)$ are those given in (48).

Following Theorem 2, it follows that $B_j(x)$ is asymptotic to $D_0^{(1)}(x)$ as $\lambda_n \to \infty$ and hence $B_{12}(x)$, $B_{22}(x)$ tend asymptotically to zero as $\lambda_n \to \infty$. Similarly, $\bar{B}_{11}(x)$, $\bar{B}_{21}(x)$ also tend asymptotically to zero as $\lambda_n \to \infty$. It then follows that

$$a_{1n} = \int_0^\pi s(x)U_{21}^2(x)dx,$$

$$a_{2n} = \int_0^\pi t(x)\bar{U}_{22}^2(x)dx,$$

and

$$a_{3n} = 0, \quad \text{as} \quad \lambda_n \to \infty. \tag{78}$$

Since $U_j(x)$, $\bar{U}_j(x)$ behave asymptotically in the same way as y_j of Theorem 2, it follows that

$$a_{1n} = \lambda_n^{-1/3} K_2^2 (w_1(\pi))^{3/2} + 0\left(\lambda_n^{-4/3}\right),$$

and

$$a_{2n} = \lambda_n^{-1/3} K_2^2 (w_2(\pi))^{3/2} + 0\left(\lambda_n^{-4/3}\right). \tag{79}$$

Thus

$$A_n^{1/2} = \lambda_n^{-1/2} K_2^3 \left(w_1(\pi)\right)^{3/4} \left(w_2(\pi)\right)^{3/4} \left[\left(w_1(\pi)\right)^{3/2} + \left(w_2(\pi)\right)^{3/2}\right]^{1/2} + 0\left(\lambda_n^{-3/2}\right), \quad \text{as } \lambda_n \to \infty. \tag{80}$$

Therefore, we have

$$\psi(x,\lambda_n) = (\psi_1(x,\lambda_n), \psi_2(x,\lambda_n))^T = \lambda_n^{1/6} K_2^{-1} \left(w_1(\pi) w_2(\pi)\right)^{-3/4} \left\{\left(w_1(\pi)\right)^{3/2} + \left(w_2(\pi)\right)^{3/2}\right\}^{-1/2} \left(\left(w_2(\pi)\right)^{3/2} \alpha_{21}(x), - \left(w_1(\pi)\right)^{3/2} \alpha_{22}(x)\right)^T \left(1 + 0(\lambda_n^{-1})\right), \tag{81}$$

as $\lambda_n \to \infty$, where $K_2 = 3^{1/6}\pi^{-1/2}\Gamma(1/3)$.

This is the asymptotic expression for the normalized eigenvector corresponding to the eigenvalue λ_n in the case when the boundary condition is Dirichlet.

Similarly for the problem when the boundary condition is Neumann, we choose $\phi^*(x,\lambda_n) = b_{2n}U_1(x,\lambda_n) - b_{1n}(x,\lambda_n)$ an eigenvector corresponding to the eigenvalue λ_n, where

$$b_{1n} = \int_0^\pi U_1^T(x,\lambda_n)R(x)U_1(x,\lambda_n)dx,$$

and

$$b_{2n} = \int_0^\pi \bar{U}_3^T(x,\lambda_n)R(x)\bar{U}_3(x,\lambda_n)dx. \tag{82}$$

Let

$$b_{3n} = \int_0^\pi U_1^T(x,\lambda_n)R(x)\bar{U}_3(x,\lambda_n)dx,$$

and

$$B_n = b_{1n}b_{2n}\left(b_{1n} + b_{2n} - 2b_{3n}\right). \tag{83}$$

Then $\psi^*(x,\lambda_n) = B_n^{-1/2}\phi^*(x,\lambda_n)$ is a normalized eigenvector corresponding to the eigenvalue λ_n.

Following the method as described before it readily follows that the asymptotic expression for the normalized eigenvector corresponding to the eigenvalue λ_n for the Neumann boundary conditions is given by

$$\psi^*(x,\lambda n) = \lambda_n^{1/6}K_1^{-1}\left(w_1(\pi)\,w_2(\pi)\right)^{-3/4}\left\{(w_1(\pi))^{3/2} + (w_2(\pi))^{3/2}\right\}$$
$$\left((w_1(\pi))^{3/2}\alpha_{11}(x), -(w_2(\pi))^{3/2}\alpha_{12}(x)\right)^T(1 + 0(\lambda_n^{-1})), \tag{84}$$

as $\lambda_n \to \infty$ and $K_1 = 3^{1/6}\pi^{-1/2}\Gamma(2/3)$.

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