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UNIQUENESS ALGEBRAIC CONDITIONS IN THE STUDY OF SECOND ORDER ELLIPTIC SYSTEMS

Ioan Cristian Chifu[1 §], Ionut Traian Luca[2], Gabriela Petrusel[3]
[1,2,3]Department of Business
Faculty of Business
Babes-Bolyai University
7, Horea Str., Cluj-Napoca, RO-400174, ROMANIA
[1]e-mail: cristian.chifu@tbs.ubbcluj.ro
[2]e-mail: ionut.luca@tbs.ubbcluj.ro
[3]e-mail: gabi.petrusel@tbs.ubbcluj.ro

Abstract: The purpose of this paper is to give some algebraic conditions for the coefficients of a second order elliptic system in order to obtain some uniqueness and comparison results.

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1. Introduction

Let us consider the following second order elliptic system

$$Lu := \sum_{i,j=1}^{m} A_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} A_i(x) \frac{\partial u}{\partial x_i} + A_o(x) u = 0, \tag{1}$$

where A_{ij}, A_i, $A_o \in M_n(\mathbb{R})$, and the following statement holds

$$\left.\begin{array}{l} u \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right), \\ Lu = 0, \quad \text{in} \quad \Omega, \\ u = 0, \quad \text{on} \quad \partial\Omega, \end{array}\right\} \Longrightarrow u \equiv 0 \quad \text{in} \quad \overline{\Omega}, \Omega \subset \mathbb{R}^m. \tag{2}$$

In the literature, there are many results which state some conditions in which (2) is true. We can remind here the results of Somigliana [8], Giraud [4] and Rus [5]. Cioranescu in [3] studies elliptic systems in which the main part is formed

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§Correspondence author

by Laplace operators. It is well known the fact that if u satisfies a maximum principle, then the statement (2) automatically holds.

The aim of this paper is to determine effective algebraic conditions for A_{ij}, A_i, A_o such that the statement (2) to take place, without using a maximum principle.

Let $A \in M_n(\mathbb{R})$, J – the Jordan normal form of A. We know that there exists a nonsingular matrix T such that $A = TJT^{-1}$.

We shall denote by

$$\begin{aligned} \widetilde{\alpha} &= \begin{cases} \frac{1}{n}\sum\limits_{k=1}^{s} n_k\lambda_k, & \lambda_k \in \mathbb{R}, \\ \frac{1}{n}\sum\limits_{k=1}^{s} n_k \ Re\lambda_k, & \lambda_k \in \mathbb{C}\backslash\mathbb{R}, \end{cases} \\ \gamma_F &= \|T\|_F \cdot \left\|T^{-1}\right\|_F, \\ m_F &= \|J - \widetilde{\alpha} I\|_F, \end{aligned}$$

where λ_k are the eigenvalues of A, n_k is the number of λ_k which appears in Jordan blocks (generated by λ_k) and $\|\cdot\|_F$ is the euclidean norm of the matrix (see [2]).

We shall use the following result given in Chifu-Oros [2]:

Theorem 1. *Let $\varphi_{\|\cdot\|} : \mathbb{R} \to \mathbb{R}, \varphi_{\|\cdot\|}(\alpha) = \|A - \alpha I_n\|$, $\|\cdot\|$ being one of the following norms: $\|\cdot\|_F$, $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$. In these conditions*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) \leq \sqrt{n}\gamma_F m_F.$$

Remark 1. In the case of Euclidean norm $\|\cdot\|_F$ and spectral norm $\|\cdot\|_2$ we have that $\varphi_{\|\cdot\|}(\widetilde{\alpha}) \leq \gamma_F m_F$ (see [2]). Because $n \geq 2$, if $m_F \neq 0$, then

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) < \sqrt{n}\gamma_F m_F.$$

Conditions determined here will be very useful to obtain comparison results (see Sections 3 and 4 of the paper).

2. Establishing the Conditions in which the Statement (2) Holds

Because of the method of prove, the method inspired by Rus [5], we shall consider system (1) in the form

$$Lu := \delta^2 I_n \Delta u + \sum_{i=1}^{m} A_i\left(x\right) \frac{\partial u}{\partial x_i} + A_o\left(x\right) u = 0, \tag{3}$$

where $\delta > 0, A_{ij}, A_i, A_o \in C\left(\overline{\Omega}, M_n(\mathbb{R})\right), \Omega \subset \mathbb{R}^m$, bounded domain.

Let $u \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right)$, $u \neq 0$, be a solution of the system (3) with the property that $u|_{\partial\Omega} = 0$. We have

$$u^* L u = \delta^2 \sum_{i=1}^{m} u^* \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{m} u^* A_i(x) \frac{\partial u}{\partial x_i} + u^* A_o(x) u;$$

$$\delta^2 \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(u^* \frac{\partial u}{\partial x_i}\right) = u^* L u + \delta^2 \sum_{i=1}^{m} \frac{\partial u^*}{\partial x_i}\frac{\partial u}{\partial x_i} - \sum_{i=1}^{m} u^* A_i(x) \frac{\partial u}{\partial x} - u^* A_o(x) u.$$

If we integrate on $\overline{\Omega}$, we obtain

$$\int_{\Omega} \left(\delta^2 \sum_{i=1}^{m} \frac{\partial u^*}{\partial x_i}\frac{\partial u}{\partial x_i} - \sum_{i=1}^{m} u^* A_i(x) \frac{\partial u}{\partial x_i} - u^* A_o(x) u\right) dx = 0. \tag{4}$$

Denote

$$E := \delta^2 \sum_{i=1}^{m} \frac{\partial u^*}{\partial x_i}\frac{\partial u}{\partial x_i} - \sum_{i=1}^{m} u^* A_i(x) \frac{\partial u}{\partial x_i} - u^* A_o(x) u. \tag{5}$$

We shall show that under some assumptions for the coefficients $A_i(x)$ and $A_o(x)$ this expression is positive and this will lead us to a contradiction.

Remark 2. If $E \equiv 0$, then $\|u\|^2$ is a constant and because $u|_{\partial\Omega} = 0$, we shall obtain that $u \equiv 0$.

Let $u = R \cdot e$, where $R = \|u\| = \left(\sum\limits_{i=1}^{n} u_i^2\right)^{\frac{1}{2}}$, $e \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right)$, $e = (e_1, ..., e_n)^t$, $e^* = (e_1, ..., e_n)$, $\|e\| = \left(\sum\limits_{i=1}^{n} e_i^2\right)^{\frac{1}{2}} = 1$. A simple computation shows that

$$E = \delta^2 \sum_{i=1}^{m} \left(R'_{x_i}\right)^2 - \left[\sum_{i=1}^{m} \left(e^* A_i(x) e\right) R'_{x_i}\right] R - \left(e^* L e\right) R^2, \tag{6}$$

where

$$e^* L e = -\delta^2 \sum_{i=1}^{m} \left\|e'_{x_i}\right\|^2 + \sum_{i=1}^{m} e^* A_i(x) e'_{x_i} + e^* A_o(x) e.$$

From Theorem 1 we know that for every $x \in \Omega$ there exist $\widetilde{\alpha}_i(x) \in \mathbb{R}$ such that

$$\|A_i(x) - \widetilde{\alpha}_i(x) I_n\| \leq \gamma_F^i m_F^i, i = \overline{1, m}.$$

If we suppose that $m_F^i \neq 0, i = \overline{1, m}$, then

$$\|A_i(x) - \widetilde{\alpha}_i(x) I_n\| < \sqrt{n}\gamma_F^i m_F^i, i = \overline{1, m}.$$

It is easy to see that

$$e^*Le \leq \frac{1}{4\delta^2}\sum_{i=1}^{m}\|A_i(x)-\widetilde{\alpha_i}(x)I_n\|^2+e^*A_o(x)e.$$

If we suppose that

$$e^*A_o(x)e \leq -\frac{1}{4\delta^2}n\sum_{i=1}^{m}\left(\gamma_F^i m_F^i\right)^2, \forall x\in\Omega, \tag{7}$$

then

$$e^*Le \leq \frac{1}{4\delta^2}\sum_{i=1}^{m}\left(\|A_i(x)-\widetilde{\alpha_i}(x)I_n\|^2-n\left(\gamma_F^i m_F^i\right)^2\right) := -p^2(x)<0, \tag{8}$$

where

$$p^2(x) := \sum_{i=1}^{m}p_i^2(x)>0, \tag{9}$$

and

$$p_i^2(x) := -\frac{1}{4\delta^2}\left(\|A_i(x)-\widetilde{\alpha_i}(x)I_n\|^2-n\left(\gamma_F^i m_F^i\right)^2\right), \quad i=\overline{1,m}. \tag{10}$$

From (8), (9) and (10) we obtain

$$E\geq\sum_{i=1}^{m}\left\{\delta^2\left(R'_{x_i}\right)^2-\left(e^*A_i(x)e\right)R'_{x_i}R+p_i^2(x)R^2\right\}. \tag{11}$$

Let

$$E_i := \delta^2\left(R'_{x_i}\right)^2-\left(e^*A_i(x)e\right)R'_{x_i}R+p_i^2(x)R^2, \quad i=\overline{1,m}.$$

Suppose that

$$e^*A_i(x)e\leq 2\delta p_i(x), \quad i=\overline{1,m}, x\in\Omega. \tag{12}$$

If (7) and (12) take place, then the quadric form (6) is positive and that means that the integral cannot be identically null, only if $E\equiv 0$.

In this way, if $m_F^i\neq 0, i=\overline{1,m}$, we obtain the following result:

Theorem 2. *Suppose that:*

(i) $e^*A_o(x)e\leq -\frac{1}{4\delta^2}n\sum\limits_{i=1}^{m}\left(\gamma_F^i m_F^i\right)^2, \forall x\in\Omega;$

(ii) $e^*A_i(x)e\leq 2\delta p_i(x), \forall x\in\Omega, i=\overline{1,m};$
$\forall e\in C^2\left(\overline{\Omega},\mathbb{R}^n\right), \|e\|=\left(\sum\limits_{i=1}^{n}e_i^2\right)^{\frac{1}{2}}=1,$ *with* p_i *as in (10).*

In these conditions the statement (2) holds.

Example 1. If we consider system (3), in the case $n = 2$, with $A_i = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}$, $a_2, a_3 > 0$, $a_1^2 \leq a_2^2 + 4a_2a_3 + a_3^2$, we have an example of matrix A_i which verifies (12), with $p^2 = \frac{1}{4\delta^2}\left(3a_2^2 + 8a_2a_3 + 3a_3^2\right)$.

Remark 3. If $m_F^i = 0, i = \overline{1,m}$, then the statement (2) still holds if

$$e^* A_o(x)\, e \leq -\frac{1}{4\delta^2}\sum_{i=1}^{m}\left(\widetilde{\alpha}_i(x)\right)^2, \quad \forall x \in \Omega.$$

3. A Comparison Result

Let us consider the following second order differential systems

$$Lu := \lambda^2 I_n \Delta u + \sum_{i=1}^{m} A_i(x)\frac{\partial u}{\partial x_i} + A_o(x)\, u = 0, \tag{13}$$

$$Mv := \mu^2 I_n \Delta v + B_o(x)\, v = 0, \tag{14}$$

where $A_i \in C^1\left(\overline{\Omega}, M_n(\mathbb{R})\right), A_o, B_o \in C\left(\overline{\Omega}, M_n(\mathbb{R})\right), B_o$ symmetric, $\lambda > \mu > 0$.

Let $u \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right)$, $u \neq 0$, be a solution of the system (13) with the property that $u|_{\partial\Omega} = 0$. We have

$$\int_{\Omega}\left(\lambda^2\sum_{i=1}^{m}\frac{\partial u^*}{\partial x_i}\frac{\partial u}{\partial x_i} - \sum_{i=1}^{m} u^* A_i(x)\frac{\partial u}{\partial x_i} - u^* A_o(x)\, u\right)dx = 0.$$

Supposing that there exists a matrix solution S of (14) such that $\det S(x) \neq 0$ in $\overline{\Omega}$ and the matrix $\frac{\partial S}{\partial x_i}S^{-1}, i = \overline{1,m}$, is symmetric, we shall obtain that

$$\int_{\Omega}\left(\lambda^2\sum_{i=1}^{m}\frac{\partial u^*}{\partial x_i}\frac{\partial u}{\partial x_i} - \sum_{i=1}^{m} u^* A_i(x)\frac{\partial u}{\partial x_i} - u^* A_o(x)\, u\right)dx$$
$$+\int_{\Omega}\left(\mu^2\sum_{i=1}^{m}\frac{\partial^2(uS)}{\partial x_i^2}S^{-1} + u^* B_o(x)\, u\right)dx = 0,$$

and from here

$$\int_{\Omega}\left(\left(\lambda^2 - \mu^2\right)\sum_{i=1}^{m}\frac{\partial u^*}{\partial x_i}\frac{\partial u}{\partial x_i} - \sum_{i=1}^{m} u^* A_i(x)\frac{\partial u}{\partial x_i} - u^*\left(A_o(x) - B_o(x)\right)u\right)dx$$
$$+\int_{\Omega}\left(\mu^2\sum_{i=1}^{m}\left\|\frac{\partial u}{\partial x_i} - \frac{\partial S}{\partial x_i}S^{-1}u\right\|^2\right)dx = 0. \tag{15}$$

If we denote

$$E := \left(\lambda^2 - \mu^2\right) \sum_{i=1}^{m} \frac{\partial u^*}{\partial x_i} \frac{\partial u}{\partial x_i} - \sum_{i=1}^{m} u^* A_i\left(x\right) \frac{\partial u}{\partial x_i} - u^*\left(A_o\left(x\right) - B_o\left(x\right)\right) u + \mu^2 \sum_{i=1}^{m} \left\| \frac{\partial u}{\partial x_i} - \frac{\partial S}{\partial x_i} S^{-1} u \right\|^2,$$

we have

$$E \geq \left(\lambda^2 - \mu^2\right) \sum_{i=1}^{m} \frac{\partial u^*}{\partial x_i} \frac{\partial u}{\partial x_i} - \sum_{i=1}^{m} u^* A_i\left(x\right) \frac{\partial u}{\partial x_i} - u^*\left(A_o\left(x\right) - B_o\left(x\right)\right) u.$$

Let

$$E_1 := \left(\lambda^2 - \mu^2\right) \sum_{i=1}^{m} \frac{\partial u^*}{\partial x_i} \frac{\partial u}{\partial x_i} - \sum_{i=1}^{m} u^* A_i\left(x\right) \frac{\partial u}{\partial x_i} - u^*\left(A_o\left(x\right) - B_o\left(x\right)\right) u.$$

Using the same method of prove as in the case of previous theorem we shall obtain that in the case when $m_F^i \neq 0, i = \overline{1,m}$, the quadric form E_1 is positive if

$$\begin{aligned} e^*\left(A_o\left(x\right) - B_o\left(x\right)\right) e &\leq -\frac{1}{4\delta^2} n \sum_{i=1}^{m} \left(\gamma_F^i m_F^i\right)^2, \quad \forall x \in \Omega; \\ e^* A_i\left(x\right) e &\leq 2\delta p_i\left(x\right), \quad \forall x \in \Omega, \end{aligned}$$

where $\delta^2 = \lambda^2 - \mu^2 > 0$, and

$$p_i^2\left(x\right) := -\frac{1}{4\delta^2}\left(\left\|A_i\left(x\right) - \widetilde{\alpha}_i\left(x\right) I_n\right\|^2 - n\left(\gamma_F^i m_F^i\right)^2\right), \quad i = \overline{1,m}. \qquad (16)$$

Theorem 3. *Suppose*

(i) $e^*\left(A_o\left(x\right) - B_o\left(x\right)\right) e \leq -\frac{1}{4\delta^2} n \sum\limits_{i=1}^{m} \left(\gamma_F^i m_F^i\right)^2, \forall x \in \Omega;$

(ii) $e^* A_i\left(x\right) e \leq 2\delta p_i\left(x\right), \forall x \in \Omega, i = \overline{1,m}; \forall e \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right),$
$\|e\| = \left(\sum\limits_{i=1}^{n} e_i^2\right)^{\frac{1}{2}} = 1$, *with* $\delta^2 = \lambda^2 - \mu^2$ *and* p_i *as in (16).*

If the system (13) has a solution $u \in C^2\left(\overline{\Omega}, \mathbb{R}^n\right)$, $u \neq 0$ *with the property that* $u|_{\partial\Omega} = 0$ *then there exist* $x_o \in \overline{\Omega}$ *such that* $\det S\left(x_o\right) = 0$, *where* S *is a matrix solution of the system (14) with* $\frac{\partial S}{\partial x_i} S^{-1}, i = \overline{1,m}$, *symmetric.*

Remark 4. If $m_F^i = 0$, conditions (i) and (ii) from Theorem 3 are reduced to:

$$e^*\left(A_o\left(x\right) - B_o\left(x\right)\right) e \leq -\frac{1}{4\delta^2} \sum_{i=1}^{m} \left(\widetilde{\alpha}_i\left(x\right)\right)^2, \forall x \in \Omega.$$

4. Application

In what follows we shall apply Theorem 3 in the case of metaharmonic systems. Let

$$\Delta^k u + A_{k-1}(x)\,\Delta^{k-1} u + ... + A_1(x)\,\Delta u + A_o(x)\,u = 0 \tag{17}$$

be an metaharmonic differential equations system. If we denote $u = w_1, \Delta u = w_2, ..., \Delta^{k-1} u = w_k$, then the system (17) becomes

$$\begin{cases} \Delta w_1 - w_2 = 0, \\ \Delta w_2 - w_3 = 0, \\ \vdots \\ \Delta w_k + A_o w_1 + A_1 w_2 + \cdots + A_{k-1} w_k = f, \end{cases}$$

or equivalent

$$\Delta W + \widetilde{A} W = F, \tag{18}$$

where

$$W = \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}, \qquad \widetilde{A} = \begin{pmatrix} 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & -I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & -I \\ A_o & A_1 & A_2 & \cdots & A_{k-1} \end{pmatrix}.$$

Let us consider the system

$$\lambda^2 I_n \Delta V + \sum_{i=1}^{m} B_i(x)\,\frac{\partial V}{\partial x_i} + B_o(x)\,V = 0, \tag{19}$$

where $\lambda > 1, B_j \in C\left(\overline{\Omega}, M_{nk}(\mathbb{R})\right), V = (V_1, ..., V_{nk})^t \in \mathbb{R}^{nk}, B_o$ symmetric.

Using Theorem 3 we obtain the following result.

Theorem 4. *Suppose*

(i) $e^*\left(B_o(x) - \widetilde{A_o}(x)\right)e \le -\frac{1}{4\delta^2} nk \sum_{i=1}^{m} \left(\gamma_F^i m_F^i\right)^2, \forall x \in \Omega;$

(ii) $e^* B_i(x)\,e \le 2\delta p_i(x), i = \overline{1,m}, \forall x \in \Omega, \forall e \in C^2\left(\overline{\Omega}, \mathbb{R}^{nk}\right),$

$\|e\| = \left(\sum_{i=1}^{nk} e_i^2\right)^{\frac{1}{2}} = 1,$*with* $\delta^2 = \lambda^2 - \mu^2$ *and* p_i *as in* (15).

If the system (19) has a solution $V \in C^2\left(\overline{\Omega}, \mathbb{R}^{nk}\right)$, $V \neq 0$ *with the property that* $V|_{\partial\Omega} = 0$ *then there exist* $x_o \in \overline{\Omega}$ *such that* $\det S(x_o) = 0$, *where* S *is a matrix solution of the system (18) with* $\frac{\partial S}{\partial x_i} S^{-1}, i = \overline{1,m}$, *symmetric.*

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