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EXISTENCE THEORY FOR SINGLE AND MULTIPLE POSITIVE PERIODIC SOLUTIONS TO THE DELAY LOGISTIC EQUATIONS WITH IMPULSE EFFECTS

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Abstract: This paper deals with a new existence theory for single and multiple positive periodic solutions to a kind of delay logistic functional differential equations with impulse actions at fixed moments by employing a fixed point theorem in cones. Easily verifiable sufficient criteria are established. The paper extends some previous results and reports some new results about impulsive functional differential equations.

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1. Introduction

The purpose of the present paper is to present existence of positive periodic solutions for the following logistic equation with impulsive system

$$\begin{aligned} &\dot{y}(t) = y(t)[a(t) - f(t, y(t-\tau_1(t)), ..., y(t-\tau_n(t)))], \\ &\qquad\qquad\qquad\qquad t \in (-\infty, \infty),\ t \neq t_j,\ j \in Z, \\ &y(t_j^+) = y(t_j^-) - I_j(y(t_j)), \quad t = t_j, \end{aligned} \tag{1.1}$$

where $y(t_j^+)$ and $y(t_j^-)$ represent the right and the left limit of $y(t_j)$, y is left continuous at $t_j; a(t) \in C((-\infty,\infty),(0,\infty))$, $\tau_m(t) \in C((-\infty,\infty),(-\infty,\infty))$, and $a(t) = a(t+T)$, $\tau_m(t) = \tau_m(t+T), m = 1, ..., n$, $f(t, u_1, u_2, ..., u_n) = f(t + T, u_1, u_2, ..., u_n), T > 0$. There exists a positive integer p such that $t_{j+p} = t_j + T$, $I_{j+p} = I_j$, $j \in Z$. We also assume that $[0,T)\bigcap\{t_j : j \in Z\} = \{t_1, t_2, \cdots, t_p\}$. Here $f : (-\infty,\infty) \times [0,\infty)^n \to [0,\infty)$ and $I_j : [0,\infty) \to [0,\infty)$ are continuous.

It is well known that equation (1.1) without impulse effects includes many mathematical ecological equations. For example, equation (1.1) includes the single species periodic population models (see [6], [12], [4], [16], [11], [7]):

$$y'(t) = a(t)y(t)[1 - \frac{y(t-\tau(t))}{K(t)}], \quad t \in (-\infty, \infty); \tag{1.2}$$

and

$$y'(t) = y(t)[a(t) - \sum_{j=1}^{n} b_j(t)y(t-\tau_j(t))], \quad t \in (-\infty, \infty). \tag{1.3}$$

Also equation (1.1) includes:

(i) the multiplicative delay periodic logistic equation (see [6], [12], [4], [5], [16], [13])

$$y'(t) = a(t)y(t)[1 - \prod_{j=1}^{n} \frac{y(t-\tau_j(t))}{K(t)}], \quad t \in (-\infty, \infty); \tag{1.4}$$

and

(ii) the periodic Michaelis-Menton model (see [12], [9])

$$y'(t) = a(t)y(t)[1 - \sum_{j=1}^{n} \frac{a_j(t)y(t-\tau_j(t))}{1 + c_j(t)y(t-\tau_j(t))}], \quad t \in (-\infty, \infty). \tag{1.5}$$

Many authors have investigated the existence of one positive periodic solutions for delay differential equations, for example (see [6], [12], [7]) and the references therein. The existence of one positive periodic solution to Equation (1.1) without impulse was discussed in [12] using Mawhin's continuation theorem, and in [6], [7] using Krasnoselskii fixed point theorem in cones. However, the study for the functional differential equation with impulse is still in an initial stage of its development. As far as authors known, there are few works on the existence of multiple periodic solutions for the function differential equation (1.1) with impulse effects.

In fact, many physical systems whose states are subjects to sudden change at certain moments, for example, in population biology, the diffusion of chemicals, the spread of heat, the radiation of electromagnetic waves, the maintenance of a species through instantaneous stocking and harvesting, etc. The impulsive differential equation is also an adequate apparatus for the mathematical simulation of such processes and phenomena. Moreover, their theory is considerably richer than that of ordinary differential equations without impulses. There has been increasing interest in the investigation for such equations during the past few years (see for example, the monographs of Samoilenko and Perestyuk [18] and Lakshmikantham et al [10]). Now some qualitative properties such as oscillation, asymptotic behavior, and stability are investigated extensively by many authors (see [1], [2], [19], [20]). However, little has been done for the periodicity of nonautonomous impulsive differential equations, especially the study based on the Krasnoselskii fixed point theorem. As far as known, three of the most common techniques to approach the periodic problems of impulsive differential equations are: (1) to obtain an a priori bounds for the possible solutions and then the application of topological degree arguments (see [21], [8]) and (2) the theory of upper and lower solutions [17] and (3) the theory of Schaeffers (see [14], [17]). These techniques can be interconnected and have proved to be very strong and fruitful and became very popular in this research area. However, any method of proof has some limitations and in fact, for practical purposes, serious difficulties arise frequently in the search for a priori bounds or upper and lower solutions.

In this paper, we choose another strategy of proof which rely essentially on a fixed point theorem due to Krasnoselskii for completely continuous operators on a Banach space that exhibit a cone compression and expansion of norm type. This result has been extensively employed in the related literature, specially to study several kinds of separated boundary value problems, while for the periodic problem it is more difficult to find references. The reason is that in order to apply this fixed point theorem, it is necessary to find Green's function, we overcome this problem with some ideas from [15].

The paper is organized as follows: Section 2 is devoted to finding Green's function and deriving easily verifiable sufficient criteria for the existence of periodic solutions with strictly positive components of (1.1). In Section 3, we apply the main result to study some examples which have some biological background.

To conclude this section, we state a fixed point theorem in cones which will be needed in this paper.

Theorem 1.1. (see [3]) *Let X be a Banach space and K is a cone in X. Assume $\Omega_1,\ \Omega_2$ are open subsets of X with $0 \in \Omega_1,\ \bar{\Omega}_1 \subset \Omega_2$. Let*

$$\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a continuous and completely continuous operator such that:

(i) $\|\Phi x\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$,

(ii) there exists $\psi \in K\backslash\{0\}$ such that $x \neq \Phi x + \lambda\psi$ for $x \in K \cap \partial\Omega_2$ and $\lambda > 0$.

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Remark 1.1. In Theorem 1.1, if (i) and (ii) are replaced by:

(i)* $\|\Phi x\| \leq \|x\|$ for $x \in K \cap \partial\Omega_2$,

and

(ii)* there exists $\psi \in K\backslash\{0\}$ such that $x \neq \Phi x + \lambda\psi$ for $x \in K \cap \partial\Omega_1$ and $\lambda > 0$, then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. Positive Periodic Solutions

In this section we establish the existence of positive periodic solutions to equation (1.1).

We now consider the "linear problem"

$$\begin{cases} \dot{y}\,(t) = a(t)y(t) - \sigma(t), \quad t \neq t_j, \\ y(t_j^+) = y(t_j^-) - I_j(y(t_j)), \quad j \in Z,\ t = t_j, \end{cases} \tag{2.1}$$

where $\sigma(t) \in C(R, [0, +\infty))$ is a T-periodic function and the rest parameters satisfy the hypothesis about (1.1).

Note that (2.1) is not really a linear problem since the impulse functions are not necessarily linear. However, if $I_j,\ j = 1, 2, \cdots m$ are linear, then (2.1) is a linear impulsive problem.

In the proofs of our main results, the next result is fundamental in our discussion. Since the method is similar as that in [15], we omit the proof.

Lemma 2.1. *$y(t)$ is an T-periodic solution of equation* (2.1), *is equivalent to $y(t)$ is an T-periodic solution of the integral equation*

$$y(t) = \int_t^{t+T} G(t,s)\sigma(s)ds + \sum_{j:t_j\in[t,t+T)} G_i(t,t_j)I_j(y(t_j)), \tag{2.2}$$

where

$$G(t,s) := \frac{\exp\left[\int_0^T a(\xi)d\xi - \int_t^s a(\xi)d\xi\right]}{\exp[\int_0^T a(\xi)d\xi] - 1}. \tag{2.3}$$

Lemma 2.2. *$y(t)$ is an T-periodic solution of equation* (1.1), *is equivalent to $y(t)$ is an T-periodic solution of the integral equation*

$$y(t) = \int_t^{t+T} G(t,s)[y(s)f(s,y(s-\tau_1(s)),...,y(s-\tau_n(s)))]ds + \sum_{j:t_j\in[t,t+T)} G_i(t,t_j)I_j(y(t_j)), \tag{2.4}$$

where $G(t,s)$ is as (2.3) in Lemma 2.1.

Let X be a Banach space, and K a closed, nonempty subset of X. K is a cone provided: (i) $\alpha u + \beta v \in K$, for all $u, v \in K$ and all $\alpha, \beta \geq 0$. (ii) $u, -u \in K$ imply $u = 0$.

Define

$$PCB(R) = \{y : R \to R | y \in C(t_j, t_{j+1}),\ \ y(t_j^-) = y(t_j),\ \exists\ y(t_j^+),\ j \in Z,\ y(t) \text{ is a bounded function on } R\}.$$

Let

$$X = \{y(t) : y(t) \in PCB(R), y(t+T) = y(t)\} \tag{2.5}$$

with the norm

$$\|y\| = \sup_{t\in[0,T]} \{|y(t)| : y \in X\}.$$

Then X with the norm $\|\cdot\|$ is a Banach space.

Define an operator on X as following

$$(\Phi y)(t) = \int_t^{t+T} G(t,s)y(s)f(s,y(s-\tau_1(s)),...,y(s-\tau_n(s)))ds + \sum_{j:t_j\in[t,t+T)} G(t,t_j)I_j(y(t_j)), \tag{2.6}$$

for $y \in X$.

Let

$$K = \{y \in X : y(t) \geq 0 \quad \text{and} \quad y(t) \geq \sigma\|y\|\},$$
$$A \overset{def}{=} G(t, t+T) \leq G(t,s) \leq G(t,t) \overset{def}{=} B, \tag{2.7}$$

where

$$A = \frac{1}{\exp(\int_0^T a(\xi)d\xi) - 1}, \quad B = \frac{\exp(\int_0^T a(\xi)d\xi)}{\exp(\int_0^T a(\xi)d\xi) - 1}.$$

Thus $0 < \sigma = A/B < 1$.

It is not difficult to verify that K is a cone in X.

Lemma 2.3. *Assume that* (P) *holds. Then* $\Phi(K) \subseteq K$.

Proof. For any $y \in K$, we have

$$\|\Phi y\| \leq B \int_0^T y(s) f(s, y(s-\tau_1(s)), ..., y(s-\tau_n(s)))ds + B\sum_{j=1}^{p} I_j(y(t_j)),$$

and

$$(\Phi y)(t) \geq A \int_0^T y(s) f(s, y(s-\tau_1(s)), ..., y(s-\tau_n(s)))ds + A\sum_{j=1}^{p} I_j(y(t_j)).$$

Thus

$$(\Phi y)(t) \geq \frac{A}{B}\|\Phi y\| = \sigma\|\Phi y\|,$$

so $\Phi y \in K$. $\square$

For convenience and simplicity in the following discussion, we always use the notations:

$$f_0 = \liminf_{u_1,...,u_n \downarrow 0} \min_{t\in[0,T]} \frac{f(t,u_1,...,u_n)}{a(t)}, \qquad I_0(j) = \liminf_{u\downarrow 0} \frac{I_j(u)}{u},$$
$$f^\infty = \limsup_{u_1,...,u_n \uparrow \infty} \max_{t\in[0,T]} \frac{f(t,u_1,...,u_n)}{a(t)}, \qquad I^\infty(j) = \limsup_{u\uparrow \infty} \frac{I_j(u)}{u},$$
$$f^0 = \limsup_{u_1,...,u_n \downarrow 0} \max_{t\in[0,T]} \frac{f(t,u_1,...,u_n)}{a(t)u}, \qquad I^0(j) = \limsup_{u\downarrow 0} \frac{I_j(u)}{u},$$
$$f_\infty = \liminf_{u_1,...,u_n \uparrow \infty} \min_{t\in[0,T]} \frac{f(t,u_1,...,u_n)}{a(t)}, \qquad I_\infty(j) = \liminf_{u\uparrow \infty} \frac{I_j(u)}{u};$$

and for $q > 0$, we define

$$I_{(q)}(j) = \min_{\sigma q \leq u \leq q} \frac{I_j(u)}{u}, \qquad I^{(q)}(j) = \max_{\sigma q \leq u \leq q} \frac{I_j(u)}{u}.$$

$$f_{(q)} = \min_{t\in[0,T], \sigma q \leq u_i \leq q} \frac{f(t,u_1,...,u_n)}{a(t)}, \qquad f^{(q)} = \max_{t\in[0,T], \sigma q \leq u_i \leq q} \frac{f(t,u_1,...,u_n)}{a(t)}.$$

In this paper, some of the following hypotheses are satisfied:

(H_1) $f_0 + A\sum_{j=1}^{p} I_0(j) > 1$ and $f_\infty + A\sum_{j=1}^{p} I_\infty(j) > 1$.

(H_2) $f^0 + B\sum_{j=1}^{p} I^0(j) < 1$ and $f^\infty + B\sum_{j=1}^{p} I^\infty(j) < 1$.

(H_3) There is a $q > 0$ such that

$$f^{(q)} + B\sum_{j=1}^{p} I^{(q)}(j) < 1.$$

(H_4) There is a $q > 0$ such that

$$f_{(q)} + A\sum_{j=1}^{p} I_{(q)}(j) > 1.$$

The main result of the present paper is as follows.

Theorem 2.1. *Assume that (H_1) and (H_3) are satisfied. Then equation (1.1) has at least two T-periodic positive solutions y_1 and y_2 such that*

$$0 < \|y_1\| < p < \|y_2\|.$$

Proof. Suppose that (H_1) and (H_3) hold. By using the first inequality of (H_1), i.e., $f_0 + A\sum_{j=1}^{p} I_0(j) > 1$, one can find $0 < \epsilon < 1$ such that

$$(1-\epsilon)[f_0 + A\sum_{j=1}^{p} I_0(j)] > 1,$$

and there exists a constant $0 < r < q$ such that

$$f(t, u_1, ..., u_n) \geq (1-\epsilon)f_0 a(t), \quad I_j(u) \geq (1-\epsilon)I_0(j)u, \quad j = 1, 2, \cdots, p,$$

whenever $0 \leq u_m,\ u \leq r, m = 1, ..., n$.

Let $\Omega_r = \{x \in X, ||x|| < r\}$. Thus, if $y \in K$ with $\|y\| = r$, then $\sigma r \leq y(t) \leq r$. Let $\psi \equiv 1$ for $t \in R$ and we prove that

$$y \neq \Phi y + \lambda\psi \quad \text{for} \quad y \in K \cap \partial\Omega_r \text{ and } \lambda \geq 0. \tag{2.8}$$

If not, there exists $y_0 \in K \cap \partial\Omega_r$ and $\lambda_0 \geq 0$ such that

$$y_0 = \Phi y_0 + \lambda_0\psi.$$

Let $\mu = \min_{t\in[0,T]} y_0(t)$. Then for $t \in R$ we have

$$y_0(t) = (\Phi y_0)(t) + \lambda_0$$

$$
\begin{aligned}
&= \int_t^{t+T} G(t,s)y_0(s)f(s, y_0(s-\tau_1(s)), ..., y_0(s-\tau_n(s)))ds \\
&\qquad + \sum_{j:t_j\in[t,t+T)} G(t,t_j)I_j(y_0(t_j)) + \lambda_0 \\
&\geq (1-\epsilon)f_0 \int_t^{t+T} G(t,s)a(s)y_0(s)ds + A\sum_{j=1}^{p} I_j(y_0(t_j)) + \lambda_0 \\
&\geq (1-\epsilon)f_0\mu \int_t^{t+T} G(t,s)a(s)ds + (1-\epsilon)A\sum_{j=1}^{p} I_0(j)\mu + \lambda_0 \\
&= (1-\epsilon)[f_0 + A\sum_{j=1}^{p} I_0(j)]\mu + \lambda_0 > \mu + \lambda_0,
\end{aligned}
$$

and this implies $\mu > \mu + \lambda_0$, a contradiction.

On the other hand, since (H_3) holds, there is a $q > 0$ such that

$$f(t, u_1, ..., u_n) \leq f^q a(t), \quad I_j(u) \leq I^q(j)u, \quad j = 1, 2, \cdots, p,$$

whenever $\sigma q \leq u_m,\ u \leq q, m = 1, ..., n.$

Since

$$\sigma q = \sigma\|u\| \leq u(t) \leq \|u\| = q, \quad \text{for} \quad u \in K \cap \partial\Omega_q,$$

then for $y \in K$ with $||y|| = q$, we have

$$
\begin{aligned}
(\Phi y)(t) &= \int_t^{t+T} G(t,s)y(s)f(s, y(s-\tau_1(s)), ..., y(s-\tau_n(s)))ds \\
&\qquad + \sum_{j:t_j\in[t,t+T)} G(t,t_j)I_j(y(t_j)) \\
&\leq f^q \int_t^{t+T} G(t,s)a(s)y(s)ds + B\sum_{j=1}^{p} I_j(y(t_j)) \\
&\leq f^q q \int_t^{t+T} G(t,s)a(s)ds + Bq\sum_{j=1}^{p} I^q(j) = q[f^q + B\sum_{j=1}^{p} I^q(j)] < q.
\end{aligned}
$$

This implies that

$$\|\Phi y\| < \|y\| \quad \text{for } \ y \in K \cap \partial\Omega_q. \tag{2.9}$$

In view of (2.8) and (2.9), by Theorem 1.1, it follows that Φ has a fixed point $y_1 \in K \cap (\bar{\Omega}_q \setminus \Omega_r)$. Furthermore, $r < \|y_1\| < q$ and $y_1(t) \geq \sigma r > 0$, which means that $y_1(t)$ is an T-periodic positive solution of (1.1).

Next, by using the second inequality of (H_1), i.e., $f_\infty + A\sum_{j=1}^{p} I_\infty(j) > 1$, one can find that for $0 < \varepsilon < 1$ such that

$$(1-\epsilon)[f_\infty + A\sum_{j=1}^{p} I_\infty(j)] > 1,$$

and there exists a constant $r_1 > q$ such that

$$f(t,u_1,...,u_n) \geq (1-\epsilon)f_\infty a(t), \qquad I_j(u) \geq (1-\epsilon)I_\infty(j)u,$$

whenever $u_m,\ u \geq r_1, m = 1,...,n.$

Let $R = \frac{r_1}{\sigma}$, so we have,

$$u(t) \geq \sigma\|u\| = \sigma R = r_1 \quad \text{for} \quad u \in K \cap \partial\Omega_R.$$

Let $\psi \equiv 1$ for $t \in R$ and we prove that

$$y \neq \Phi y + \lambda\psi \quad \text{for} \quad y \in K \cap \partial\Omega_R \text{ and } \lambda \geq 0. \tag{2.10}$$

If not, there exists $y_0 \in K \cap \partial\Omega_R$ and $\lambda_0 \geq 0$ such that

$$y_0 = \Phi y_0 + \lambda_0\psi.$$

Let $\mu = \min_{t\in[0,T]} y_0(t)$. Then for $t \in R$ we have

$$\begin{aligned}
y_0(t) &= (\Phi y_0)(t) + \lambda_0 \\
&= \int_t^{t+T} G(t,s)y_0(s)f(s,y_0(s-\tau_1(s)),...,y_0(s-\tau_n(s)))ds \\
&\qquad + \sum_{j:t_j\in[t,t+T)} G(t,t_j)I_j(y_0(t_j)) + \lambda_0 \\
&\geq (1-\epsilon)f_\infty \int_t^{t+T} G(t,s)a(s)y_0(s)ds + A\sum_{j=1}^{p} I_j(y_0(t_j)) + \lambda_0 \\
&\geq (1-\epsilon)f_\infty\mu \int_t^{t+T} G(t,s)a(s)ds + (1-\epsilon)A\sum_{j=1}^{p} I_\infty(j)\mu + \lambda_0 \\
&= (1-\epsilon)[f_\infty + A\sum_{j=1}^{p} I_\infty(j)]\mu + \lambda_0 > \mu + \lambda_0,
\end{aligned}$$

and this implies $\mu > \mu + \lambda_0$, a contradiction.

Therefore, in view of (2.9) and (2.10), by Theorem 1.1, it follows that Φ has a fixed point $y_2 \in K\cap(\bar{\Omega}_R \backslash \Omega_q)$. Furthermore, $q < \|y_2\| < R$ and $y_2(t) \geq \sigma q > 0$, which means that $y_2(t)$ is an ω-periodic positive solution of (1.1).

This completes the proof of Theorem 2.1. □

Remark 2.1. To reduce the existence of y_1 in Theorem 2.1, we need only to assume (H_3) and $f_0 + A\sum\limits_{j=1}^{p} I_0(j) > 1$. A similar remark applies for y_2.

Corollary 2.1. *Using the following (H_1^*) instead of (H_1), the conclusion of Theorem 2.1 is true*

$$(H_1^*) \quad f_0 = \infty \ \text{ or } \ \sum_{j=1}^{p} I_0(j) = \infty, \ \text{ and } \ f_\infty = \infty \ \text{ or } \ \sum_{j=1}^{p} I_\infty(j) = \infty.$$

Theorem 2.2. *Assume that (H_2) and (H_4) are satisfied. Then equation (1.1) has at least two T-periodic positive solutions y_1 and y_2 such that*

$$0 < \|y_1\| < p < \|y_2\|.$$

Proof. Suppose that (H_2) and (H_4) hold. By using the first inequality of (H_2), i.e., $f^0 + B\sum\limits_{j=1}^{p} I^0(j) < 1$, one can find that for

$$0 < \epsilon < \frac{1 - (f^0 + B\sum\limits_{j=1}^{p} I^0(j)}{pB + 1},$$

there exist a constant $0 < r < q$ such that

$$f(t, u_1, ..., u_n) \le (f^0 + \epsilon)a(t), \qquad I_j(u) \le (I^0(j) + \epsilon)u, \quad j = 1, 2, \cdots, p,$$

whenever $0 \le u, u_m \le r$, $m = 1, 2, \cdots, n$. Thus, if $y \in K$ with $\|y\| = r$, then $\sigma r \le y(t) \le r$. Then for $y \in K$ with $||y|| = r$, we have

$$\begin{aligned}
(\Phi y)(t) = &\int_t^{t+T} G(t,s)y(s)f(s, y(s-\tau_1), ...y(s-\tau_n(s)))ds \\
&+ \sum_{j:t_j \in [t, t+T)} G(t, t_j)I_j(y(t_j)) \\
\le &\int_t^{t+T} (f^0 + \epsilon)G(t,s)a(s)y(s)ds + B\sum_{j=1}^{p} I_j(y(t_j)) \\
\le &(f^0 + \epsilon)\int_t^{t+T} G(t,s)a(s)ds||y|| + B\sum_{j=1}^{p}(I^0(j) + \epsilon)||y|| \\
= &[f^0 + B\sum_{j=1}^{p} I^0(j) + (pB+1)\epsilon]||y|| < \|y\|.
\end{aligned}$$

This implies that

$$\|\Phi y\| < \|y\| \tag{2.11}$$

for $y \in K \cap \partial\Omega_r$.

On the other hand, since (H_4) holds, we know that there is a $q > 0$ such that

$$f(t, u_1, ..., u_n) \geq f_q a(t), \quad I_j(u) \leq I_q(j)u, \quad j = 1, 2, \cdots, p,$$

whenever $\sigma q \leq u_m,\ u \leq q, m = 1, ..., n$. Thus, if $y \in K$ with $\|y\| = q$, then $\sigma q \leq y(t) \leq q$.

Let $\psi \equiv 1$ for $t \in R$ and we prove that

$$y \neq \Phi y + \lambda\psi \text{ for } y \in K \cap \partial\Omega_q \text{ and } \lambda \geq 0. \tag{2.12}$$

If not, there exist $y_0 \in K \cap \partial\Omega_q$ and $\lambda_0 \geq 0$ such that

$$y_0 = \Phi y_0 + \lambda_0\psi.$$

Let $\mu = \min\limits_{t\in[0,T]} y_0(t)$. Then for $t \in R$ we have

$$\begin{aligned} y_0(t) = (\Phi y_0)(t) + \lambda_0 = &\int_t^{t+T} G(t, s)y_0(s)f(s, y_0(s - \tau_1(s)), ..., y_0(s - \tau_n(s)))ds \\ &+ \sum_{j:t_j\in[t,t+T)} G(t, t_j)I_j(y_0(t_j)) + \lambda_0 \\ &\geq f_q \int_t^{t+T} G(t, s)a(s)y_0(s)ds + A\sum_{j=1}^{p} I_j(y_0(t_j)) + \lambda_0 \end{aligned}$$

$$\geq f_q\mu \int_t^{t+T} G(t, s)a(s)ds + A\sum_{j=1}^{p} I_q(j)\mu + \lambda_0 = [f_q + A\sum_{j=1}^{p} I_q(j)]\mu + \lambda_0 > \mu + \lambda_0,$$

and this implies $\mu > \mu + \lambda_0$, a contradiction.

In view of (2.11) and (2.12), by Theorem 1.1, it follows that Φ has a fixed point $y_1 \in K \cap (\bar{\Omega}_q \setminus \Omega_r)$. Furthermore, $r < \|y_1\| < q$ and $y_1(t) \geq \sigma r > 0$, which means that $y_1(t)$ is an ω-periodic positive solution of (1.1).

Next, by using the second inequality of (H_2), i.e.,

$$f^\infty + B\sum_{j=1}^{p} I^\infty(j) < 1,$$

one can find that for $0 < \varepsilon < \frac{1-(f^\infty + B\sum_{j=1}^{p} I^\infty(j))}{Bp+1}$, there exists a constant $r_1 > q$

such that

$$f(t,u_1,u_2,...,u_n) \le (f^\infty+\epsilon)a(t), \qquad I_j(u) \le (I^\infty(j)+\epsilon)u, \quad j=1,2,\cdots,p,$$

for $u, u_m \ge r_1, \quad m=1,2,\cdots,n.$

Let $R = \frac{r_1}{\sigma}$, so we have,

$$u(t) \ge \sigma\|u\| = \sigma R = r_1 \quad \text{for} \quad u \in K \cap \partial\Omega_R. \tag{2.13}$$

Then for $y \in K$ with $||y|| = R$, we have

$$\begin{aligned}
(\Phi y)(t) = & \int_t^{t+T} G(t,s)y(s)f(s,y(s-\tau_1(s)),...,y(s-\tau_n(s)))ds \\
& + \sum_{j:t_j\in[t,t+T)} G(t,t_j)I_j(y(t_j)) \\
& \le \int_t^{t+T}(f^\infty+\epsilon)G(t,s)a(s)y(s)ds + B\sum_{j=1}^{p} I_j(y(t_j)) \\
& \le (f^\infty+\epsilon)\int_t^{t+T} G(t,s)a(s)ds\|y\| + B\sum_{j=1}^{p}(I^\infty(j)+\epsilon)\|y\| \\
& = [f^\infty + B\sum_{j=1}^{p}(I^\infty(j)+(Bp+1)\epsilon]\|y\| < \|y\|.
\end{aligned}$$

This implies that

$$\|\Phi y\| \le \|y\| \tag{2.14}$$

for $y \in K \cap \partial\Omega_R$.

Therefore, in view of (2.11) and (2.14), by Theorem 1.1, it follows that Φ has a fixed point $y_2 \in K \cap (\bar{\Omega}_R \setminus \Omega_q)$. Furthermore, $q < \|y_2\| < R$ and $y_2(t) \ge \sigma q > 0$, which means that $y_2(t)$ is an T-periodic positive solution of (1.1).

This completes the proof Theorem 2.2. □

Remark 2.2. To reduce the existence of y_1 in Theorem 2.1, we need only to assume (H_4) and $f^0 + B\sum_{j=1}^{p} I^0(j) < 1$. A similar remark applies for y_2.

Corollary 2.2. *Using the following (H_2^*) instead of (H_2), the conclusion of Theorem 2.2 is true.*

(H_2^*) $\qquad f^0=0,\ I_0=0,\ f^\infty=0,\ I^\infty(j)=0, \quad j=1,2,\cdots,p.$

Theorem 2.3. *Equation (1.1) has at least one T-periodic positive solution*

y_1 and y_2, provided one of the following conditions holds:

$$(i) \quad f_0 + A\sum_{j=1}^{p} I_0(j) > 1 \quad \text{and} \quad f^\infty + B\sum_{j=1}^{p} I^\infty(j) < 1;$$

$$(ii) \quad f^0 + B\sum_{j=1}^{p} I^0(j) < 1 \quad \text{and} \quad f_\infty + A\sum_{j=1}^{p} I_\infty(j) > 1.$$

Proof. The proof follows the ideas in the proofs of Theorem 2.1 and Theorem 2.2. □

Corollary 2.3. *Equation (1.1) has at least one ω-periodic positive solution, provided one of the following conditions holds:*

(i) $f_0 = \infty$, or $\sum_{j=1}^{p} I_0(j) = \infty$ and $f\infty = 0$, $I^\infty(j) = 0$, $j = 1, 2, \cdots, p$ (sublinear).

(ii) $f^0 = 0$, $I^0(j) = 0$, $j = 1, 2, \cdots, p$, and $f_\infty = \infty$ or $\sum_{j=1}^{p} I_\infty(j) = \infty$ (superlinear).

3. Applications

In this section, we apply the main results obtained in previous sections to examples modelling biological phenomena.

It is well known that equation (1.1) with impulse effects includes many mathematical ecological equations. For example, equation (1.1) includes the single species periodic population models with impulse effects:

$$\begin{cases} y'(t) &= a(t)y(t)[1 - \frac{y(t-\tau(t))}{K(t)}], \quad t \in (-\infty, \infty); \quad t \neq t_j, \quad j \in Z, \\ y(t_j^+) &= y(t_j^-) - I_j(y(t_j)), \quad t = t_j, \end{cases} \tag{3.1}$$

and

$$\begin{cases} y'(t) &= y(t)[a(t) - \sum_{j=1}^{n} b_j(t)y(t - \tau_j(t))], \quad t \in (-\infty, \infty) \quad t \neq t_j, \quad j \in Z, \\ y(t_j^+) &= y(t_j^-) - I_j(y(t_j)), \quad t = t_j. \end{cases} \tag{3.2}$$

Also equation (1.1) includes:

(i) the multiplicative delay periodic logistic equation

$$\begin{cases} y'(t) &= a(t)y(t)[1 - \prod_{j=1}^{n} \frac{y(t-\tau_j(t))}{K(t)}], \quad t \in (-\infty, \infty); \quad t \neq t_j, \quad j \in Z, \\ y(t_j^+) &= y(t_j^-) - I_j(y(t_j)), \quad t = t_j, \end{cases} \tag{3.3}$$

and

(ii) the periodic Michaelis-Menton model

$$\begin{cases} y'(t) &= a(t)y(t)[1-\sum_{j=1}^{n}\frac{a_j(t)y(t-\tau_j(t))}{1+c_j(t)y(t-\tau_j(t))}], \quad t\in(-\infty,\infty).\ t\neq t_j, \quad j\in Z,\\ y(t_j^+) &= y(t_j^-)-I_j(y(t_j)), \quad t=t_j. \end{cases} \tag{3.4}$$

Here $I_j : [0,\infty) \to [0,\infty)$ is continuous, $y(t_j^+)$ and $y(t_j^-)$ represent the right and the left limit of $y(t_j)$, y is left continuous at t_j. There exists a positive integer p such that $t_{j+p} = t_j + T$, $I_{j+p} = I_j$, $j \in Z$. We also assume that $[0,T)\bigcap\{t_j :\ j\in Z\}=\{t_1,t_2,\cdots,t_p\}$.

It follows from Theorem 2.3 that the following results hold.

Corollary 3.1. *Assume that:*

(H_1): $a(t)\in C((-\infty,\infty),(0,\infty))$, $\tau(t)\in C((-\infty,\infty),(-\infty,\infty))$, $K(t)\in C((-\infty,\infty),(0,\infty))$, and $a(t)=a(t+T)$, $K(t)=K(t+T)$, $\tau(t)=\tau(t+T)$.

(H_2): $B\sum\limits_{j=1}^{p}I^0(j)<1$.

Then equation (3.1) has at least one T-periodic positive solution.

Corollary 3.2. *Assume that:*

(H_1): $a(t)\in C((-\infty,\infty),(0,\infty))$, $b_m(t)\in C((-\infty,\infty),(0,\infty))$, $\tau_m(t)\in C((-\infty,\infty),(-\infty,\infty))$, and $a(t)=a(t+T)$,$b_m(t)=b_m(t+T)$, $\tau_m(t)=\tau_m(t+T)$, $m=1,...,n$.

(H_2): $B\sum\limits_{j=1}^{p}I^0(j)<1$.

Then equation (3.2) has at least one T-periodic positive solution.

Corollary 3.3. *Assume that:*

(H_1): $a(t),K(t)\in C((-\infty,\infty),(0,\infty))$, $b_m(t)\in C((-\infty,\infty),(0,\infty))$, $\tau_m(t)\in C((-\infty,\infty),(-\infty,\infty))$, and $a(t)=a(t+T),K(t)=K(t+T)$,$b_m(t)=b_m(t+T)$, $\tau_m(t)=\tau_m(t+T)$, $m=1,...,n$.

(H_2): $B\sum\limits_{j=1}^{p}I^0(j)<1$.

Then equation (3.3) has at least one T-periodic positive solution.

Corollary 3.4. *Assume that:*

(H_1): $a(t),a_m(t),c_m(t)\in C((-\infty,\infty),(0,\infty))$, $\tau_m(t)\in C((-\infty,\infty),(-\infty,\infty))$, and $a(t)=a(t+T)$, $a_m(t)=a_m(t+T)$, $c_m(t)=c_m(t+T)$, $\tau_m(t)=\tau_m(t+T)$, $m=1,...,n$.

(H_2): $B\sum\limits_{j=1}^{p} I^0(j) < 1, \quad \min_{t\in[0,T]}\sum_{j=1}^{n}\frac{a_j(t)}{c_j(t)} + A\sum\limits_{j=1}^{p} I_\infty(j) > 1.$

Then equation (1.5) has at least one T-periodic positive solution.

Corollary 3.1, Corollary 3.2 and Corollary 3.3 can be checked easily, since $f^0 = 0$ and $f_\infty = \infty$. For Corollary 3.4, notice

$$\lim_{u_1,\dots,u_n\uparrow\infty}\sum_{j=1}^{n}\frac{a_j(t)u_j}{1+c_j(t)u_j} \quad = \quad \sum_{j=1}^{n}\frac{a_j(t)}{c_j(t)}, \quad t\in[0,T],$$

and

$$\lim_{u_1,\dots,u_n\downarrow 0^+}\sum_{j=1}^{n}\frac{a_j(t)u_j}{1+c_j(t)u_j} = 0, \quad t\in[0,T],$$

then $f^0 = 0$ and $f_\infty = \min_{t\in[0,T]}\sum_{j=1}^{n}\frac{a_j(t)}{c_j(t)}$, so the result follows from Theorem 2.3.

Corollary 3.1-Corollary 3.4 extend and improve the results of [6], [12].

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