International Journal of Pure and Applied Mathematics
Volume 63 No. 1 2010, 39-49

MERIT FUNCTIONS AND EQUIVALENT DIFFERENTIABLE OPTIMIZATION PROBLEMS FOR THE EXTENDED GENERAL VARIATIONAL INEQUALITIES

Min Sun
Department of Mathematics and Information Science
Zaozhuang University
Shandong, 277160, P.R. CHINA
e-mail: sunmin_2008@yahoo.com.cn

Abstract: In this paper, we study some merit functions and equivalent differentiable optimization problems for the extended general variational inequalities involving three operators, which was introduced and studied by M.A. Noor (see [7]) very recently. Using the projection technique, we obtain some error bounds for the solution of extended general variational inequalities under some mild conditions, and then formulate the extended general variational inequalities as differentiable constrained (or unconstrained) optimization problems. We also prove that the extended general variational inequalities have a unique solution under some conditions.

AMS Subject Classification: 90C30, 90C33
Key Words: extended general variational inequalities, merit functions, error bounds

1. Introduction

The classical variational inequality problem, denoted by VI(T, K), is to find a vector $u^* \in K$ such that

$$\langle Tu^*, v - u^* \rangle \geq 0, \quad \forall\, v \in K,$$

where $K \subseteq R^n$ is a nonempty closed convex subset of R^n and T is a mapping

Received: May 4, 2010

from R^n into itself. When K is nonnegative orthant R^n_+, VI(T, K) reduces to nonlinear complementarity problem: find $u \in R^n$ such that

$$u \geq 0, \quad Tu \geq 0, \quad \langle u, Tu \rangle \geq 0.$$

When $K = R^n$, VI(T, K) simply reduces to solving the system of nonlinear equations

$$Tu = 0.$$

Variational inequalities and complementarity problems are of fundamental importance in a wide range of mathematical and applied sciences problems, such as mathematical programming, traffic engineering, economics and equilibrium problems. We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed convex set in H. In recent years, VI(T, K) has been extended and generalized in different directions, both for their own sake and for their applications. Most recently, a new class of variational inequality problem was introduced by Noor [7], which involves three operators: for given three operators $T, g, h : H \to H$, finding $u \in H, h(u) \in K$ such that

$$\langle Tu, g(v) - h(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K. \tag{1}$$

Problem (1) is called extended general variational inequalities in [7], and is denoted by EGVI in the following analysis. Obviously, by choosing some special forms of g or h, EGVI reduces to the general variational inequalities [4], the classical variational inequalities [10], the general complementarity problems [2], respectively. Thus, the extended general variational inequalities include several previously known classes of variational inequalities and related optimization problems as special cases.

From the above discussion, it is very useful to study the extended general variational inequalities. Using the projection technique, Noor [7] shows that the extended general variational inequalities are equivalent to the fixed point and the extended general Wiener-Hopf equations, and some projection iterative methods were proposed for the extended general variational inequalities.

Another approach for solving VI(T, K) is to transform it into an equivalent minimization problem using some merit (gap) functions. In recent years, various merit (gap) functions for VI(T, K) have been proposed by many authors, see [1], [9], [8]. Using merit functions, VI(T, K) can be cast as unconstrained minimization problem, which is maybe easier to be solved than VI(T, K). Noor [5], [6] extended the merit (gap) functions for VI(T, K), and studied some merit (gap) functions for general variational inequalities and quasi-variational inequalities, and the paper [3] recently extended these functions to deal with general mixed quasi-variational inequalities. In this paper, we will further extend the merit (gap) functions proposed by Noor, and present some new merit (gap) functions for the extended general variational inequalities.

As pointed by Noor [7], EGVI is equivalent to finding $u \in H, h(u) \in K$ such that

$$\langle \rho Tu + h(u) - g(u), g(v) - h(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2}$$

where $\rho > 0$ is a constant. This equivalent formulation is very useful in the following analysis.

The paper is organized as follows. In Section 2 we give some preliminaries, some basic definitions and results which will be used in the paper. In Section 3 we present some merit (gap) functions for EGVI, and then formulate EGVI as constrained (or unconstrained) minimization problem.

2. Preliminaries

Now we give some basic concepts and results which will be used in latter analysis.

Definition 1. $\forall\, u, v \in H$, the operator $T : H \to H$ is said to be:

(1) strongly $g-$monotone, iff, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, g(u) - g(v) \rangle \geq \alpha \|g(u) - g(v)\|^2.$$

(2) $g-$Lipschitz continuous, iff, there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta \|g(u) - g(v)\|.$$

(3) strongly nonexpansive, iff, there exists a constant $\gamma > 0$ such that

$$\|Tu - Tv\| \geq \gamma \|u - v\|.$$

Remark 1. If the operator T is $g-$strongly monotone with a constant $\alpha > 0$, then

$$\|Tu - Tv\| \geq \alpha \|g(u) - g(v)\|, \quad \forall\, u, v \in H.$$

For a given vector $u \in H$, the projection of u onto the set K, is defined as the nearest vector $v \in K$ to u, i.e.,

$$P_K[u] = \arg\min_{v \in K} \|u - v\|.$$

The projection mapping $P_K[\cdot]$ has the following important property, which will be used in the following.

Lemma 1. *Let K be a closed convex set in H.*

(i) For a given $z \in H, u \in K$ satisfies the equality

$$\langle u - z, v - u\rangle \geq 0, \quad \forall\, v \in K, \tag{3}$$

iff

$$u = P_K[z].$$

(ii)

$$\|P_K[u] - P_K[v]\| \leq \|u - v\|, \quad \forall\, u, v \in H.$$

Definition 2. A function $M : H \to R \cup \{+\infty\}$ is called a merit (gap) function for the extended general variational inequalities, iff:

(i) $M(u) \geq 0, \forall\, u \in H$; (ii). $M(\bar{u}) = 0$, iff, $\bar{u} \in H$ solves (1).

In the following of this paper, we need the following assumptions.

(i) H is a finite dimension space.

(ii) g is homeomorphism on H, i.e., g is bijective, continuous and g^{-1} is continuous.

(iii) g is Lipschitz continuous and h is Lipschitz continuous and strongly nonexpanding on H; T is strongly h (or g)-monotone and h (or g)-Lipschitz continuous.

(iv) The solution set of EGVI is nonempty.

3. Main Results

Firstly, we study those conditions under which EGVI has a unique solution.

Theorem 3. *Let T be a strongly $g-$monotone with constant $\bar{\alpha} > 0$ and $g-$Lipschitz continuous operator with constant $\beta > 0$, and T is also strongly $h-$monotone with constant $\alpha > 0$. Let g be Lipschitz continuous operator with constant $\gamma > 0$ and h be a strongly nonexpanding operator with constant $\sigma > 0$. If there exists a $\rho > 0$ such that*

$$|\rho - \frac{\bar{\alpha}}{\beta^2}| < \frac{\sqrt{\bar{\alpha}^2\gamma^2 - \gamma(\gamma - \sigma^2)}}{\beta^2\gamma}, \tag{4}$$

$$\bar{\alpha}\gamma \geq \sqrt{\gamma(\gamma - \sigma^2)}, \tag{5}$$

then EGVI has a unique solution.

Proof. (a) *Uniqueness.* Let $u_1 \neq u_2 : h(u_1) \in K, h(u_2) \in K$ be two solutions of EGVI. Then, from (1) we have

$$\langle Tu_1, h(u_2) - h(u_1)\rangle \geq 0, \tag{6}$$

$$\langle Tu_2, h(u_1) - h(u_2)\rangle \geq 0. \tag{7}$$

Adding (6) and (7), we obtain

$$\langle Tu_1 - Tu_2, h(u_1) - h(u_2)\rangle \leq 0.$$

Since T is strongly $h-$monotone and h is strongly nonexpanding, we get

$$\alpha\sigma^2\|u_1 - u_2\|^2 \leq \alpha\|h(u_1) - h(u_2)\|^2 \leq \langle Tu_1 - Tu_2, h(u_1) - h(u_2)\rangle \leq 0,$$

which implies that $u_1 = u_2$, the uniqueness of the solution of EGVI.

(b) *Existence.* For a given $u \in H : h(u) \in K$, we consider the problem of finding a unique $w \in H : h(w) \in K$ such that

$$\langle \rho Tu + h(w) - g(u), g(v) - h(w)\rangle \geq 0, \quad \forall\, v \in H : g(v) \in K, \tag{8}$$

where $\rho > 0$ is a constant. It is clear that the relation (8) defines a mapping $u \to w$. Since EGVI is equivalent to (2), it is enough to show the mapping $u \to w$ defined by the relation (8) has a fixed point belonging to H. Let $w_1 \neq w_2$ be two solution of (8) related to $u_1, u_2 \in H$, respectively. It is sufficient to show that for a well chosen $\rho > 0$,

$$\|w_1 - w_2\| \leq \theta\|u_1 - u_2\|,$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $g(v) = h(w_2)$ (respectively $h(w_1)$) in (8) related to u_1 (respectively u_2), adding the resultant, we have

$$\begin{aligned}&\langle h(w_1) - h(w_2), h(w_1) - h(w_2)\rangle \\ &\qquad\leq \langle h(w_1) - h(w_2), g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2)\rangle,\end{aligned}$$

from which we have

$$\begin{aligned}&\|h(w_1) - h(w_2)\|^2 \leq \|g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2)\|^2 \\ &\quad= \|g(u_1) - g(u_2)\|^2 - 2\rho\langle g(u_1) - g(u_2), Tu_1 - Tu_2\rangle + \rho^2\|Tu_1 - Tu_2\|^2 \\ &\quad\leq \|g(u_1) - g(u_2)\|^2 - 2\rho\bar{\alpha}\|g(u_1) - g(u_2)\|^2 + \rho^2\beta^2\|g(u_1) - g(u_2)\|^2 \\ &\quad= (1 - 2\rho\bar{\alpha} + \rho^2\beta^2)\|g(u_1) - g(u_2)\|^2 \leq \gamma(1 - 2\rho\bar{\alpha} + \rho^2\beta^2)\|u_1 - u_2\|^2.\end{aligned}$$

Now using the strongly nonexpandicity with constant $\sigma > 0$ of h, we get

$$\|w_1 - w_2\| \leq \theta\|u_1 - u_2\|,$$

where

$$\theta = \sqrt{\frac{\gamma}{\sigma^2}(1 - 2\rho\bar{\alpha} + \rho^2\beta^2)}.$$

From (4) and (5), it follows that $0 < \theta < 1$, which shows that the mapping defined by (8) has a fixed point belonging to H, which is also the solution of problem (1). The proof is complete. □

In the following, we give some merit functions for EGVI. Using these merit functions, we obtain some error bounds for EGVI. One can prove that EGVI is equivalent to the fixed point problem by invoking Lemma 1.

Lemma 2. *The function $u \in H : h(u) \in K$ is a solution of EGVI iff $u \in H : h(u) \in K$ satisfies the relation*

$$h(u) = P_K[g(u) - \rho Tu], \tag{9}$$

where $\rho > 0$ is an arbitrary but fixed parameter.

Lemma 2 implies that EGVI and fixed point equation (9) are equivalent.

Let

$$r(u, \rho) := h(u) - P_K[g(u) - \rho Tu] \tag{10}$$

denote the residual error of the projection equation, then solving EGVI is equivalent to finding the zero points of the residual function $r(u, \rho)$. Then the normal residual function $\|r(u, \rho)\|$ is a merit function of EGVI. Next we use this merit function to derive a global error bound for the solution of EGVI, which is the main motivation of our next result.

Theorem 4. *Let $\bar{u}$ be a solution of EGVI. Let g be Lipschitz continuous with constant $\gamma > 0$ and let h be both strongly monexpanding and Lipschitz continuous with constants $\sigma > 0$ and $\delta > 0$, respectively. If the operator T is both strongly $h-$monotone and $h-$Lipschitz continuous with constants $\alpha > 0$ and $\beta > 0$, respectively, then for $\rho > (\delta\gamma - \sigma^2)/(\alpha\sigma^2)$, we have*

$$c_1\|r(u, \rho)\| \le \|u - \bar{u}\| \le c_2\|r(u, \rho)\|, \quad \forall\ u \in H, \tag{11}$$

where c_1, c_2 are two positive constants.

Proof. Since $\bar{u} \in H$ is a solution of EGVI, and EGVI is equivalent to (2), we get

$$\langle \rho T\bar{u} + h(\bar{u}) - g(\bar{u}), g(v) - h(\bar{u})\rangle \ge 0, \quad \forall\ v \in H : g(v) \in K. \tag{12}$$

Taking $g(v) = P_K[g(u) - \rho Tu]$ in (12), we have

$$\langle \rho T\bar{u} + h(\bar{u}) - g(\bar{u}), P_K[g(u) - \rho Tu] - h(\bar{u})\rangle \ge 0. \tag{13}$$

Letting $u = P_K[g(u) - \rho Tu], z = g(u) - \rho Tu$ and $v = h(\bar{u})$ in (3), we obtain

$$\langle P_K[g(u) - \rho Tu] - g(u) + \rho Tu, h(\bar{u}) - P_K[g(u) - \rho Tu]\rangle \geq 0. \quad (14)$$

Adding (13) and (14), we can get

$$\langle h(\bar{u}) - P_K[g(u) - \rho Tu], P_K[g(u) - \rho Tu] - g(u) + \rho Tu - \rho T\bar{u} - h(\bar{u}) + g(\bar{u})\rangle \geq 0,$$

which implies that

$$\begin{aligned} &\langle T\bar{u} - Tu, h(\bar{u}) - P_K[g(u) - \rho Tu]\rangle \\ &\quad \leq \frac{1}{\rho}\langle P_K[g(u) - \rho Tu] - h(\bar{u}), g(u) - g(\bar{u}) + h(\bar{u}) - P_K[g(u) - \rho Tu]\rangle. \quad (15) \end{aligned}$$

Since T is strongly $g-$monotone and g is non-expansive, there exists a constant $\alpha > 0$, such that

$$\begin{aligned} &\alpha\sigma^2\|\bar{u} - u\|^2 \leq \alpha\|h(\bar{u}) - h(u)\|^2 \leq \langle T\bar{u} - Tu, h(\bar{u}) - h(u)\rangle \\ &= \langle T\bar{u} - Tu, h(\bar{u}) - P_K[g(u) - \rho Tu] + P_K[g(u) - \rho Tu] - h(u)\rangle \\ &= \langle T\bar{u} - Tu, h(\bar{u}) - P_K[g(u) - \rho Tu]\rangle + \langle T\bar{u} - Tu, P_K[g(u) - \rho Tu] - h(u)\rangle \\ &\leq \frac{1}{\rho}\langle P_K[g(u) - \rho Tu] - h(\bar{u}), g(u) - g(\bar{u}) + h(\bar{u}) - P_K[g(u) - \rho Tu]\rangle \\ &\qquad + \langle T\bar{u} - Tu, P_K[g(u) - \rho Tu] - h(u)\rangle \\ &= \frac{1}{\rho}\langle h(u) - h(\bar{u}) - r(u,\rho), g(u) - g(\bar{u}) + h(\bar{u}) - h(u) + r(u,\rho)\rangle + \langle Tu - T\bar{u}, r(u,\rho)\rangle \\ &= -\frac{1}{\rho}\|r(u,\rho)\|^2 + \frac{1}{\rho}\langle r(u,\rho), 2h(u) - 2h(\bar{u}) + g(\bar{u}) - g(u)\rangle - \frac{1}{\rho}\|h(u) - h(\bar{u})\|^2 \\ &\qquad + \frac{1}{\rho}\langle h(u) - h(\bar{u}), g(u) - g(\bar{u})\rangle + \langle Tu - T\bar{u}, r(u,\rho)\rangle \\ &\leq \frac{2}{\rho}\|r(u,\rho)\|\|h(u) - h(\bar{u})\| + \frac{1}{\rho}\|r(u,\rho)\|\|g(u) - g(\bar{u})\| - \frac{\sigma^2}{\rho}\|u - \bar{u}\|^2 \\ &\qquad + \frac{1}{\rho}\|h(u) - h(\bar{u})\|\|g(u) - g(\bar{u})\| + \|r(u,\rho)\|\|Tu - T\bar{u}\| \\ &\leq \frac{2\delta}{\rho}\|r(u,\rho)\|\|u - \bar{u}\| + \frac{\gamma}{\rho}\|r(u,\rho)\|\|u - \bar{u}\| + \frac{\delta\gamma - \sigma^2}{\rho}\|u - \bar{u}\|^2 + \beta\delta\|r(u,\rho)\|\|u - \bar{u}\|, \end{aligned}$$

which implies that

$$\|u - \bar{u}\| \leq c_2\|r(u,\rho)\|,$$

where $c_2 = (2\delta + \gamma + \beta\delta\rho)/(\rho\alpha\sigma^2 + \sigma^2 - \delta\gamma)$, and it is obvious that c_1 is a positive constant for $\rho > (\delta\gamma - \sigma^2)/(\alpha\sigma^2)$.

Now from the nonexpansivity of the projection operator and $h-$Lipschitz continuity of T, we have

$$\begin{aligned}\|r(u,\rho)\| &= \|h(u) - P_K[g(u) - \rho Tu]\| \\ &= \|h(u) - h(\bar{u}) + P_K[g(\bar{u}) - \rho T\bar{u}] - P_K[g(u) - \rho Tu]\| \\ &\le \|h(u) - h(\bar{u}) + \|g(\bar{u}) - \rho T\bar{u} - g(u) + \rho Tu\| \\ &\le \|h(u) - h(\bar{u}) + \|g(\bar{u}) - g(u)\| + \rho\|T\bar{u} - Tu\| \\ &\le \delta\|u - \bar{u}\| + \gamma\|u - \bar{u}\| + \rho\beta\delta\|u - \bar{u}\| = (\delta + \gamma + \rho\beta\delta)\|u - \bar{u}\|,\end{aligned}$$

from which we have

$$\|u - \bar{u}\| \ge c_1\|r(u,\rho)\|,$$

where $c_1 = 1/(\delta + \gamma + \rho\beta\delta)$ is a positive constant. The proof is complete. □

Now, we consider the second merit function for EGVI

$$M_\rho(u) = \langle Tu, r(u,\rho)\rangle - \frac{1}{2\rho}\|r(u,\rho)\|^2 - \frac{1}{\rho}\langle r(u,\rho), g(u) - h(u)\rangle. \tag{16}$$

We now show that the function $M_\rho(u)$ defined by (16) is a merit function and this is the main motivation of our next result.

Theorem 5. *For any $u \in H : h(u) \in K$, we have*

$$M_\rho(u) \ge \frac{1}{2\rho}\|r(u,\rho)\|^2.$$

Clearly, $M_\rho(u) \ge 0, \forall\ u \in H : h(u) \in K$. In particular, we have $M_\rho(u) = 0$, iff, u is a solution of EGVI.

Proof. Setting $z = g(u) - \rho Tu, v = h(u)$ in (3), we have

$$\langle \rho Tu - (g(u) - P_K[g(u) - \rho Tu]), h(u) - P_K[g(u) - \rho Tu]\rangle \ge 0,$$

which implies that

$$\langle Tu, r(u,\rho)\rangle \ge \frac{1}{\rho}\langle r(u,\rho), g(u) - P_K[g(u) - \rho Tu]\rangle. \tag{17}$$

Combing (16) and (17), we have

$$\begin{aligned}&M_\rho(u) \\ &\ge \frac{1}{\rho}\langle r(u,\rho), g(u) - P_K[g(u) - \rho Tu]\rangle - \frac{1}{2\rho}\|r(u,\rho)\|^2 - \frac{1}{\rho}\langle r(u,\rho), g(u) - h(u)\rangle \\ &= \frac{1}{\rho}\langle r(u,\rho), g(u) - h(u) + r(u,\rho)\rangle - \frac{1}{2\rho}\|r(u,\rho)\|^2 - \frac{1}{\rho}\langle r(u,\rho), g(u) - h(u)\rangle \\ &= \frac{1}{\rho}\|r(u,\rho)\|^2 - \frac{1}{2\rho}\|r(u,\rho)\|^2 = \frac{1}{2\rho}\|r(u,\rho)\|^2.\end{aligned}$$

Clearly, we have $M_\rho(u) \ge 0, \forall u \in H : h(u) \in K$.

Now, if $M_\rho(u) = 0, u \in H : h(u) \in K$, then clearly $r(u,\rho) = 0$. Hence from Lemma 2, we see that u is a solution of EGVI. Conversely, if $u \in H : h(u) \in K$

is a solution of EGVI, then from Lemma 2, we have $r(u, \rho) = 0$. Consequently, from (16), we see that $M_\rho(u) = 0$. The assertion of the theorem is proved. □

From Theorem 5, EGVI is equivalent to the following constrained optimization problem

$$\min M_\rho(u),$$
$$\text{s.t.} \quad u \in H, h(u) \in K.$$

where $\rho > 0$ is a constant.

In the following, we consider another merit function for EGVI, which is denoted by $D\rho, \mu(u)$:

$$D\rho, \mu(u) = \langle Tu, r(u, \rho) - r(u, \mu) \rangle + \frac{1}{2\mu}\|r(u, \mu)\|^2 - \frac{1}{2\rho}\|r(u, \rho)\|^2 + \frac{1}{\rho}\langle r(u, \mu) - r(u, \rho), g(u) - h(u) \rangle. \quad (18)$$

We now show that the function $D\rho, \mu(u)$ defined by (18) is indeed a merit function for EGVI and this is the main motivation of our next result.

Theorem 6. *For any $u \in H, \rho > \mu > 0$, we have*

$$(\rho - \mu)\|r(u, \mu)\|^2 \leq 2\rho\mu D\rho, \mu(u) \leq (\rho - \mu)\|r(u, \rho)\|^2.$$

Clearly, $D\rho, \mu(u) \geq 0, \forall\ u \in H$. In particular, we have $D\rho, \mu(u) = 0$, iff, u is a solution of EGVI.

Proof. Setting $z = g(u) - \rho Tu, v = P_K[g(u) - \mu Tu]$ in (3), we have

$$\langle \rho Tu - (g(u) - P_K[g(u) - \rho Tu]), P_K[g(u) - \mu Tu] - P_K[g(u) - \rho Tu] \rangle \geq 0,$$

which implies that

$$\langle Tu, r(u, \rho) - r(u, \mu) \rangle \geq \frac{1}{\rho}\langle r(u, \rho) - r(u, \mu), g(u) - P_K[g(u) - \rho Tu] \rangle. \quad (19)$$

Combing (18) and (19), we have

$$\begin{aligned}
&D\rho, \mu(u) \\
&\geq \frac{1}{\rho}\langle r(u, \rho) - r(u, \mu), g(u) - P_K[g(u) - \rho Tu] \rangle + \frac{1}{2\mu}\|r(u, \mu)\|^2 - \frac{1}{2\rho}\|r(u, \rho)\|^2 \\
&+ \frac{1}{\rho}\langle r(u, \mu) - r(u, \rho), g(u) - h(u) \rangle \\
&= \frac{1}{\rho}\langle r(u, \rho) - r(u, \mu), g(u) - h(u) + r(u, \rho) \rangle + \frac{1}{2\mu}\|r(u, \mu)\|^2 - \frac{1}{2\rho}\|r(u, \rho)\|^2 \\
&+ \frac{1}{\rho}\langle r(u, \mu) - r(u, \rho), g(u) - h(u) \rangle \\
&= \frac{1}{\rho}\langle r(u, \rho) - r(u, \mu), r(u, \rho) \rangle + \frac{1}{2\mu}\|r(u, \mu)\|^2 - \frac{1}{2\rho}\|r(u, \rho)\|^2
\end{aligned}$$

$$= (\frac{1}{2\mu} - \frac{1}{2\rho})\|r(u,\mu)\|^2 + \frac{1}{\rho}\langle r(u,\rho), r(u,\rho) - r(u,\mu)\rangle - \frac{1}{2\rho}\|r(u,\rho) - r(u,\mu)\|^2$$
$$- \frac{1}{\rho}\langle r(u,\mu), r(u,\rho) - r(u,\mu)\rangle$$
$$= (\frac{1}{2\mu} - \frac{1}{2\rho})\|r(u,\mu)\|^2 + \frac{1}{2\rho}\|r(u,\rho) - r(u,\mu)\|^2 \geq (\frac{1}{2\mu} - \frac{1}{2\rho})\|r(u,\mu)\|^2,$$

which implies that

$$(\rho - \mu)\|r(u,\mu)\|^2 \leq 2\rho\mu D_{\rho,\mu}(u). \tag{20}$$

In a similar way, we obtain

$$2\rho\mu D_{\rho,\mu}(u) \leq (\rho - \mu)\|r(u,\rho)\|^2. \tag{21}$$

Combing (20) and (21), we have the required result. Clearly, $D\rho,\mu(u) \geq 0, \forall\ u \in H : h(u) \in K$.

Now, if $D_{\rho,\mu}(u) = 0, u \in H$, then from (20), we have $r(u,\mu) = 0$. Hence from Lemma 1, we see that u is a solution of EGVI. Conversely, if $u \in H : h(u) \in K$ is a solution of EGVI, then from Lemma 2, we have $r(u,\rho) = 0, r(u,\mu) = 0$. Consequently, from (18), we see that $D_{\rho,\mu}(u) = 0$. The proof is complete. □

From Theorem 6, EGVI is equivalent to the following unconstrained optimization problem

$$\min D_{\rho,\mu}(u),$$
$$\text{s.t.}\ \ u \in H,$$

where $\rho > \mu > 0$ are two constants.

References

[1] M. Fukushima, Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, *Math. Programming*, **53** (1992), 99-110.

[2] F. Giannessi, A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum, New York (1995).

[3] L.L. Huan, B. Qu, J.G. Jiang, Merit functions for general quasi-variational inequalities, *Journal of Applied Mathematics and Computing*, DOI 10.1007/s12190-009-0293-y (2009).

[4] M.A. Noor, General variational inequalities, *Appl. Math. Lett.*, **1** (1988), 119-121.

[5] M.A. Noor, Merit functions for general variational inequalities, *J. Math. Anal. Appl.*, **316** (2006), 736-752.

[6] M.A. Noor, On merit functions for quasi-variational inequalities, *J. Math. Inequal.*, **1**, No. 2 (2007), 259-268.

[7] M.A. Noor, Projection iterative methods for extended general variational inequalities, *Journal of Applied Mathematics and Computing*, DOI 10.1007/s12190-009-0234-9 (2009).

[8] J.M. Peng, Equivalence of variational inequality problems to unconstrained minimization, *Math. Programming*, **78** (1997), 347-355.

[9] B. Qu, C.Y. Wang, J.Z. Zhang, Convergence and error bound of method for solving variational inequality problem via the generalized D-gap function, *J. Optim. Theory Appl.*, **119** (2002), 535-552.

[10] G. Stampacchia, Formes bilineaires coercivites sur les ensembles convexes, *C.R. Acad. Sci.*, Paris, **258** (1964), 4413-4416.