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SLANT WEIGHTED TOEPLITZ OPERATOR

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Abstract: Motivated by the definition of a weighted Toeplitz operator and that of a slant Toeplitz operator, we introduce and study the notion of a slant weighted Toeplitz operator.

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1. Introduction

Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$ and $r \leq \frac{\beta_n}{\beta_{n+1}} \leq R$ where $0 < r, R \leq 1$.

Consider the spaces (see [5], [6])

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \,\middle|\, a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

and

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \,\middle|\, a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}.$$

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Then $(L^2(\beta), \|\cdot\|_\beta)$ is a Hilbert space [5] with an orthonormal basis given by $\left\{e_k(z) = \frac{z^k}{\beta_k}\right\}_{k\in Z}$ and with inner product defined by

$$\left\langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \right\rangle = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n \beta_n^2 .$$

Further, $(H^2(\beta), \|\cdot\|_\beta)$ is a subspace [5] of $L^2(\beta)$. Let $P : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. Now, let

$$\begin{aligned} L^\infty(\beta) &= \left\{ \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n | \phi L^2(\beta) \subseteq L^2(\beta) \text{ and there is some} \right. \\ &\qquad \left. c \in \mathbb{R} \text{ such that } \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for all } \ f \in L^2(\beta) \right\}. \end{aligned}$$

Then $L^\infty(\beta)$ is a Banach space (see [5]) with respect to the norm defined by

$$\|\phi\|_\infty = \inf\{c \,|\, \|\phi f\|_\beta \leq c\|f\|_\beta \ \forall \ f \in L^2(\beta)\}.$$

Let $\phi \in L^\infty(\beta)$ and let $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$. Then the weighted Toeplitz operator $T_\phi^{(\beta)}$ on $H^2(\beta)$ with the symbol $\phi \in L^\infty(\beta)$ is defined [5] as:

$$T_\phi^{(\beta)}(f) = P(\phi f) .$$

The above mapping is well defined, if $f \in H^2(\beta) \subset L^2(\beta)$, then by definition, $\phi f \in L^2(\beta)$ and hence $P(\phi f) \in H^2(\beta)$.

Now the matrix of $T_\phi^{(\beta)}$ is given by the following:

$$\begin{aligned} T_\phi^{(\beta)} e_0(z) &= P\left(\sum_{n=-\infty}^{\infty} a_n z^n \frac{z^0}{\beta_0} \right) = P\left(\sum_{n=-\infty}^{\infty} \left(\frac{1}{\beta_0} a_n \beta_n \right) e_n(z) \right) \\ &= \frac{1}{\beta_0} \sum_{n=0}^{\infty} a_n \beta_n e_n(z) , \\ T_\phi^{(\beta)} e_1(z) &= P\left(\sum_{n=-\infty}^{\infty} a_n z^n \frac{z}{\beta_1} \right) = P\left(\sum_{n=-\infty}^{\infty} \left(\frac{a_n \beta_{n+1}}{\beta_1} \right) e_{n+1}(z) \right) \\ &= \frac{1}{\beta_1} \sum_{n=-1}^{\infty} a_n \beta_{n+1} e_{n+1}(z) . \end{aligned}$$

Similarly

$$T_\phi^{(\beta)} e_2(z) = P\left(\sum_{n=-\infty}^{\infty} \left(\frac{a_n \beta_{n+2}}{\beta_2} \right) e_{n+2}(z) \right) \quad \text{and so on.}$$

So the matrix of $T_\phi^{(\beta)}$ is

$$\begin{bmatrix} a_0\frac{\beta_0}{\beta_0} & a_{-1}\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & \cdots \\ a_1\frac{\beta_1}{\beta_0} & a_0\frac{\beta_1}{\beta_1} & a_{-1}\frac{\beta_1}{\beta_2} & \cdots \\ a_2\frac{\beta_2}{\beta_0} & a_1\frac{\beta_2}{\beta_1} & a_0\frac{\beta_2}{\beta_2} & \cdots \\ a_3\frac{\beta_3}{\beta_0} & a_2\frac{\beta_3}{\beta_1} & a_1\frac{\beta_3}{\beta_2} & a_0\frac{\beta_3}{\beta_3} \\ \cdots & a_3\frac{\beta_4}{\beta_1} & a_2\frac{\beta_4}{\beta_2} & a_1\frac{\beta_4}{\beta_3} \end{bmatrix}.$$

Further, if $M_\phi^{(\beta)}$ is the weighted multiplication operator $M_\phi^{(\beta)} : L^2(\beta) \to L^2(\beta)$ defined as

$$M_\phi^{(\beta)} e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_n \beta_{n+k} e_{n+k}(z),$$

then the matrix of $M_\phi^{(\beta)}$ is as follows

$$\left[\begin{array}{c|cccc} \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & a_0\frac{\beta_0}{\beta_0} & a_{-1}\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & \cdots \\ \cdots & a_1\frac{\beta_1}{\beta_0} & a_0\frac{\beta_1}{\beta_1} & a_{-1}\frac{\beta_1}{\beta_2} & \cdots \\ \cdots & a_2\frac{\beta_2}{\beta_0} & a_1\frac{\beta_2}{\beta_1} & a_0\frac{\beta_2}{\beta_2} & \cdots \\ \cdots & a_3\frac{\beta_3}{\beta_0} & a_2\frac{\beta_3}{\beta_2} & a_1\frac{\beta_3}{\beta_2} & \cdots \end{array}\right].$$

2. Slant Weighted Toeplitz Operator

Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be a bounded measurable function on the unit circle $\mathbb{T}$ [4]. We introduce the notion of slant weighted Toeplitz operator as follows:

The slant weighted Toeplitz operator $A_\phi^{(\beta)}$ is an operator on $L^2(\beta)$ defined as

$$A_\phi^{(\beta)} : L^2(\beta) \to L^2(\beta)$$

such that

$$A_\phi^{(\beta)} e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_{2n-k}\beta_n e_n(z)\,.$$

This function is well defined as an everywhere defined matrix transformation is bounded, see [2].

The matrix of $A_\phi^{(\beta)}$ with respect to the basis $\left\{e_k(z) = \frac{z^k}{\beta_k}\right\}_{k\in Z}$ of $L^2(\beta)$ is obtained in the following way:

$$\begin{aligned} A_\phi^{(\beta)} e_0(z) &= \frac{1}{\beta_0} \sum_{n=-\infty}^{\infty} a_{2n}\beta_n e_n(z), \\ A_\phi^{(\beta)} e_1(z) &= \frac{1}{\beta_1} \sum_{n=-\infty}^{\infty} a_{2n-1}\beta_n e_n(z), \\ A_\phi^{(\beta)} e_2(z) &= \frac{1}{\beta_2} \sum_{n=-\infty}^{\infty} a_{2n-2}\beta_n e_n(z) \text{ and so on.} \end{aligned}$$

Hence the matrix of $A_{(\phi)}^{(\beta)}$ is the infinite matrix

$$\left[\begin{array}{c|ccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & a_0\frac{\beta_0}{\beta_0} & a_{-1}\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & a_{-3}\frac{\beta_0}{\beta_3} & \cdots \\ \cdots & a_2\frac{\beta_1}{\beta_0} & a_1\frac{\beta_1}{\beta_1} & a_0\frac{\beta_1}{\beta_2} & a_{-1}\frac{\beta_1}{\beta_3} & \cdots \\ \cdots & a_4\frac{\beta_2}{\beta_0} & a_3\frac{\beta_2}{\beta_1} & a_2\frac{\beta_2}{\beta_2} & a_1\frac{\beta_2}{\beta_3} & \cdots \\ \cdots & a_6\frac{\beta_3}{\beta_0} & a_5\frac{\beta_3}{\beta_1} & a_4\frac{\beta_3}{\beta_2} & a_3\frac{\beta_3}{\beta_3} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right].$$

Thus we see that this is the matrix obtained by simply eliminating every alternate row of the matrix of the weighted multiplication operator $M_\phi^{(\beta)}$ and multiplying each ith row by $\frac{\beta_i}{\beta_{2i}}$. Now consider

$$W : L^2(\beta) \to L^2(\beta)$$

such that

$$We_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z)$$

and

$$We_{2n-1}(z) = 0 \quad \text{for all} \quad n \in Z\,.$$

Then the matrix of W is

$$\left[\begin{array}{c|cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & \frac{\beta_0}{\beta_0} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \frac{\beta_1}{\beta_2} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \frac{\beta_2}{\beta_4} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right]$$

and $||W|| = \sup_n \left|\frac{\beta_n}{\beta_{2n}}\right| \le 1$. We see the following

Theorem 2.1. *(i)* $A_\phi^{(\beta)} = WM_\phi^{(\beta)}$.

(ii) $W = A_1^{(\beta)}$.

Proof. (i) One can easily check that

$$\begin{aligned} WM_\phi^{(\beta)} e_k(z) &= W\left(\frac{1}{\beta_k}\sum_{n=-\infty}^{\infty} a_n \beta_{n+k} e_{n+k}(z)\right) \\ &= W\frac{1}{\beta_k}\sum_{n=-\infty}^{\infty} a_{n-k}\beta_n e_n(z) \\ &= \frac{1}{\beta_k}\sum_{n=-\infty}^{\infty} a_{2n-k}\beta_n e_n(z) \\ &= A_\phi^{(\beta)} e_k(z). \end{aligned}$$

(ii) Take $\phi(z) = 1$ for all $z \in \mathbb{T}$, in the above result, then

$$A_1^{(\beta)} = WM_1^{(\beta)} = W\,.$$

□

Hence we can give now an alternate definition of the slant weighted Toeplitz operator $A_\phi^{(\beta)}$ as

$$A_\phi^{(\beta)}(f) = WM_\phi^{(\beta)}(f) = W\phi f\,.$$

Let W^* denote the Hilbert adjoint operator (see [4]) of W. Then for each f in $L^2(\beta)$ and for each ϕ in $L^\infty(\beta)$,

$$\begin{aligned}\langle W^* e_j(z), e_{2i-1}(z)\rangle &= \langle e_j(z), We_{2i-1}\rangle \\ &= \langle e_j(z), 0\rangle \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\langle W^* e_j(z), e_{2i}(z)\rangle &= \langle e_j(z), We_{2i}(z)\rangle \\ &= \frac{\beta_i}{\beta_{2i}}\langle e_j(z), e_i(z)\rangle \\ &= \begin{cases}\dfrac{\beta_i}{\beta_{2i}} & \text{if } \ i=j, \\ 0 & \text{otherwise}\,.\end{cases}\end{aligned}$$

Therefore

$$\begin{aligned}W^* e_n(z) &= \sum_{k=-\infty}^{\infty}\langle W^* e_n(z), e_{2k}(z)\rangle e_{2k}(z) \\ &= \sum_{k=-\infty}^{\infty}\langle e_n(z), We_{2k}(z)\rangle e_{2k}(z) \\ &= \sum_{k=-\infty}^{\infty}\left\langle e_n(z), \frac{\beta_k}{\beta_{2k}}\, e_k(z)\right\rangle e_{2k}(z) \\ &= \frac{\beta_n}{\beta_{2n}} e_{2n}(z) \text{ for each } \ n\in Z.\end{aligned}$$

Thus $W^* e_n(z) = \dfrac{\beta_n}{\beta_{2n}}\, e_{2n}(z)$.

Theorem 2.2. *(i) P reduces W.*

(ii) $WM_z^{(\beta)}W^* = 0$.

Proof. (i)

$$\begin{aligned}PWe_{2n}(z) &= P\frac{\beta_n}{\beta_{2n}} e_n(z) \\ &= \frac{\beta_n}{\beta_{2n}} e_n(z)\end{aligned}$$

$$\begin{aligned} &= We_{2n}(z) \\ &= WPe_{2n}(z) \quad \text{for all} \ \ n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Also,

$$\begin{aligned} PW_{e_{2n+1}}(z) = PO = O &= We_{2n+1}(z) \\ &= WPe_{2n+1}(z) \text{ for all } \ n \in \mathbb{N}\{0\}. \end{aligned}$$

For negative integers n,

$$\begin{aligned} PWe_n(z) &= P(We_n(z)) \\ &= 0 \\ &= W(0) = WPe_n(z). \end{aligned}$$

Therefore $PW = WP$.

Again for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} PW^*e_n(z) = P\frac{\beta_n}{\beta_{2n}}e_{2n}(z) = \frac{\beta_n}{\beta_{2n}}e_{2n}(z) &= W^*e_n(z) \\ &= W^*Pe_n(z). \end{aligned}$$

For negative integers n,

$$\begin{aligned} PW^*e_n(z) &= P\left(\frac{\beta_n}{\beta_{2n}}e_{2n}(z)\right) \\ &= 0 = W^*(0) \\ &= W^*Pe_n(z). \end{aligned}$$

Thus $PW^* = W^*P$.

Hence P reduces W.

(ii) Let $f \in L^2(\beta)$. Then W^*f lies in the closed span of $\{e_{2k}(z) : k \in \mathbb{Z}\}$ and so $z(W^*f)$ belongs to the closed span of $\{e_{2k-1}(z) : k \in \mathbb{Z}\}$. Therefore, $W(zW^*f) = 0$ for each f in $L^2(\beta)$. Hence $WM_zW^* = 0$. □

Further, since $M_\phi^{(\beta)}$ is a bounded operator on $L^2(\beta)$ and $A_\phi^{(\beta)} = WM_\phi^{(\beta)}$, we have

$$\|A_\phi^{(\beta)}\| \leq \|W\|\|M_\phi^{(\beta)}\| \leq \|M_\phi^{(\beta)}\|\,.$$

Hence $A_\phi^{(\beta)}$ is also a bounded operator on $L^2(\beta)$.

Lemma. *If $h(z)$ is an $L^2(\beta)$ function, then $h(z^2)$ is also an $L^2(\beta)$ function. Further,*

$$||h(z^2)||_\beta \leq ||h(z)||\beta.$$

Proof. Let $h(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^n$ be an $L^2(\beta)$ function. Hence

$$||h(z)\|_\beta^2 = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty. \tag{1}$$

Also, then

$$h(z^2) = \sum_{n=-\infty}^{\infty} \alpha_n z^{2n} = \sum_{n=-\infty}^{\infty} \alpha_n \beta_{2n} e_{2n}.$$

Hence

$$\begin{aligned}\|h(z^2)\|_\beta^2 &= \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_{2n}^2 \\ &\leq \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_{2n}^2 \times \frac{\beta_n^2}{\beta_{2n}^2} \quad \text{as} \quad \frac{\beta_n}{\beta_{n+1}} \leq 1 \ \forall\ n. \\ &= \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty \quad \text{by} \ \ (1).\end{aligned}$$

Therefore $h(z^2)$ is also an $L^2(\beta)$ function.

Further we observe that

$$\|h(z^2)\|_\beta^2 \leq \|h(z)\|_\beta^2,$$

or equivalently,

$$\|h(z^2)\|_\beta \leq \|h(z)\|_\beta. \tag{2}$$

□

Theorem 2.3. *If $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a sequence of positive numbers with $\beta_0 = 1$ and $r \leq \frac{\beta_n}{\beta_{n+1}} \leq R$ where $0 < r, R \leq 1$, then a bounded operator T on $L^2(\beta)$ is a slant weighted Toeplitz operator if and only if*

$$M_z^{(\beta)} T = T M_{z^2}^{(\beta)}.$$

Proof. Let $T = A_\phi^{(\beta)}$ be a slant weighted Toeplitz operator. Then

$$\begin{aligned}M_z^{(\beta)} T e_k(z) &= M_z^{(\beta)} A_\phi^{(\beta)} e_k(z) \\ &= M_z^{(\beta)} \left(\sum_{n=-\infty}^{\infty} \frac{a_{2n-k}}{\beta_k} \beta_n e_n(z) \right) \\ &= \sum_{n=-\infty}^{\infty} \frac{a_{2n-k}}{\beta_k} \beta_{n+1} e_{n+1}(z).\end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned} TM_{z^2}^{(\beta)}e_k(z) = A_\phi^{(\beta)}M_{z^2}^{(\beta)}e_k(z) &= A_\phi^{(\beta)}\left(\frac{1}{\beta_k}\beta_{k+2}e_{k+2}(z)\right) \\ &= \frac{\beta_{k+2}}{\beta_k}\frac{1}{\beta_{k+2}}\sum_{n=-\infty}^{\infty} a_{2n-k-2}\beta_n e_n(z) \\ &= \frac{1}{\beta_k}\sum_{n=-\infty}^{\infty} a_{2n-k}\beta_{n+1}e_{n+1}(z). \end{aligned} \tag{4}$$

From (3) and (4) we get

$$M_z^{(\beta)}T_{e_k}(z) = TM_{z^2}^{(\beta)}e_k(z) \text{ for all } k \in \mathbb{Z}.$$

Hence $M_z^{(\beta)}T = TM_{z^2}^{(\beta)}$.

Now for sufficiency, suppose T satisfies $M_z^{(\beta)}T = TM_{z^2}^{(\beta)}$. Then we claim that $T = A_\phi^{(\beta)}$ for some ϕ in $L^\infty(\beta)$.

Let $f(z) = (T1)(z^2)$ and

$$g(z) = \bar{z}(Tz)(z^2).$$

Then [4] for any h in $L^2(\beta)$,

$$M_{h(z)}T = TM_{h(z^2)}.$$

Therefore for all f in $L^2(\beta)$,

$$M_{h(z)}Tf = TM_{h(z^2)}f.$$

Thus

$$h \cdot (Tf) = T(h(z^2))f. \tag{5}$$

Taking $f = 1$ in (5) we get

$$h \cdot (T1) = T(h(z^2)1). \tag{6}$$

Therefore,

$$\begin{aligned} \|h \cdot (T1)\|_\beta &= \|T(h(z^2))\|_\beta \\ &\leq \|T\|\|h(z^2)\|_\beta \\ &\leq \|T\|\|h(z)\|_\beta \quad \text{by (2)}. \end{aligned}$$

Hence the multiplication on $L^2(\beta)$ by the function $T1$ is bounded, which means that $T1$ is a bounded function. Hence $f = (T1)(z^2)$ is bounded by the lemma above.

Similarly, we get that

$$\|\bar{z}h(z)(Tz)\|_\beta = \|\bar{z}(Tz)h(z^2)\|_\beta \leq \|T\|\|h\|_\beta. \tag{7}$$

So $g = \bar{z}(Tz)(z^2)$ is bounded. Hence $\phi = f + g$ is an $L^\infty(\beta)$ function.

Now we show that $T = A_\phi^{(\beta)}$. Let F be a function in $L^2(\beta)$. We can write

$$F = P(z^2) + zQ(z^2) \quad \text{for some } P \text{ and } Q \text{ in } L^2(\beta).$$

Therefore,

$$\begin{aligned} A_\phi^{(\beta)} F &= W M_\phi^{(\beta)} F = W(\phi F) \\ &= W\left\{[(T1)(z^2) + \bar{z}(Tz)(z^2)][P(z^2) + zQ(z^2)]\right\} \\ &= W\left\{[(T1)(z^2)P(z^2) + (T1)(z^2)zQ(z^2) + \bar{z}(Tz)(z^2)P(z^2) \right. \\ &\quad \left. +(Tz)(z^2)Q(z^2)]\right\}. \end{aligned}$$

As W eliminates terms with odd Fourier coefficients, we get that

$$\begin{aligned} A_\phi^{(\beta)} F &= W[(T1)(z^2)P(z^2) + (z^2)(Tz)Q(z^2)] \\ &= (T1)P(z^2) + (Tz)Q(z^2) \\ &= T(P(z^2) + zQ(z^2)) \\ &= TF. \end{aligned}$$

Thus $T = A_\phi^{(\beta)}$. □

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