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STRONGLY REVERSIBLE RINGS RELATIVE TO MONOID

Amit Bhooshan Singh[1 §], Prachi Juyal[2], M.R. Khan[3]
[1,2,3]Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia (Central University)
Delhi, 110025, INDIA
[1]e-mail: amit.bhooshan84@gmail.com
[2]e-mail: prachijuyal@yahoo.com
[3]e-mail: rais.mt@jmi.ac.in

Abstract: For a monoid M, we introduce strongly M-reversible rings, which are generalization of strongly reversible rings, and we investigate their properties. We show that if G be a finitely generated Abelian group, then G is torsionfree if and only if there exists a ring R with $|R| \geq 2$ such that R is strongly M-reversible. We also show that if R is right or ring with right classical quotient ring Q, then R is strongly M-reversible iff Q is strongly M-reversible.

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1. Introduction

Throughout this article, R and M denote an associative ring with identity and a monoid, respectively. In [5], Cohn introduced the notion of reversible ring. A ring R is said to be reversible, whenever $a, b \in R$ satisfy $ab = 0$ then $ba = 0$. Anderson and Camillo [2] used the term of ZC_2 for what is called reversible. A ring R is called symmetric, whenever $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. A ring R is called reduced, whenever $a^2 = 0$ implies $a = 0$ for all $a \in R$. A ring R is called semicomuutative, whenever $ab = 0$ implies $aRb = 0$ for all $a, b \in R$.

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§Correspondence author

The following implication holds:

reduced (resp. commutative)$\Rightarrow$ symmetric $\Rightarrow$ reversible $\Rightarrow$ semicommutative.

In [8], Kim and Lee showed that polynomial rings over reversible rings need not be reversible. In [16], Yang and Liu introduced the notation of strongly reversible. A ring R is called strongly reversible, whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$ implies $g(x)f(x) = 0$. All reduced rings are strongly reversible but converse is not true.

Rage and Chhawchharia [15], introduced the notion of an Armendariz ring. A ring R is called Armendariz, whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 \ldots + a_nx^n, g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$ then $a_ib_j = 0$ for all i, j. In [10], Z. Liu studied a generalization of Armendariz rings, which is called M-Armendariz rings, where M is monoid. A ring R is called M-Armendariz, whenever $\alpha = a_1g_1 + a_2g_2 + \ldots + a_ng_n, \beta = b_1h_1 + b_2h_2 + \ldots + b_mg_m \in R[M]$, with $g_i, h_j \in M$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$, for all i, j. In this paper, we introduce strongly M-reversible. A ring R is called strongly M-reversible, whenever $\alpha\beta = 0$ implies $\beta\alpha = 0$, where $\alpha, \beta \in R[M]$. Let $M = (N \cup \{0\}, +)$. Then a ring R is strongly M-reversible if and only if R is strongly reversible.

Recall that a monoid M is called a u.p. monoid (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely in the form ab where $a \in A$ and $b \in B$. The class of u.p. monoid is quite large and important (see [4, 13, 14]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p. monoid M has no nonunity element of finite order.

Motivated by the results of Z. Liu [10], G. Yang and Z.K. Liu [16], we investigate a generalization of strongly reversible rings which we call strongly M-reversible rings.

2. Main Results

Lemma 1. *Let M be u.p. monoid and R a reduced, then $R[M]$ is reduced.*

Proof. Suppose $\alpha = a_1g_1 + a_2g_2 + a_3g_3 + \ldots + a_ng_n$ in $R[M]$, where $a_i \in R, g_i \in M$ for all i, such that $\alpha^2 = 0$. Since R is reduced so by [10, Proposition 1.1] R is M-Armendariz. Thus $a_ia_j = 0$ for all $1 \leq i, j \leq n$. In particular $a_i^2 = 0$ for all $1 \leq i \leq n$. Since R is reduced so $a_i = 0$ for all $1 \leq i \leq n$. Hence $R[M]$ is reduced. □

Proposition 1. *Let M be u.p. monoid and R a reduced. Then R is strongly M-reversible.*

Proof. Suppose $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j$ are in $R[M]$ with $a_i, b_j \in R$ and $g_i, h_j \in M$ for all i, j. Take $\alpha\beta = 0$. So $(\beta\alpha)^2 = (\beta\alpha)(\beta\alpha) = \beta(\alpha\beta)\alpha = 0$. Since R is reduced and Lemma 1, thus $\beta\alpha = 0$. Hence R is strongly M-reversible ring. □

Lemma 2. *Subrings and direct products of strongly M-reversible ring are strongly M-reversible.*

Proposition 2. *Let M be a commutative, cancelative monoid and N and ideal of M. If R is strongly N-reversible ring, then R is strongly M-reversible ring.*

Proof. Suppose that $\alpha = a_1g_1 + a_2g_2 + \ldots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \ldots + b_mh_m$ are in $R[M]$ such that $\alpha\beta = 0$. Take $g \in N$. Then $gg_1, gg_2, \ldots, gg_n, h_1g, h_2g, \ldots, h_mg \in N$ and $gg_i \neq gg_j$ and $h_ig \neq h_jg$ for all $i \neq j$. So

$$\alpha_1\beta_1 = \left(\sum_{i=1}^{n} a_i g g_i\right)\left(\sum_{j=1}^{m} b_j h_j g\right) = 0.$$

Since R is strongly N-reversible so $\beta_1\alpha_1 = 0$. Thus $\beta\alpha = 0$. Therefore R is strongly M-reversible. □

Lemma 3. *Let M be a cyclic group of order $n \geq 2$ and R a ring with unity. Then R is not strongly M-reversible.*

Proof. Suppose that $M = \{e, g, g^2, \ldots, g^{n-1}\}$. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g + \ldots + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g^{n-1}$ and $\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g \in R[M]$. Then $\alpha\beta = 0$. But $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$, so $\beta\alpha \neq 0$. Thus R is not strongly M-reversible. □

Lemma 4. *Let M be a monoid and N a submonoid of M. If R is strongly M-reversible ring, then R is strongly N-reversible.*

Lemma 5. *Let M and N be u.p. monoids. Then so is a monoid $M \times N$.*

Proof. See [10, Lemma 1.13]. □

Let $T(G)$ be set of elements of finite order in an Abelian group G. Then $T(G)$ is fully invariant subgroup of G. G is said to be torsion-free if $T(G) = \{e\}$.

Theorem 1. *Let G be a finitely generated Abelian group. Then the following conditions on G are equivalent:*

(i) G is torsion-free.

(ii) There exists a ring R with $|R| \geq 2$ such that R is strongly G-reversible.

Proof. (2)⇒(1) If $g \in T(G)$ and $g \neq e$, then $N = \langle g \rangle$ is cyclic group of finite order. If a ring $R \neq \{0\}$ is strongly M-reversible. Then by Lemma 4 R is strongly N-reversible, a contradiction with Lemma 3. Thus every ring $R \neq \{0\}$ is not strongly M-reversible. □

(1) ⇒ (2) Let G be a finitely generated Abelian group with $T(G) = \{e\}$. Then $G = \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$ a finite direct product of group $\mathbb{Z}$. By Lemma 5 G is u.p. monoid. Let R be a commutative reduced ring. Then by Proposition 1, R is strongly G-reversible.

Example 1. (see [16], Example 3.7) Let S be a division ring and $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in S \right\}$. Then R is not strongly M-reversible, since it is not reversible. Let M be a monoid with $|M| \geq 2$. Take an ideal $I = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which is strongly M-reversible non-zero proper ideal of R. Take

$$\alpha = \sum_{i=1}^{n} \begin{pmatrix} a_i & b_j & 0 \\ 0 & a_i & c_i \\ 0 & 0 & a_i \end{pmatrix} g_i, \qquad \beta = \sum_{j=1}^{m} \begin{pmatrix} u_j & v_j & 0 \\ 0 & u_j & w_j \\ 0 & 0 & u_j \end{pmatrix} h_j$$

are in $R/I[M]$ satisfying $\alpha\beta = 0$. Then we have that

$$\begin{pmatrix} \Sigma a_i g_i & \Sigma b_i g_i & 0 \\ 0 & \Sigma a_i g_i & \Sigma c_i g_i \\ 0 & 0 & \Sigma a_i g_i \end{pmatrix} \begin{pmatrix} \Sigma u_j h_j & \Sigma v_j h_j & 0 \\ 0 & \Sigma u_j h_j & \Sigma w_j h_j \\ 0 & 0 & \Sigma u_j h_j \end{pmatrix} = 0$$

which implies $\sum_{i=1}^{n} a_i g_i \sum_{j=1}^{m} u_j h_j = 0$, hence $\sum_{i=1}^{n} a_i g_i = 0$ or $\sum_{j=1}^{m} u_j h_j = 0$ since S is a division ring, and it is easy to prove that $\beta\alpha = 0$. There by we get that for any strongly M-reversible nonzero proper ideal I of R, R/I is strongly M-reversible.

Proposition 3. *Suppose that R/I is strongly M-reversible for some ideal I of a ring R. If I is reduced, then R is strongly M-reversible.*

Proof. Suppose that $\alpha = \sum_{i=1}^{n} a_i g_i, \beta = \sum_{j=1}^{m} b_j h_j$ are in $R[M]$ such that

$\alpha\beta = 0$, then we have $\beta\alpha \in I[M]$. Hence $(\beta\alpha)^2 = (\beta\alpha)(\beta\alpha) = \beta(\alpha\beta)\alpha = 0$ implies $\beta\alpha = 0$ since $I[M]$ is reduced, therefore R is strongly M-reversible ring. □

In [6], a right classical quotient ring for a ring R is a right ring of fractions for R with respect to the set of all regular elements in R; the classical left quotient ring is defined symmetrically. The situation when R has a classical right quotient ring Q is also denoted by saying that 'R is right order in Q'. A ring R is called right ore if for each $x, y \in R$ with y regular there exists $r, s \in R$ with r is regular such that $xr = ys$. By [6, Theorem 6.2] R has right quotient ring iff the set of regular elements in R is a right ore set.

Theorem 2. *Let M be a monoid and R a right ore ring with classical right quotient ring Q of a ring R. The ring R is strongly M-reversible if and only if Q is strongly M-reversible.*

Proof. Suppose $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j$ are in $Q[M]$ such that $\alpha\beta = 0$, where $a_i, b_j \in R$ and $g_i, h_j \in M$ for all i, j. Since R is right ore ring with right classical quotient ring Q, by [12, Proposition 2.1.16], we assume that $a_i = p_i u^{-1}, b_j = q_j v^{-1}$ with $p_i, q_j \in R$ for all i, j and regular elements $u, v \in R$. Also by [12, Proposition 2.1.16], for each j, there exists $c_j \in R$ and regular element $s \in R$ such that $u^{-1} q_j = c_j s^{-1}$. Put $\alpha_1 = \sum_{i=1}^{n} p_i q_i, \beta_1 = \sum_{j=1}^{m} q_j h_j$, $\beta_2 = \sum_{j=1}^{m} c_j h_j$, then we have that

$$0 = \alpha\beta = \Big(\sum_{i=1}^{n} a_i g_i\Big)\Big(\sum_{j=1}^{m} b_j h_j\Big) = \Big(\sum_{i=1}^{n} p_i u^{-1} g_i\Big)\Big(\sum_{j=1}^{m} q_j v^{-1} h_j\Big)$$
$$= \Big(\sum_{i=1}^{n} p_i g_i\Big)\Big(\sum_{j=1}^{m} (u^{-1} q_j) v^{-1} h_j\Big) = \Big(\sum_{i=1}^{n} p_i g_i\Big)\Big(\sum_{j=1}^{m} c_j h_j\Big)(vs)^{-1}$$
$$= \alpha_1 \beta_2 (vs)^{-1}.$$

Hence $\alpha_1\beta_2 = 0$, consequently $\alpha_1\beta_1 = 0$ in $R[M]$. Again by [12, Proposition 2.1.16], for each i there exist $d_i \in R$ and regular element $t \in R$ such that $v^{-1} p_i = d_i t^{-1}$. Put $\alpha_2 = \sum_{i=1}^{n} d_i g_i \in R[M]$, then we have

$$0 = \alpha_1 t \beta_1 = \Big(\sum_{i=1}^{n} p_i g_i\Big) t \Big(\sum_{j=1}^{m} q_j h_j\Big) = \Big(\sum_{i=1}^{n} (p_i t) g_i\Big)\Big(\sum_{j=1}^{m} q_j h_j\Big)$$

$$= \left(\sum_{i=1}^{n}(vd_i)g_i\right)\left(\sum_{j=1}^{m}q_jh_j\right) = v\alpha_2\beta_1,$$

thus $\alpha_2\beta_1 = 0$ in $R[M]$. Since R is strongly M-reversible so $\beta_1\alpha_2 = 0$. Now we have that

$$\begin{aligned}\beta\alpha &= \left(\sum_{i=1}^{m}b_jh_j\right)\left(\sum_{i=1}^{n}a_ig_i\right) = \left(\sum_{j=1}^{m}q_jv^{-1}h_j\right)\left(\sum_{i=1}^{n}p_iu^{-1}g_i\right)\\ &= \left(\sum_{j=1}^{m}q_jh_j\right)\left(\sum_{i=1}^{n}v^{-1}p_iu^{-1}g_i\right) = \left(\sum_{j=1}^{m}q_jh_j\right)\left(\sum_{i=1}^{n}d_ig_i\right)(uv)^{-1}\\ &= \beta_1\alpha_2(uv)^{-1} = 0.\end{aligned}$$

Thus Q is strongly M-reversible.

Conversely, suppose Q is strongly M-reversible then by Lemma 2 R is strongly M-reversible. □

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