International Journal of Pure and Applied Mathematics
Volume 62 No. 4 2010, 443-452

EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract: In this paper, we establish the existence of a positive solution to a nonlinear fractional differential equation with nonlinear boundary conditions by means of a fixed point theorem in a cone.

AMS Subject Classification: 34B15, 26A33, 34A08
Key Words: boundary value problem, fractional derivative, nonlinear boundary conditions, positive solution, fixed-point theorem

1. Introduction

Theory of fractional derivatives and integrals has gained much attention during the past few decades. It is found that derivatives and integrals of fractional-order are very suitable in modeling the behavior of premotor neurons in the vestibulo-ocular reflex and also in modeling diffusion in a specific type of porous medium. It has been shown that fractional differential equations provide excellent tools in description of the properties of materials used in industry. Fractional-order models are found to be more accurate than classical integer-order models in mathematical treatment of many problems arising in the fields of physics, electrochemistry, signal processing, electromagnetic and so forth.

Received: June 22, 2010

Recently, there are some papers studying the boundary value problems for nonlinear fractional differential equations; see [2] and [4]-[6].

Zhang [6] has proved the existence of positive solutions for a nonlinear fractional differential equation boundary value problem involving Caputo's derivative:

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad 0 < t < 1,$$
$$u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0,$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the Caputo's fractional derivative, and $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

Qiu and Bai [4] gave the existence of a positive solution to boundary value problem of the following fractional differential equation:

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(1) = u''(0) = 0\,,$$

where $2 < \alpha \leq 3$, D_{0+}^{α} is the Caputo's differentiation, and $f : (0, 1] \times [0, \infty) \to [0, \infty)$ with $\lim\limits_{t \to 0^+} f(t, \cdot) = +\infty$ (that is f is singular at $t = 0$).

Tian and Chen [5] proved an existence result for the problem involving Riemann-Liouville fractional derivative:

$$\mathbf{D}^{q}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \quad u(1) = \alpha \mathbf{D}^{\frac{(q-1)}{2}}u(t)\,|_{t=\xi},$$

where $1 < q \leq 2$ is a real number, α and ξ satisfy certain conditions.

Motivated by the above results and methods, we discuss in this paper the existence of a positive solution to the following nonlinear fractional differential equation with nonlinear boundary conditions:

$$\begin{aligned} &D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ &u'(0) = 0, \quad u''(0) = 0, \quad u(0) = g(u'(1)), \end{aligned} \tag{1}$$

where $2 < \alpha \leq 3$ is a real number, D_{0+}^{α} is the Caputo's differentiation, $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous. We impose growth conditions on the functions f and g, and apply fixed-point theorem in a cone to prove the existence of a positive solution.

We only found one paper [2] dealing with the existence theorems for nonlinear fractional differential equations with nonlinear boundary conditions, in which the authors study a different problem using Amann Theorem and the method of upper and lower solutions.

2. Preliminaries

We present here the definitions and some fundamental properties of Caputo's derivative which can be found in the literature.

Definition 1. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 3. (see [3] and [6]) *Let $n - 1 < \alpha \leq n$, $u \in C^n[0, 1]$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - C_1 - C_2 t - \cdots - C_n t^{n-1},$$

where $C_i \in \mathbb{R}$, $i = 1, 2, \ldots n$.

Lemma 4. (see [1]) *Let E be a Banach space, $P \subseteq E$ a cone, and Ω_1, Ω_2 are two bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$, or

(ii) $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Before proceeding to the next section, we first prove the following lemma.

Lemma 5. *Given $h(t) \in C[0, 1]$ and a continuous function $g : [0, +\infty) \to [0, +\infty)$, let $2 < \alpha \leq 3$, then the boundary value problem*

$$\begin{gathered} D_{0+}^{\alpha} u(t) = h(t), \quad 0 < t < 1, \\ u'(0) = 0, \quad u''(0) = 0, \quad u(0) = g(u'(1)), \end{gathered} \tag{2}$$

has a unique solution

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + g\left(\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s) ds\right).$$

Proof. With Lemma 3, we can reduce equation (2) to an equivalent integral equation

$$u(t) = I_{0+}^{\alpha} h(t) + C_1 + C_2 t + C_3 t^2.$$

We have

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} h(s)ds + C_2 + 2C_3 t,$$

$$u''(t) = \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} h(s)ds + 2C_3.$$

Then from the boundary conditions $u'(0) = 0,\ u''(0) = 0$, we have $C_2 = C_3 = 0$, and

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + C_1.$$

Since $u(0) = g(u'(1))$, we have $C_1 = g(\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s)ds)$. Therefore, the unique solution of (2) is

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + g\left(\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s)ds\right). \quad \square$$

3. Main Results

Throughout this section, we make the following assumptions about the functions f and g :

(A1) $f:\ [0,1] \times [0,+\infty) \to [0,+\infty)$ is continous, $g:\ [0,+\infty) \to [0,+\infty)$ is continous and non-decreasing.

(A2) There exists a positive constant L, such that $|g(x) - g(y)| \le L\,|x-y|$, for any $x, y \in [0,+\infty)$.

(A3) $\lim_{x \to +\infty} \frac{g(x)}{x} = 0, \quad \lim_{x \to 0+} \frac{g(x)}{x} = +\infty.$

Throughout this section, we use the following notations:

$$L_1 = \frac{1}{\alpha\Gamma(\alpha)}, \tag{3}$$

$$L_2 = \frac{1}{\Gamma(\alpha)}, \tag{4}$$

$$F_1 = \frac{1}{\alpha\Gamma(\alpha)} + \frac{L}{\Gamma(\alpha)}, \tag{5}$$

where L is the constant in assumption (A2).

From (A3), there exists a positive constant N_1, such that if $x < N_1$, then $\frac{g(x)}{x} \geq 1$, i.e., $g(x) \geq x$.

Similarly, there exists a positive constant $N_2 (N_2 > N_1)$, such that if $x > N_2$, then $\frac{g(x)}{x} \leq 1$, i.e., $g(x) \leq x$.

Then we choose a positive constant R, such that $R > \frac{N_2(L_1+L_2)}{L_2}$. Similarly, we can choose a positve constant r, such that $r < \frac{N_1(L_1+L_2)}{L_2}$. Then $r < R$.

With the constant r, R chosen as above, we impose additional conditions on the function f:

(A4) $f(t,x) \leq \frac{R}{L_1+L_2}$, for $(t,x) \in [0,1] \times [0,R]$;

(A5) $f(t,x) \geq \frac{r}{L_1+L_2}$, for $(t,x) \in [0,1] \times [0,r]$,

where L_1, L_2 are constants in (3) and (4).

We construct a cone $P = \{u \in E : \ u(t) \geq 0, \ 0 \leq t \leq 1\}$, where $E = C[0,1]$ and $\|u\| = \max\limits_{0 \leq t \leq 1} |u(t)|$. E is a Banach space. We define an operator $T : P \to P$ as follows:

$$\begin{aligned} Tu(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))ds \\ & + g\left(\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s))ds \right). \end{aligned} \tag{6}$$

We state the main existence results in the following theorem.

Theorem 6. *Let $2 < \alpha \leq 3$, the functions f and g satisfy assumptions (A1)-(A5), then problem (1) has a positive solution.*

Proof. The proof is divided into three steps.

Step 1. We prove that T maps P to P.

From (6) and (A1), it is clear that $Tu(t)$ is non-negative. Next, we prove that for $u \in P$, we have $Tu(t) \in C[0,1]$. Since $u(t) \in C[0,1]$, there exists a constant $M > 0$ such that $|u(t)| \leq M$ for $t \in [0,1]$. With assumption (A1), let

$$F = \max_{(t,u) \in [0,1] \times [0,M]} f(t,u(t)) + 1,$$

we have

$$0 \leq f(t,u(t)) < F. \tag{7}$$

We consider the following three cases:

Case 1. $t_0 \in (0,1),\ t \in (t_0, 1]$. By mean value theorem, we have

$$(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1} = (\alpha-1)(\bar{t}-s)^{\alpha-2}(t-t_0) \tag{8}$$

for some $\bar{t} \in (t_0, t)$. It follows from (7) and (8) that

$$\begin{aligned}
|Tu(t) - Tu(t_0)| &= \frac{1}{\Gamma(\alpha)} \int_0^{t_0} \left[(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right] f(s,u(s))ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s,u(s))ds \\
&< \frac{F}{\Gamma(\alpha)} \int_0^{t_0} (\alpha-1)(\bar{t}-s)^{\alpha-2}(t-t_0)ds + \frac{F}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \\
&= \frac{(t-t_0)(\alpha-1)F}{\Gamma(\alpha)} \int_0^{t_0} (\bar{t}-s)^{\alpha-2} ds + \frac{F}{\alpha\Gamma(\alpha)}(t-t_0)^{\alpha} \\
&= \frac{(t-t_0)(\alpha-1)F}{\Gamma(\alpha)} \frac{(\bar{t}^{\alpha-1} - (\bar{t}-t_0)^{\alpha-1})}{\alpha-1} + \frac{F}{\alpha\Gamma(\alpha)}(t-t_0)^{\alpha} \\
&\le \frac{(t-t_0)(\alpha-1)F}{\Gamma(\alpha)} \frac{1}{\alpha-1} + \frac{F}{\alpha\Gamma(\alpha)}(t-t_0)^{\alpha} \\
&= \frac{F}{\Gamma(\alpha)}(t-t_0) + \frac{F}{\alpha\Gamma(\alpha)}(t-t_0)^{\alpha}.
\end{aligned}$$

Given $\epsilon > 0$, let

$$\delta = \min\left(\frac{\epsilon\Gamma(\alpha)}{2F},\ \left(\frac{\epsilon\alpha\Gamma(\alpha)}{2F}\right)^{\frac{1}{\alpha}}\right).$$

Then when $0 < t - t_0 < \delta$, we have

$$\begin{aligned}
|Tu(t) - Tu(t_0)| &< \frac{F}{\Gamma(\alpha)}(t-t_0) + \frac{F}{\alpha\Gamma(\alpha)}(t-t_0)^{\alpha} \\
&\le \frac{F}{\Gamma(\alpha)} \frac{\epsilon\Gamma(\alpha)}{2F} + \frac{F}{\alpha\Gamma(\alpha)} \frac{\epsilon\alpha\Gamma(\alpha)}{2F} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Case 2. $t_0 \in (0,1],\ t \in [0, t_0)$

$$\begin{aligned}
&|Tu(t_0) - Tu(t)| \\
&= \frac{1}{\Gamma(\alpha)} \int_0^{t} \left[(t_0-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] f(s,u(s))ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t_0} (t_0-s)^{\alpha-1} f(s,u(s))ds \\
&< \frac{F}{\Gamma(\alpha)} \int_0^{t} (\alpha-1)(\bar{t}-s)^{\alpha-2}(t_0-t)ds + \frac{F}{\Gamma(\alpha)} \int_t^{t_0} (t_0-s)^{\alpha-1} ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(t_0-t)(\alpha-1)F}{\Gamma(\alpha)}\int_0^t(\bar t-s)^{\alpha-2}ds+\frac{F}{\alpha\Gamma(\alpha)}(t_0-t)^\alpha\\
&= \frac{(t_0-t)(\alpha-1)F}{\Gamma(\alpha)}\left(\frac{\bar t^{\alpha-1}-(\bar t-t)^{\alpha-1}}{\alpha-1}\right)+\frac{F}{\alpha\Gamma(\alpha)}(t_0-t)^\alpha\\
&\le \frac{(t_0-t)(\alpha-1)F}{\Gamma(\alpha)}\frac{1}{\alpha-1}+\frac{F}{\alpha\Gamma(\alpha)}(t_0-t)^\alpha\\
&= \frac{F}{\Gamma(\alpha)}(t_0-t)+\frac{F}{\alpha\Gamma(\alpha)}(t_0-t)^\alpha,
\end{aligned}
$$

where $t<\bar t<t_0$.

Given $\epsilon>0$, let

$$\delta=\min(\frac{\epsilon\Gamma(\alpha)}{2F},\ (\frac{\epsilon\alpha\Gamma(\alpha)}{2F})^{\frac{1}{\alpha}}).$$

Then when $0<t_0-t<\delta$, we have

$$
\begin{aligned}
|Tu(t_0)-Tu(t)| &< \frac{F}{\Gamma(\alpha)}(t_0-t)+\frac{F}{\alpha\Gamma(\alpha)}(t_0-t)^\alpha\\
&\le \frac{F}{\Gamma(\alpha)}\frac{\epsilon\Gamma(\alpha)}{2F}+\frac{F}{\alpha\Gamma(\alpha)}\frac{\epsilon\alpha\Gamma(\alpha)}{2F}=\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}
$$

Case 3. $t_0=0,\ t\in(0,1]$

$$
\begin{aligned}
|Tu(t)-Tu(0)| &= \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))ds\\
&< \frac{F}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds=\frac{F}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha}.
\end{aligned}
$$

Given $\epsilon>0$, let

$$\delta=\left(\frac{\epsilon\alpha\Gamma(\alpha)}{F}\right)^{\frac{1}{\alpha}}.$$

Then when $0<t<\delta$, we have

$$|Tu(t)-Tu(0)|<\frac{F}{\alpha\Gamma(\alpha)}\frac{\epsilon\alpha\Gamma(\alpha)}{F}=\epsilon.$$

Hence, we have proved that for $u\in P$, we have $Tu(t)\in C[0,1]$. Therefore, $T:\ P\to P$.

Step 2. Next, we prove that the operator $T:\ P\to P$ is completely continuous.

Let $u_0\in P$ and $\|u_0\|=a_0$. For $u\in P$ and $\|u-u_0\|<1$, we have $\|u\|<1+a_0:=a$. By assumption (A1), we know that $f(t,u(t))$ is uniformly continuous on $[0,1]\times[0,a]$.

Therefore, given any $\epsilon > 0$, there exists $\delta > 0$ and $\delta < 1$, such that when $|u(t) - u_0(t)| < \delta$, we have $|f(t, u(t)) - f(t, u_0(t))| < \frac{\epsilon}{F_1}$, where F_1 is the constant in (5).

If $\|u - u_0\| < \delta$, then $u(t), u_0(t) \in [0, a]$ and $|u(t) - u_0(t)| < \delta$, for any $t \in [0, 1]$, then we have

$$|f(t, u(t)) - f(t, u_0(t))| < \frac{\epsilon}{F_1}, \quad \forall\, t \in [0, 1]. \tag{9}$$

It follows from (9) and assumption (A2) that

$$\begin{aligned}
\big|Tu(t) - Tu_0(t)\big| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big|f(s, u(s)) - f(s, u_0(s))\big| ds \\
&+ \Big| g\left(\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds\right) \\
&- g\left(\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u_0(s)) ds\right) \Big| \\
&< \frac{\epsilon}{F_1}\left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{L}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} ds\right] \\
&= \frac{\epsilon}{F_1}\left[\frac{t^\alpha}{\alpha\Gamma(\alpha)} + \frac{L}{(\alpha-1)\Gamma(\alpha-1)}\right] \\
&\leq \frac{\epsilon}{F_1}\left[\frac{1}{\alpha\Gamma(\alpha)} + \frac{L}{\Gamma(\alpha)}\right] = \frac{\epsilon}{F_1} F_1 = \epsilon.
\end{aligned}$$

Then

$$\|Tu - Tu_0\| = \max_{t \in [0,1]} \big|Tu(t) - Tu_0(t)\big| < \epsilon.$$

Therefore, $T: \ P \to P$ is continuous.

Let Ω be a bounded set in P, i.e., $\Omega := \{u \in P \ : \|u\| < M\}$, where M is a positive constant. Let $F = \max_{(t,u) \in [0,1] \times [0,M]} f(t, u(t)) + 1$.

Since g is non-decreasing, we have

$$\begin{aligned}
\big|Tu(t)\big| &< \frac{F}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + g\left(\frac{F}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} ds\right) \\
&= \frac{F}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} + g\left(\frac{F}{\Gamma(\alpha-1)} \frac{1}{\alpha-1}\right) \\
&\leq \frac{F}{\alpha\Gamma(\alpha)} + g\left(\frac{F}{\Gamma(\alpha)}\right)
\end{aligned}$$

Then

$$\|Tu\| = \max_{t \in [0,1]} \big|Tu(t)\big| < \frac{F}{\alpha\Gamma(\alpha)} + g\left(\frac{F}{\Gamma(\alpha)}\right)$$

for any $u \in \Omega$.

Hence $T(\Omega)$ is bounded.

From the proof in Step 1, we can easily see that for each $u \in \Omega$, given $\epsilon > 0$, there exists $\delta > 0$, such that, if $t_1, t_2 \in [0,1]$, and $0 < t_2 - t_1 < \delta$, then $\big|Tu(t_2) - Tu(t_1)\big| < \epsilon$. Therefore, $T(\Omega)$ is equicontinuous. By the Arzela-Ascoli Theorem, $T: \ P \to P$ is completely continuous.

Step 3. Let $\Omega_1 = \{u \in P \ : \|u\| < r\}$, where r is the constant chosen in the beginning of this section. For $u \in \partial\Omega_1$, we have $\|u\| = r$ and $0 \le u(t) \le r$, for $t \in [0,1]$. It follows from assumption (A1), (A3) and (A5) that

$$\begin{aligned}
Tu(1) =& \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,u(s))ds \\
&+ g\left(\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s))ds\right) \\
\ge& \frac{r}{(L_1+L_2)} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds + g\left(\frac{r}{(L_1+L_2)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} ds\right) \\
=& \frac{r}{(L_1+L_2)\alpha\Gamma(\alpha)} + g\left(\frac{r}{(L_1+L_2)(\alpha-1)\Gamma(\alpha-1)}\right) \\
=& L_1 \frac{r}{L_1+L_2} + g\left(L_2 \frac{r}{L_1+L_2}\right).
\end{aligned}$$

It follows from the discussion after assumption (A3) that $L_2 \frac{r}{L_1+L_2} < N_1$, and we have $g\left(L_2 \frac{r}{L_1+L_2}\right) \ge L_2 \frac{r}{L_1+L_2}$.

Therefore,

$$Tu(1) \ge L_1 \frac{r}{L_1+L_2} + g\left(L_2 \frac{r}{L_1+L_2}\right) \ge L_1 \frac{r}{L_1+L_2} + L_2 \frac{r}{L_1+L_2} = r.$$

Then $\|Tu\| \ge \|u\|$ on $\partial\Omega_1$.

Similarly, let $\Omega_2 = \{u \in P \ : \|u\| < R\}$, where R is the constant chosen in the beginning of this section. For $u \in \partial\Omega_2$, we have $\|u\| = R$ and $0 \le u(t) \le R$, for $t \in [0,1]$. It follows from assumptions (A1), (A3) and (A4) that, for any $t \in [0,1]$,

$$\begin{aligned}
\big|Tu(t)\big| \le& \frac{1}{\Gamma(\alpha)} \frac{R}{L_1+L_2} \int_0^t (t-s)^{\alpha-1} ds \\
&+ g\left(\frac{R}{L_1+L_2} \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} ds\right) \\
=& \frac{1}{\Gamma(\alpha)} \frac{R}{(L_1+L_2)} \frac{t^\alpha}{\alpha} + g\left(\frac{R}{L_1+L_2} \frac{1}{\Gamma(\alpha-1)} \frac{1}{\alpha-1}\right)
\end{aligned}$$

$$\le \frac{L_1 R}{L_1 + L_2} + g\left(\frac{L_2 R}{L_1 + L_2}\right).$$

It follows from the discussion after assumption (A3) that $L_2 \frac{R}{L_1+L_2} > N_2$, and we have $g\left(L_2 \frac{R}{L_1+L_2}\right) \le L_2 \frac{R}{L_1+L_2}$.

Therefore,

$$\left|Tu(t)\right| \le L_1 \frac{R}{L_1 + L_2} + g\left(L_2 \frac{R}{L_1 + L_2}\right) \le L_1 \frac{R}{L_1 + L_2} + L_2 \frac{R}{L_1 + L_2} = R.$$

Then $\|Tu\| \le \|u\|$ on $\partial\Omega_2$.

By Lemma 4, we complete the proof of theorem. □

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