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RANKS OF REDUCIBLE SUBVARIETIES OF PROJECTIVE SPACES

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Abstract: Here we define several different ranks for points of projective spaces with respect to reducible subvarieties.

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1. Introduction

Let $X \subset \mathbb{P}^n$ be a closed and reduced algebraic subset over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal integer k such that there is $S \subset X$ with $\sharp(S) = k$ and $P \in \langle S \rangle$. Since X is non-degenerate, $r_X(P) \le n+1$ for all $P \in \mathbb{P}^n$. If X is integral and $\mathrm{char}(\mathbb{K}) = 0$, then $r_X(P) \le n - \dim(X) + 1$ for all P (see [5], 5.1). In positive characteristic if X is integral, then $r_X(P) \le n - \dim(X) + 2$ for all P and strict inequality holds, unless P is a strange point of X (see [2], Theorem 1).

Here we introduce two generalizations of the X-rank. Both are related to the join of varieties (exactly as the X-rank is related to the secant varieties of X which give the so-called *border rank*). Before introducing them we look at the "semi-classical" case described above with X equidimensional, but not integral. We prove the following general results for 2-components algebraic sets, one of the irreducible components being a linear space.

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Theorem 1. *Assume char*$(\mathbb{K}) = 0$. *Fix integers* n, m *such that* $n \geq 2m + 1 \geq 3$. *Let* $Y \subset \mathbb{P}^n$ *be an integral and non-degenerate* m*-dimensional variety. Let* $L \subset \mathbb{P}^n$ *be a general* m*-dimensional linear subspace. Set* $X := Y \cup L$. *Then* $r_X(P) \leq n - m$ *for all* $P \in \mathbb{P}^n$.

Theorem 2. *Assume char*$(\mathbb{K}) = 0$. *Fix an integer* $n \geq 3$. *Let* $Y \subset \mathbb{P}^n$ *be an integral and non-degenerate curve. Let* $L \subset \mathbb{P}^n$ *be a line such that* $L \cap Y = \emptyset$. *If* $n \geq 4$ *assume that the restriction to* X *of the linear projection* $\ell_L : \mathbb{P}^n \backslash L \to \mathbb{P}^{n-2}$ *induces a morphism birational onto its image. Set* $X := Y \cup L$. *Then* $r_X(P) \leq n - 1$ *for all* $P \in \mathbb{P}^n$.

To extend Theorem 2 to positive characteristic for some curves Y (but not all curves Y) one may use [6]. For instance, it would be sufficient to assume the reflexivity of Y.

Let U, V be reduced closed algebraic subsets of $\mathbb{P}^n$. We recall the notion of *join* $[U; V]$ of U and V (see [1]). If $U = V = \{P\}$ for some $P \in \mathbb{P}^n$, then set $[U; V] := \{P\}$. In all other cases let $[U; V]$ denote the closure in $\mathbb{P}^n$ of the union of all lines $\langle\{P, Q\}\rangle$ with $P \in U$, $Q \in V$ and $P \neq V$. We have $[U; V] = [V; U]$. If $U = U_1 \cup \cdots U_s$, then $[U; V] = \cup_{I=1}^{s}[U_i; V]$. Fix an integer $x \geq 2$. We may define the join $[U_1; \ldots; U_x]$ inductively by the formula $[U_1; \ldots; U_x] := [[U_1; \ldots; U_{x-1}]; U_x]$. In this way the join of x closed algebraic subsets satisfies the usual rules (associativity, commutativity and biadditivity with respect to finite unions). Fix a sequence $\{X_i\}_{i\geq 1}$ of integral and non-degenerate subvarieties of $\mathbb{P}^n$ (repetitions are allowed). Fix $P \in \mathbb{P}^n$. The $\{X_i\}_{i\geq 1}$-rank $r_{\{X_k\}}(P)$ of P is the minimal integer k such that there is $P_i \in X_i$, $1 \leq i \leq k$, such that $P \in \langle\{P_1, \ldots, P_k\}\rangle$. Notice that we prescribe that P_1 in in the first variety X_1 of the string $\{X_i\}_{i\geq 1}$, P_2 is the second, and so on. Hence in several cases P may be contained in the linear span of a proper subset of $\{P_1, \ldots, P_k\}$ (e.g. take $k = 2$, $P \in X_2 \backslash X_2 \cap X_1$) even if k is minimal.

Theorem 3. *Fix a sequence* $\{X_i\}_{i\geq 1}$ *of integral and non-degenerate subvarieties of* $\mathbb{P}^n$. *For every integer* $i \geq 1$ *set* $m_i := \dim(X_i)$ *and* $\mu_i := \min_{1\leq j\leq i} m_j$. *Fix* $P \in \mathbb{P}^n$. *Then:*

(a) $r_{\{X_i\}}(P) \leq n + 2 - \mu_{n+1}$.

(b) If none of the varieties X_i, $1 \leq i \leq n$, *is strange with* P *as one of its strange points, then* $r_{\{X_i\}}(P) \leq n + 1 - \mu_n$.

(c) If either char$(\mathbb{K}) = 0$ *or char*$(\mathbb{K}) > \deg(X_i)$, *for all* $i \in \{1, \ldots, n\}$, *then* $r_{\{X_i\}}(P) \leq n + 1 - \mu_n$.

Now we define a refining of the rank with respect to reducible varieties.

For any closed reduced algebraic set A let $\mathcal{B}(A)$ denote the set of all irreducible components of A. Let $X \subset \mathbb{P}^n$ be a reduced and closed algebraic subset spanning $\mathbb{P}^n$. Set $s := \sharp(\mathcal{B}(X))$. We fix an ordering η of $\mathcal{B}(X)$ and call $X_1, \dots, X_s$ the ordered sequence of the irreducible components of X. Fix $\underline{a} = (a_1, \dots, a_s) \in \mathbb{N}^{\oplus s} \backslash \{(0, \dots, 0)\}$ and $P \in \mathbb{P}^n$. We say that $\underline{a}$ is one of the (X, η)-ranks of P and write $\underline{a} \in \rho_{X,\eta}(P)$ if there are $S_i \in X_i$, $1 \le i \le s$, such that $\sharp(S_i) = a_i$ for all i and $P \in \langle S_1 \cup \cdots \cup S_s \rangle$. Thus $\underline{a} \in \rho_{X,\eta}(P)$ is a non-empty open subset of $\mathbb{N}^{\oplus s} \backslash \{(0, \dots, 0)\}$. Consider the following partial ordering $\le$ of the semigroup $\mathbb{N}^{\oplus s}$. We write $(a_1, \dots, a_s) \le (b_1, \dots, b_s)$ if and only if $a_i \le b_i$ for all i. If $\underline{a} \in \rho_{X,\eta}(P)$ and $\underline{a} \le \underline{b}$ then $\underline{b} \in \rho_{X,\eta}(P)$. Thus to describe $\rho_{X,\eta}(P)$ it is sufficient to describe its minimal element with respect to this partial ordering.

2. The Proofs

For any linear space $A \subset \mathbb{P}^n$ let $\ell_A : \mathbb{P}^n \backslash A \to \mathbb{P}^{n-\dim(A)-1}$ denote the linear projection from A.

Proposition 1. *Let $Y \subset \mathbb{P}^3$ be any closed algebraic subset with pure dimension 1 such that $\langle Y \rangle = \mathbb{P}^3$. Let $L \subset \mathbb{P}^3$ be any line such that $Y \cap L = \emptyset$. Set $X := Y \cup L$. Then $r_X(P) = 1$ for all $P \in X$ and $r_X(P) = 2$ for all $P \in \mathbb{P}^3 \backslash X$.*

Proof. If $P \in X$, then $r_X(P) = 1$. If $P \notin X$, then use that the two plane curves $\ell_P(L)$ and $\ell_P(Y)$ meet. □

Proposition 2. *Fix an integer $n \ge 4$. Let $Y \subset \mathbb{P}^n$ be an integral and non-degenerate curve. Let $L \subset \mathbb{P}^n$ be a line such that $L \cap Y = \emptyset$. Set $X := Y \cup L$. Assume $(\langle L \cup \{P_1, \dots, P_{n-3}\} \rangle \cap Y)_{red} = \{P_1, \dots, P_{n-3}\}$ for general $P_1, \dots, P_{n-3} \in Y$. Then $r_X(P) \le n - 1$ for all $P \in \mathbb{P}^n$.*

Proof. If $P \notin X$, then $r_X(P) = 1$. Assume $P \notin X$. Fix $n-3$ general points $P_1, \dots, P_{n-3} \in X$ and set $V := \langle \{P_1, \dots, P_{n-3} \rangle$. Since X is non-degenerate and each P_i is general, $\dim(V) = n - 4$. If $P \in V$, then $r_X(P) \le n - 3$. Hence we may assume $P \notin V$. By assumption $V \cap L = \emptyset$ and $R := \ell_V(L)$ is a line. Let C denote the closure in $\mathbb{P}^{n-3}$ of $\ell_V(Y \backslash V \cap Y)$. By assumption $C \cap R = \emptyset$. Hence $r_{C \cup R}(\ell_V(P)) = 2$ (Proposition 1). Hence $r_X(P) \le 2 + (n - 3) = n - 1$. □

Proof of Theorem 2. If $n = 3$, then use Proposition 1. Now assume $n \ge 4$. We only need to prove that the assumption of Proposition 2 is satisfied. Fix general $P_1, \dots, P_{n-3} \in Y$ and set $Q_i := \ell_L(P_i) \in \mathbb{P}^{n-2}$, $1 \le i \le n-3$. Set $V := \langle \{P_1, \dots, P_{n-3}\} \rangle$ and $M := \langle \{Q_1, \dots, Q_{n-3}\} \rangle$. Since $P_1, \dots, P_{n-3}$ are general, $Q_1, \dots, Q_{n-3}$ are general in $C := \ell_L(Y)$. Hence M is a codimension 2 linear subspace of $\mathbb{P}^{n-2}$ spanned by $n-3$ general points of C. Since we are in characteristic zero, a general hyperplane section of C is in linearly general position. Hence $C \cap M = \{Q_1, \dots, Q_{n-3}\}$ (even scheme-theoretically). Since $\ell_L|Y$ is birational onto its image, a general fiber of $\ell_L|Y$ is a singleton. Thus the generality of the points $Q_1, \dots, Q_{n-3}$ gives $\langle V \cup L \rangle \cap Y = \{P_1, \dots, P_{n-3}\}$. Hence the assumption of Proposition 2 is satisfied. □

Proof of Theorem 1. If $m = 1$, then we may apply Theorem 2, because the last assumption on L of Theorem 2 is satisfied (use that a general hyperplane section of an integral and non-degenerate curve is in linearly general position). If $m \ge 2$, then a use of Bertini's Theorem (as in the proof of [5], 5.1) reduces to the case $m = 1$ just done. □

Proof of Theorem 3. Part (c) follows from part (b), because a generically finite inseparable morphism has degree at least char($\mathbb{K}$). Hence it is sufficient to prove parts (a) and (b). Let H be a general hyperplane containing P.

(i) Since X_i is connected, the cohomology exact sequence of the exact sequence

$$0 \to \mathcal{I}_{X_i} \to \mathcal{I}_{X_i}(1) \to \mathcal{I}_{X_i \cap H}(1) \to 0$$

gives that the scheme $X_i \cap H$ spans H.

(ii) Here we assume $\mu_{n+1} = 1$ (to get part (a)) or $\mu_n = 1$ (to get part (b)). If $P \in X_i$, then $r_{\{X_i\}}(P) \le i$. Hence to prove parts (a), (b) or (c) under our assumptions we may assume $P \notin X_i$ for all $i \le n+1$ (resp. $P \notin X_i$ for all $i \le n$). First assume char($\mathbb{K}$) $= 0$. By Bertini's Theorem each $X_i \cap H$, $1 \le i \le i+1$, is reduced. Each set $X_i \cap H$ spans H by step (i). It is sufficient to take one point of $X_i \cap H$ for all $i \le n$ to get a set spanning H. Now assume $p :=$ char($\mathbb{K}$) > 0 and that $X_i \cap H$ is not reduced for some i. Since H is a general hyperplane containing P and $P \notin X_i$, this is equivalent to the strangeness of X_i with respect to P. Hence we get part (b) in this case.

(iii) Here we assume $\mu_{n+1} \ge 2$ (to get part (a)) or $\mu_n \ge 2$ (to get part (b)). First assume $P \notin X_i$ for all $i \le n+1$ (to get part (a)) or for all $i \le n$ (to get part (b)). Fix an index $i \le n+1$ (to get (a)) or $i \le n$ (to get (b)). First assume char($\mathbb{K}$) $= 0$. Since $P \notin X_i$ and $\dim(X_i) \ge 2$, Bertini's Theorem gives that $X_i \cap H$ is geometrically integral. Step (i) gives that $X_i \cap H$ spans H. Hence it is sufficient to apply induction on μ_{n+1} or μ_n to H and the string of non-degenerate subvarieties $\{X_i \cap H\}_{i \ge 1}$ in which we take a point as $X_i \cap H$ for all

$i \geq n+2$ (or $i \geq n+1$ for (b)). Now assume $p := \mathrm{char}(\mathbb{K}) > 0$ and that P is not a strange point of X_i for any $i \in \{1, \dots, \}$. Fix $i \in \{1, \dots, n\}$. Since $P \in X_i$, the linear projection $\ell_P : \mathbb{P}^n \setminus \{P\} \to \mathbb{P}^{n-1}$ induces a finite morphism $\ell_P|X_i : X_i \to \mathbb{P}^{n-1}$. Since $m_i \geq 2$, Bertini's Theorem gives that $X_i \cap H$ is geometrically integral (see [4], part 4) of Theorem I.6.3). Fix a general $Q \in (X_i \cap H)_{reg}$. For general H we may take as Q a general point of X_i. Hence $P \notin T_Q X_i$. Hence $P \notin (T_Q X_i) \cap H = T_Q(X_i \cap H)$. Thus P is not a strange point of $X_i \cap H$. The inductive assumption gives $r_{\{X_i \cap H\}}(P) \leq (n-1) - (\mu_n - 1) + 1 = n - \mu_n + 1$. Since $r_{\{\{X_i\}}(P) \leq r_{\{X_i \cap H\}}(P)$, we are done. □

Proposition 3. *Fix a reduced curve $X \subset \mathbb{P}^n$ spanning $\mathbb{P}^n$ and an ordering $\eta = (X_1, \dots, X_s)$ of the irreducible components of X. Assume that each connected component of X spans $\mathbb{P}^n$. Fix any $P \in \mathbb{P}^n$. Assume either char$(\mathbb{K}) = 0$ or char$(\mathbb{K}) > \deg(X_i)$ for all i or that P is not a strange point of some of the curves X_i, $1 \leq i \leq s$. Then there is $(a_1, \dots, a_s) \in \rho_{X,\eta}(P)$ such that $a_1 + \cdots + a_s \leq n$.*

Proof. Adapt steps (i) and (ii) of the proof of Theorem 3. □

Remark 1. Proposition 3 is not true for disconnected curves. For instance, take $n = 3$ and as X the union of two disjoint lines. In this case for any $P \notin X$, the pairs $(2, 1)$ and $(1, 2)$ are the minimal elements of $\rho_{X,\eta}(P)$.

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