International Journal of Pure and Applied Mathematics
Volume 63 No. 1 2010, 21-30

PROPERTIES OF WRONSKIAN AND PARTIAL WRONSKIAN

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Abstract: In this paper, the authors discuss some properties of Wronskian. We also introduced a new concept partial Wronskian. Using this we investigate the independence of two variable functions.

AMS Subject Classification: 26A33
Key Words: Wronskian, determinant of matrix

1. Introduction

Differential equations play an important role in science, engineering and social sciences. It is a bridge link between mathematics and science. The importance of differential equations lies in the abdundane of their occurances and their utility in understanding the sciences. It occurs quit frequently in our day to day life.

The concept of Wronskian was first introduced by Hone Wronski (1778-1853). It has a special role in the study of relations to linear dependence and independence of functions (see [1], [2], [7], [5]). Some properties of Wronskian and the solution of Wronskian differential equation was discussed in [3], [4].

In this paper, the authors discuss, some new properties of Wronskian function,

Received: April 23, 2010

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formula for n-th derivative of Wronskian function. Actually, the definition of Wronskian function and its properties are discussed only for the ordinary differentiation. There is no definition of Wronskian for the partial differentiation. To break this, we introduce a new concept called *partial Wronskian* that is, Wronskian for partial differention, and using this we investigate the independence of two variable functions.

2. Properties of Wronskian

Definition 2.1. (see [1]) Let ϕ_1, ϕ_2 be any two differential functions of x, then

$$W[\phi_1, \phi_2] = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}, \tag{2.1}$$

which is called the Wronskian of ϕ_1, ϕ_2. It is a function, and its value at x is denoted by $W[\phi_1, \phi_2](x)$.

Lemma 2.2. *The Wronskian defined in (2.1) satisfies the following properties:*

(a) $W[\phi_1, \phi_2] = -W[\phi_2, \phi_1]$.

(b) $W[\alpha\phi_1, \beta\phi_2] = \alpha\beta W[\phi_1, \phi_2]$.

Here α *and* β *are constants.*

Proof. (a) From the definition of Wronskian, we get

$$W[\phi_2, \phi_1] = \phi_2\phi_1' - \phi_2'\phi_1 = -W[\phi_1, \phi_2].$$

(b) By the definition of Wronskian with simple steps, our result is trivial. □

Lemma 2.3. *Let* ϕ_1, ϕ_2 *be any two differential functions of* x, *then*

$$W[\phi_1 + \alpha, \phi_2 + \alpha] = W[\phi_1, \phi_2] + \alpha\frac{d}{dx}[\phi_2 - \phi_1], \tag{2.2}$$

where α *is a constant.*

Proof. From definition of Wronskian, we obtain

$$\begin{aligned} W[\phi_1 + \alpha, \phi_2 + \alpha] &= (\phi_1 + \alpha)\phi_2' - \phi_1'(\phi_2 + \alpha) \\ &= W[\phi_1, \phi_2] + \alpha\frac{d}{dx}(\phi_2 - \phi_1). \end{aligned}$$

Hence the lemma is proved. □

Theorem 2.4. *Let ϕ_1, ϕ_2 be any two differential functions of x, then*

$$\phi_1 W\left[\frac{\phi_2}{\phi_1}, \phi_1\right] + \phi_2 W\left[\frac{\phi_1}{\phi_2}, \phi_2\right] = \frac{d}{dt}(\phi_1\phi_2). \tag{2.3}$$

Proof. Using the definition of Wronskian, we get

$$W\left[\frac{\phi_2}{\phi_1}, \phi_1\right] = \frac{2\phi_2\phi_1' - \phi_1\phi_2'}{\phi_1} = \frac{-W[\phi_1, \phi_2] + \phi_2\phi_1'}{\phi_1}.$$

Hence

$$\phi_1 W\left[\frac{\phi_2}{\phi_1}, \phi_1\right] = -W[\phi_1, \phi_2] + \phi_2\phi_1'. \tag{2.4}$$

Similarly,

$$\phi_2 W\left[\frac{\phi_1}{\phi_2}, \phi_2\right] = W[\phi_1, \phi_2] + \phi_1\phi_2'. \tag{2.5}$$

Adding (2.4) and (2.5), we arrive to (2.3). □

Theorem 2.5. *Let ϕ_i, φ_j be differential functions of x for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, then*

$$W\left[\sum_{i=1}^{n}\phi_i, \sum_{j=1}^{m}\varphi_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} W\left[\phi_i, \varphi_j\right]. \tag{2.6}$$

Proof. By the properties of the determinant our result is trival. □

Theorem 2.6. *If ϕ_i, φ_j are differential functions of x for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$, then*

$$W\left[\prod_{i=1}^{n}\phi_i, \prod_{j=1}^{n}\varphi_j\right] = \sum_{i=1}^{n}\left\{\left(\prod_{j=1 j\neq i}^{n}\phi_j\varphi_j\right) W\left[\phi_i, \varphi_i\right]\right\}. \tag{2.7}$$

Proof. For $i = 1, 2$ and $j = 1, 2$, it follows that

$$\begin{aligned} W\left[\phi_1\phi_2, \varphi_1\varphi_2\right] &= \begin{vmatrix} \phi_1\phi_2 & \varphi_1\varphi_2 \\ \phi_1\phi_2' + \phi_1'\phi_2 & \varphi_1\varphi_2' + \varphi_1'\varphi_2 \end{vmatrix} \\ &= \phi_1\varphi_1 W\left[\phi_2, \varphi_2\right] + \phi_2\varphi_2 W\left[\phi_1, \varphi_1\right] \\ &= \sum_{i=1}^{2}\left\{\left(\prod_{j=1\ j\neq i}^{2}\phi_j\varphi_j\right) W\left[\phi_i, \varphi_i\right]\right\}. \end{aligned}$$

Similarly, for $i = 1, 2, 3$ and $j = 1, 2, 3$, we obtain

$$\begin{aligned} & W\left[\phi_1\phi_2\phi_3, \varphi_1\varphi_2\varphi_3\right] \\ &= \begin{vmatrix} \phi_1\phi_2\phi_3 & \varphi_1\varphi_2\varphi_3 \\ \phi_1'\phi_2\phi_3 + \phi_1\phi_2'\phi_3 + \phi_1\phi_2\phi_3' & \varphi_1'\varphi_2\varphi_3 + \varphi_1\varphi_2'\varphi_3 + \varphi_1\varphi_2\varphi_3' \end{vmatrix} \end{aligned}$$

$$= \phi_1\phi_2\varphi_1\varphi_2 W[\phi_3,\varphi_3] + \phi_2\phi_3\varphi_2\varphi_3 W[\phi_1,\varphi_1] + \phi_1\phi_3\varphi_1\varphi_3 W[\phi_2,\varphi_2]$$
$$= \sum_{i=1}^{3}\left\{\left(\prod_{j=1\ j\neq i}^{3} \phi_j\varphi_j\right) W[\phi_i,\varphi_i]\right\}.$$

Proceeding like this, for $i = 1, 2\cdots, n$ and $j = 1, 2, \cdots, n$, one can get the result (2.11). □

The following theorem gives the general formula for n-th derivative of Wronskian.

Theorem 2.7. *Let ϕ, φ be differential functions of x, then*

$$\frac{d^n W[\phi,\varphi]}{dt^n} = \begin{vmatrix} \phi & \varphi \\ \phi^{(n+1)} & \varphi^{(n+1)} \end{vmatrix} + (n-1)\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(n)} & \varphi^{(n)} \end{vmatrix} + \sum_{i=1}^{p}\left(g_i^{n-2i}\begin{vmatrix} \phi^{(i+1)} & \varphi^{(i+1)} \\ \phi^{(n-i)} & \varphi^{(n-i)} \end{vmatrix}\right), \tag{2.8}$$

where $p = \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd;} \\ \dfrac{n-2}{2} & \text{if } n \text{ is even,} \end{cases}$

$$g_m^n = \left(\sum_{i=1}^{n} g_{m-1}^{i+2}\right) - g_{m-1}^2 \tag{2.9}$$

for all $n = 1, 2, 3\cdots$, $m = 2, 3, 4\cdots$. *For* $m = 1$,

$$g_1^n = \left(\sum_{i=1}^{n} i\right) - 1. \tag{2.10}$$

Here $\phi^{(n)}, \varphi^{(n)}$ denote the n-th derivative of the functions ϕ and φ, respectively.

Proof. By the definition of Wronskian function, we have

$$W[\phi,\varphi] = \begin{vmatrix} \phi & \varphi \\ \phi^{(1)} & \varphi^{(1)} \end{vmatrix}.$$

Now

$$\frac{d(W[\phi,\varphi])}{dt} = \begin{vmatrix} \phi & \varphi \\ \phi^{(2)} & \varphi^{(2)} \end{vmatrix} = 1\begin{vmatrix} \phi & \varphi \\ \phi^{(2)} & \varphi^{(2)} \end{vmatrix} + 0\begin{vmatrix} \phi^{(1)} & \varphi(1) \\ \phi^{(1)} & \varphi^{(1)} \end{vmatrix}.$$

$$\frac{d^2(W[\phi,\varphi])}{dt^2} = \begin{vmatrix} \phi & \varphi \\ \phi^{(3)} & \varphi^{(3)} \end{vmatrix} + 1\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(3)} & \varphi^{(3)} \end{vmatrix}$$

$$= \begin{vmatrix} \phi & \varphi \\ \phi^{(3)} & \varphi^{(3)} \end{vmatrix} + 1\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(3)} & \varphi^{(3)} \end{vmatrix}.$$

$$\begin{aligned}
\frac{d^3\left(W\left[\phi,\varphi\right]\right)}{dt^3} &= \begin{vmatrix} \phi & \varphi \\ \phi^{(4)} & \varphi^{(4)} \end{vmatrix} + 2\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(3)} & \varphi^{(3)} \end{vmatrix} \\
&= \begin{vmatrix} \phi & \varphi \\ \phi^{(4)} & \varphi^{(4)} \end{vmatrix} + 2\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(3)} & \varphi^{(3)} \end{vmatrix} + g_1^1\begin{vmatrix} \phi^{(2)} & \varphi^{(2)} \\ \phi^{(2)} & \varphi^{(2)} \end{vmatrix}. \\
\frac{d^4\left(W\left[\phi,\varphi\right]\right)}{dt^4} &= \begin{vmatrix} \phi & \varphi \\ \phi^{(5)} & \varphi^{(5)} \end{vmatrix} + 3\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(4)} & \varphi^{(4)} \end{vmatrix} + 2\begin{vmatrix} \phi^{(2)} & \varphi^{(2)} \\ \phi^{(3)} & \varphi^{(3)} \end{vmatrix} \\
&= \begin{vmatrix} \phi & \varphi \\ \phi^{(5)} & \varphi^{(5)} \end{vmatrix} + 3\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(4)} & \varphi^{(4)} \end{vmatrix} + g_1^2\begin{vmatrix} \phi^{(2)} & \varphi^{(2)} \\ \phi^{(3)} & \varphi^{(3)} \end{vmatrix}. \\
\frac{d^5\left(W\left[\phi,\varphi\right]\right)}{dt^5} &= \begin{vmatrix} \phi & \varphi \\ \phi^{(6)} & \varphi^{(6)} \end{vmatrix} + 4\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(5)} & \varphi^{(5)} \end{vmatrix} + 5\begin{vmatrix} \phi^{(2)} & \varphi^{(2)} \\ \phi^{(4)} & \varphi^{(4)} \end{vmatrix} \\
&= \begin{vmatrix} \phi & \varphi \\ \phi^{(6)} & \varphi^{(6)} \end{vmatrix} + 4\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(5)} & \varphi^{(5)} \end{vmatrix} + g_1^3\begin{vmatrix} \phi^{(2)} & \varphi^{(2)} \\ \phi^{(4)} & \varphi^{(4)} \end{vmatrix} \\
&\quad + g_2^1\begin{vmatrix} \phi^{(3)} & \varphi^{(3)} \\ \phi^{(3)} & \varphi^{(3)} \end{vmatrix}.
\end{aligned}$$

Continue in this manner the n-th derivative of $W\left[\phi,\varphi\right]$ can be obtained as follows.

Case (i). When n is odd,

$$\begin{aligned}
\frac{d^nW\left[\phi,\varphi\right]}{dt^n} &= \begin{vmatrix} \phi & \varphi \\ \phi^{(n+1)} & \varphi^{(n+1)} \end{vmatrix} + (n-1)\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(n)} & \varphi^{(n)} \end{vmatrix} \\
&\quad + \sum_{i=1}^{\frac{n-1}{2}}\left(g_i^{n-2i}\begin{vmatrix} \phi^{(i+1)} & \varphi^{(i+1)} \\ \phi^{(n-i)} & \varphi^{(n-i)} \end{vmatrix}\right).
\end{aligned}$$

Case (ii). When n is even,

$$\begin{aligned}
\frac{d^nW\left[\phi,\varphi\right]}{dt^n} &= \begin{vmatrix} \phi & \varphi \\ \phi^{(n+1)} & \varphi^{(n+1)} \end{vmatrix} + (n-1)\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(n)} & \varphi^{(n)} \end{vmatrix} \\
&\quad + \sum_{i=1}^{\frac{n-2}{2}}\left(g_i^{n-2i}\begin{vmatrix} \phi^{(i+1)} & \varphi^{(i+1)} \\ \phi^{(n-i)} & \varphi^{(n-i)} \end{vmatrix}\right).
\end{aligned}$$

Combining case (i) and case (ii) together, we arrive to (2.8). □

Example 2.8. The derivatives

$$\frac{d^8W\left[\phi,\varphi\right]}{dt^8}, \frac{d^9W\left[\phi,\varphi\right]}{dt^9}$$

can be obtained from equation (2.8) when $n = 8$ and $n = 9$ as follows

$$\begin{aligned}
\frac{d^8W[\phi,\varphi]}{dt^8} &= \begin{vmatrix} \phi & \varphi \\ \phi^{(9)} & \varphi^{(9)} \end{vmatrix} + 7\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(8)} & \varphi^{(8)} \end{vmatrix} \\
&\quad + g_1^6\begin{vmatrix} \phi^{(2)} & \varphi^{(2)} \\ \phi^{(7)} & \varphi^{(7)} \end{vmatrix} + g_2^4\begin{vmatrix} \phi^{(3)} & \varphi^{(3)} \\ \phi^{(6)} & \varphi^{(6)} \end{vmatrix} \\
&\quad + g_3^2\begin{vmatrix} \phi^{(4)} & \varphi^{(4)} \\ \phi^{(5)} & \varphi^{(5)} \end{vmatrix}, \\
\frac{d^9W[\phi,\varphi]}{dt^9} &= \begin{vmatrix} \phi & \varphi \\ \phi^{(10)} & \varphi^{(10)} \end{vmatrix} + 8\begin{vmatrix} \phi^{(1)} & \varphi^{(1)} \\ \phi^{(9)} & \varphi^{(9)} \end{vmatrix} \\
&\quad + g_1^7\begin{vmatrix} \phi^{(2)} & \varphi^{(2)} \\ \phi^{(8)} & \varphi^{(8)} \end{vmatrix} + g_2^5\begin{vmatrix} \phi^{(3)} & \varphi^{(3)} \\ \phi^{(7)} & \varphi^{(7)} \end{vmatrix} \\
&\quad + g_3^3\begin{vmatrix} \phi^{(4)} & \varphi^{(4)} \\ \phi^{(6)} & \varphi^{(6)} \end{vmatrix} + g_4^1\begin{vmatrix} \phi^{(5)} & \varphi^{(5)} \\ \phi^{(5)} & \varphi^{(5)} \end{vmatrix}.
\end{aligned}$$

The above derivatives are easily verified by differentiating of the Wronskian function.

The following corollary is the immediate consequence of Theorem 2.7.

Corollary 2.9. *If ϕ, φ are differential functions of x, then*

$$\begin{aligned}
\frac{d^nW[\phi,\varphi]}{dt^n} &= \phi\varphi^{(n+1)} + (n-1)\phi^{(1)}\varphi^{(n)} + \sum_{i=1}^{p} g_i^{n-2i}\phi^{(i+1)}\varphi^{(n-i)} \\
&- \varphi\phi^{(n+1)} - (n-1)\varphi^{(1)}\phi^{(n)} - \sum_{i=1}^{p} g_i^{n-2i}\varphi^{(i+1)}\phi^{(n-i)}, \qquad (2.11)
\end{aligned}$$

where $p = \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd,} \\ \dfrac{n-2}{2} & \text{if } n \text{ is even,} \end{cases}$ $g_m^n = \left(\sum_{i=1}^{n} g_{m-1}^{i+2}\right) - g_{m-1}^2$ *for all* $n = 1, 2, 3\cdots$, $m = 2, 3, 4\cdots$. *For* $m = 1$, $g_1^n = \left(\sum_{i=1}^{n} i\right) - 1$.

3. Partial Wronskian

In this section, we introduce the new concept of partial Wronskian. Using this we investigate the independence of two variable functions.

Definition 3.1. Let ϕ_1, ϕ_2 be any two functions of variables x and y which are defined in the region R, then the partial Wronskian of ϕ_1, ϕ_2 is defined by

$$\overline{W}[\phi_1, \phi_2] = \begin{vmatrix} \phi_1 & \phi_2 \\ D(\phi_1) & D(\phi_2) \end{vmatrix}, \tag{3.1}$$

where

$$D(\phi_i) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\phi_i \tag{3.2}$$

for $i = 1, 2$.

Theorem 3.2. *If the partial Wronskian of two functions $\phi(x, y)$ and $\psi(x, y)$ defined on region R, is non zero for at least one point of the region R, then the functions $\phi(x, y)$ and $\psi(x, y)$ are linearly independent on R.*

Proof. The proof is given by the method of contradiction. Let as assume on the contrary that the functions $\phi(x, y)$ and $\psi(x, y)$ are linearly independent on R. Then there exist constants (at least one of them non zero) c_1 and c_2 such that

$$c_1\phi(x, y) + c_2\psi(x, y) = 0, \quad \text{for all} \quad (x, y) \in R. \tag{3.3}$$

Differentiating partially with respect to x and y and adding those equations, we obtain

$$c_1 D\phi(x, y) + c_2 D\psi(x, y) = 0, \quad \text{for all} \quad (x, y) \in R. \tag{3.4}$$

By assumption there exists a point (x_0, y_0) such that

$$\begin{vmatrix} \phi(x_0, y_0) & \psi(x_0, y_0) \\ D\phi(x_0, y_0) & D\psi(x_0, y_0) \end{vmatrix} \neq 0. \tag{3.5}$$

From the relations (3.3) and (3.4), we get

$$\begin{aligned} c_1\phi(x_0, y_0) + c_2\psi(x_0, y_0) &= 0, \\ c_1 D\phi(x_0, y_0) + c_2 D\psi(x_0, y_0) &= 0. \end{aligned} \tag{3.6}$$

Assume the relation (3.6) is a system of linear equations with c_1 and c_2 as unknown quantities. From the theory of algebraic equations it is known that if the relation (3.5) holds then the system (3.6) possess only zero solution, i.e. $c_1 = 0$ and $c_2 = 0$.

This is a contradiction to the assumption. Thus the functions ϕ and ψ are linearly independend. □

Example 3.3. The functions $\phi = e^{\alpha x}\cos\beta y$ and $\psi = e^{\alpha x}\sin\beta y$, where α and β are constants with $\beta \neq 0$, are linearly independent functions. Since, $\overline{W}[\phi, \psi] = \beta e^{\alpha x} \neq 0$ for all $(x, y) \in R$.

Theorem 3.4. *If two functions $\phi(x, y)$ and $\psi(x, y)$ which are partial* *differentiable with respect to x and y on an region R are linearly dependent on R, then their partial Wronkian*

$$\overline{W}[\phi,\psi] = 0, \quad \textit{for all} \quad (x,y) \in R.$$

Proof. Suppose that $\overline{W}[\phi,\psi] \neq 0$ for $(x_0,y_0) \in R$. By the above theorem ϕ and ψ are linearly independent which is the contradiction to the hypothesis. Hence $\overline{W}[\phi,\psi] = 0$. □

Definition 3.5. Let $\phi_1,\phi_2,\phi_3\cdots\phi_n$ be any functions of variables x and y, which are defined in the region R, then the *partial Wronskian* of $\phi_1,\phi_2,\phi_3\cdots\phi_n$ is defined by

$$\overline{W}[\phi_1,\phi_2,\phi_3\cdots\phi_n] = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ D\phi_1 & D\phi_2 & \cdots & D\phi_n \\ D^2\phi_1 & D^2\phi_2 & \cdots & D^2\phi_n \\ \vdots & \vdots & \vdots & \vdots \\ D^{n-1}\phi_1 & D^{n-1}\phi_2 & \cdots & D^{n-1}\phi_n \end{vmatrix}, \tag{3.7}$$

where

$$D^n(\phi_i) = \left(\frac{\partial^n}{\partial x^n} + \frac{\partial^n}{\partial y^n}\right)\phi_i \tag{3.8}$$

for $i = 1,2,3,\cdots,n$.

The following Theorems 3.6 and 3.7 immediately follow from Theorems 3.2, 3.4 and Definition 3.5, respectively.

Theorem 3.6. *If the partial Wronskian of functions $\phi_1,\phi_2,\phi_3\cdots\phi_n$ defined on region R, is non zero for at least one point of the region R, then the functions $\phi_1,\phi_2,\phi_3\cdots\phi_n$ are linearly independent on R.*

Theorem 3.7. *If the functions $\phi_1,\phi_2,\phi_3\cdots\phi_n$ which are differentiable on region R are linearly dependent on R, then their partial Wronskian $\overline{W}[\phi_1,\phi_2,\phi_3\cdots\phi_n] = 0$ for all $(x,y) \in R$.*

Lemma 3.8. *If $\phi(x,y)$ and $\psi(x,y)$ are any two functions, then*

$$\overline{W}[\phi,\psi] = (\phi\psi_x - \psi\phi_x) + (\phi\psi_y - \psi\phi_y). \tag{3.9}$$

Proof. By the definition of partial Wronskian, we have

$$\overline{W}[\phi,\psi] = \phi D\psi - \psi D\phi = (\phi\psi_x - \psi\phi_x) + (\phi\psi_y - \psi\phi_y). \tag{3.10}$$

□

Lemma 3.9. *If $\phi(x,y)$ and $\psi(x,y)$ are any two functions, then*

$$D\overline{W}[\phi,\psi] = (\phi\psi_x x - \psi\phi_x x) + 2(\phi\psi_x y - \psi\phi_x y) + (\phi\psi_y y - \psi\phi_y y). \tag{3.11}$$

Proof. Using the equation (3.10), we have

$$D\overline{W}[\phi,\psi] = D(\phi\psi_x - \psi\phi_x) + (\phi\psi_y - \psi\phi_y)$$
$$= (\phi\psi_x x - \psi\phi_x x) + 2(\phi\psi_x y - \psi\phi_x y) + (\phi\psi_y y - \psi\phi_y y).$$

□

Theorem 3.10. *Let ϕ, ψ are two solutions of the partial differential equation*

$$M(u) = 0 \tag{3.12}$$

where

$$M(u(x,y)) = A(x,y)(u_{xx} + 2u_{xy} + u_{yy}) + B(x,y)(u_x + u_y) + C(x,y)u$$

on region R. Then $\overline{W}[\phi,\psi](x,y)$ satisfies the partial differential equation

$$u_x + u_y + \frac{B}{A}u = 0. \tag{3.13}$$

Proof. By using Lemma 3.8 and Lemma 3.9, we have

$$\phi M(\psi) - \psi M(\phi) = \frac{\partial\overline{W}}{\partial x} + \frac{\partial\overline{W}}{\partial y} + \frac{B}{A}\overline{W} \tag{3.14}$$

Since ϕ, ψ are two solutions of (3.12), it follows from (3.14) that

$$\frac{\partial\overline{W}}{\partial x} + \frac{\partial\overline{W}}{\partial y} + \frac{B}{A}\overline{W} = 0.$$

Hence $\overline{W}$ satisfies the partial differential equation (3.13). □

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