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POSTULATION OF GENERAL UNIONS OF LINES AND LINEAR SPACES WITH GOOD COHOMOLOGY IN DEGREE 2

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Abstract: Let $X \subset \mathbb{P}^n$ be a general union of linear spaces. Let $Y \subset X$ be the union of the components of dimension ≥ 2. Here we prove that if $h^0(Y, \mathcal{O}_Y(2)) < \binom{n+2}{2} - 9n$, then X has maximal rank.

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1. Introduction

Fix a linear system $\Delta \neq \emptyset$ on $\mathbb{P}^n$, i.e. the projectivization $\mathbb{P}(V_\Delta)$ of a linear subspace V_Δ of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_\Delta))$. The integer d_Δ will be called the degree of Δ. The linear system Δ induces a rational map α_Δ from $\mathbb{P}^n$ into $\mathbb{P}^r$, $r := \dim(\Delta)$. Let B_Δ denote the base locus of Δ, i.e. the closed subscheme of $\mathbb{P}^n$ defined as the scheme-theoretic intersection of all $\{f = 0\}$, $f \in V_\Delta$. Set $U_\Delta = \mathbb{P}^n \setminus B_\Delta$. If B_Δ contains no hypersurface of $\mathbb{P}^n$, then U_Δ is the maximal open subset of $\mathbb{P}^n$ on which the rational map α_Δ may be extended as a morphism. In the general case we decrease the integer d_Δ and reduce to the previous case by deleting the codimension one part of B_Δ. Let e_Δ denote the degree of the $(n-1)$-dimensional part of B_Δ. For any closed subscheme $J \subseteq \mathbb{P}^n$ set $\Delta(-J) := \mathbb{P}(V_\Delta \cap H^0(\mathbb{P}^n, \mathcal{I}_J(d_\Delta))$. In Section 2 we prove the following result.

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Theorem 1. *Fix an integer $n \geq 3$ and a linear system Δ on $\mathbb{P}^n$. Let $L \subset \mathbb{P}^n$ be a general line. Then* $\dim(\Delta(-L)) \leq \max\{0, \dim(\Delta) - 2\}$. *Assume* $r := \dim(\Delta) \geq 2$. *Then the following conditions are equivalent:*

(i) $\dim(\Delta(-L)) = \dim(\Delta) - 2$*;*

(ii) $d_\Delta - e_\Delta = 1$*;*

(iii) α_Δ *sends a general line of* $\mathbb{P}^n$ *onto a line of* $\mathbb{P}^r$.

A key tool of the proof of the following result is [3], Theorem 1 (its statement, not its proof).

Theorem 2. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a general union of an arbitrary finite number of linear subspaces with arbitrary dimension. Let $Y \subseteq X$ be the union of the irreducible components of X with dimension at least two. If $h^0(Y, \mathcal{O}_Y(2)) < \binom{n+2}{2}$ (case $n \in \{3,4,5\}$) or $h^0(Y, \mathcal{O}_Y(2)) < \binom{n+2}{2} - 9n$ (case $n \geq 6$), then X has maximal rank, i.e. for every integer $t \geq 1$ the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(X, \mathcal{O}_X(t))$ is either injective or surjective.*

By [3], Theorem 1, the condition $h^0(Y, \mathcal{O}_Y(2)) < \binom{n+2}{2}$ is equivalent to the condition $h^0(\mathcal{I}_Y(2)) > 0$ and it implies $h^1(\mathcal{I}_Y(2)) = 0$. It should be easy to extend Theorem 2 just requiring $h^1(\mathcal{I}_Y(2)) = 0$ or, equivalently $h^0(Y, \mathcal{O}_Y(2)) \leq \binom{n+2}{2}$. Until steps (e) and (f) of the proof of Theorem 2 we will only require the condition $h^0(Y, \mathcal{O}_Y(2)) < \binom{n+2}{2}$. We required the stronger inequality for $n \geq 6$ to avoid the checking of finitely many cases (each of them being a potential source of numerical errors).

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$

2. Proof of Theorem 1

Proof of Theorem 1. Fix a general $(P, Q) \in \mathbb{P}^n \times \mathbb{P}^n$. Since the pair (P, Q) is general, $\dim(\Delta(-\{P, Q\})) = \max\{-1, \dim(D) - 2\}$. Hence $\dim(\Delta(-\langle\{P, Q\}\rangle)) \leq \max\{-1, \dim(D) - 2\}$. By semicontinuity $\dim(\Delta(-L)) \leq \max\{-1, \dim(\Delta) - 2\}$. From now on we assume $r := \dim(\Delta) \geq 2$. Obviously (ii) and (iii) are equivalent and any of them implies (i). Assume (i). Let Ω be the maximal open subset of $\mathbb{P}^n$ on which α_Δ is defined. Since $\mathbb{P}^r$ is smooth and projective, $\mathbb{P}^n \setminus \Omega$ has codimension at least 2 in $\mathbb{P}^n$. Since L is general line, $L \cap (\mathbb{P}^n \setminus \Omega) = \emptyset$, i.e. α_Δ is a morphim at each point of L. Thus $\alpha_\Delta(L)$ is an integral subvariety of $\mathbb{P}^n$ with dimension at most 1. Since α_Δ is not constant and two general points of Ω are contained in a line, $T := \alpha_\Delta(L)$ is a curve. Set $s := d_\Delta - e_\Delta$. In order to obtain a contradiction we assume $s \geq 2$. Varying L the family of all curves T_L is a covering family of integral curves and two general points of $\mathbb{P}^n$ are contained in at least one such curve. Notice that $\alpha_\Delta^*(\mathcal{O}_T(1)) \cong \mathcal{O}_L(s)$. Since $s \geq 2$, there are 3 non-collinear points in T. Hence $h^0(\mathcal{I}_T(1)) \leq r+1-3$. Since $h^0(\mathcal{I}_T(1)) - 1 = \dim(\Delta(-L))$, we got a contradiction. □

3. Proof of Theorem 2

Remark 1. Fix a reducible conic $D \subset \mathbb{P}^n$, $n \geq 3$, and let P be its singular point. Let $M \subseteq \mathbb{P}^n$ be any 3-dimensional linear space containing D. The scheme $D \cup \chi_M(P)$ is a flat degeneration inside M and hence inside $\mathbb{P}^n$ of a flat family whose general element is the disjoint union of two lines (see [5]). Let $H \subset \mathbb{P}^n$ be any hyperplane containing P, but no irreducible component of D. The scheme $D \cap H$ is a tangent vector of H with P as its reduction. Fix P and H, but take as D a general reducible conic with P as its singular point. Then $D \cap H$ is the general tangent vector of H with P as its support. Now take M general. Then $(D \cup \chi_M(P)) \cap H = \chi_{M\cap H}(P)$. Thus $(D \cup \chi_M(P)) \cap H$ is a general planar length 3 subscheme of H with P as its reduction.

The first part of the following remark is often called "Horace Lemma". The last part of the following remark is a particular case of the so-called "Differential Horace Lemma".

Remark 2. Let X be any projective scheme and D any effective Cartier divisor of X. For any closed subscheme Z of X let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. For every $L \in \mathrm{Pic}(X)$ we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes (L|D) \to 0. \quad (1)$$

From (1) we get

$$h^i(X, \mathcal{I}_Z \otimes L) \leq h^i(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D)) + h^i(D, \mathcal{I}_{Z\cap D,D} \otimes (L|D))$$

for every integer $i \geq 0$. Now assume $\dim(X) \geq 3$ and both X and D integral. Hence it make sense to speak about the general tangent vector and the general planar length 3 subscheme of D_{reg}. Fix an integer $i \in \{0,1\}$. A particular case of [1], Lemma 2.1, says that to prove $h^i(X, \mathcal{I}_{Z\cup A} \otimes L) = 0$, where A is a general length 3 planar subscheme, it is sufficient to prove

$$h^i(X, \mathcal{I}_{\mathrm{Res}_D(Z)\cup v} \otimes L(-D)) = h^i(D, \mathcal{I}_{(Z\cap D)\cup\{P\}} \otimes L|D)) = 0,$$

where v is a general tangent vector of D_{reg} and P is a general point of D.

We borrow from [2] the following three lemmas.

Lemma 1. *Fix any scheme $W \subset \mathbb{P}^n$ and any integer $t \geq 0$. Then (after fixing W and t) fix a general $P \in \mathbb{P}^n$ and a general tangent vector v such that $v_{red} = \{P\}$. If $h^0(\mathcal{I}_W(t)) \leq 2$, then $h^0(\mathcal{I}_{W\cup v}(t)) = 0$. If $h^0(\mathcal{I}_W(t)) \geq 2$, then $h^0(\mathcal{I}_{W\cup v}(t)) = h^0(\mathcal{I}_W(t)) - 2$ and $h^1(\mathcal{I}_{W\cup v}(t)) = h^1(\mathcal{I}_W(t))$.*

Lemma 2. *Fix any scheme $W \subset \mathbb{P}^n$, $n \geq 2$, and any integer $t > 0$. Assume $h^0(\mathcal{I}_W(t)) \geq 3$. Let Ψ denote the rational map induced by the linear system $|\mathcal{I}_W(t)|$. Then (after fixing W and t) fix a general planar length 3 subscheme Z of $\mathbb{P}^n$.*

(a) We have $h^0(\mathcal{I}_{W\cup Z}(t)) \leq h^0(\mathcal{I}_W(t)) - 2$ and equality holds if and only if the image of Ψ has dimension 1.

(b) If $h^0(\mathcal{I}_W(t)) \geq t+3$, then $h^0(\mathcal{I}_{W\cup Z}(t)) = h^0(\mathcal{I}_W(t)) - 3$.

Lemma 3. *Take the set-up and the assumptions of Lemma 2. Fix an integer $s \geq 2$. Let $E = \sqcup_{i=1}^{s} Z_i \subset \mathbb{P}^n$ be a general disjoint union of s planar length 3 subschemes. Assume $h^0(\mathcal{I}_W(t)) \geq 3(s-1)+t+3$. Then $h^0(\mathcal{I}_{W\cup E}(t)) = h^0(\mathcal{I}_W(t)) - 3s$.*

Let W be any scheme and, for a fixed W, let W' be the union of W and a prescribed number of general points. If W has maximal rank, then W' has maximal rank. Thus we reduce to the case in which no component of X is a sigle point.

Lemma 4. *Let $Y \subsetneq \mathbb{P}^n$ be a general union of linear subspaces such that $h^1(\mathcal{I}_Y(2)) = 0$. Assume that Y contains a hyperplane and let $W \subsetneq Y$ the union of the irreducible components of Y with codimension at least 2. We have $h^1(\mathcal{I}_W(2)) = 0$ and W may be seen as a general union of linear subspaces with prescribed dimension. If Theorem 2 is true for W, then it is true for Y.*

Proof. By [3], Theorem 4.3, we have $h^1(\mathcal{I}_W(2)) = 0$. Hence Theorem 2 may be stated also for W. Let z be the degree of the $(n-1)$-dimensional part of Y. Fix a general union X of Y and a prescribed number of general lines. Notice that $h^i(\mathcal{I}_Y(t)) = h^i(\mathcal{I}_{W\cup(X\setminus Y)}(t-z))$, $i = 0, 1$. □

Since Theorem 2 is true if $\dim(Y) \leq 1$ (see [5]), from now on we assume $Y \neq \emptyset$, that every irreducible component of Y has dimension at most $n-2$ and that no connected component of X is a point. Since $Y \neq \emptyset$, we are also assuming $n \geq 4$.

Lemma 5. *We have $h^i(Y, \mathcal{O}_Y(t+1-i)) = 0$ for all integers $i \geq 1$ and $t \geq 2$.*

Proof. First assume $i = 1$. Use that $h^0(\mathcal{I}_Y(2)) > 0$ and the description of all cases with that condition given in [3], Theorem 4.2. Then use the exact sequence

$$0 \to \mathcal{O}_Y(x-1) \to \mathcal{O}_Y(x) \to \mathcal{O}_{Y\cap H}(x) \to 0 \tag{2}$$

and the lemma for the general union $Y \cap H$ of linear subspaces of H. □

Since X is the union of Y and a prescribed number, σ, of general lines, no such line intersects Y. Since $h^0(Y, \mathcal{O}_Y(2)) \leq \binom{n+2}{2}$, [3], Theorem 1, gives $h^1(Y, \mathcal{O}_Y(2)) = 0$. Hence $h^0(\mathcal{I}_Y(2)) = \binom{n+2}{2} - h^0(Y, \mathcal{O}_Y(2))$. Set $h^0(\mathcal{I}_Y(2)) = 3y + e$ with $0 \leq e \leq 2$ and $y \in \mathbb{N}$. Let $U \subset \mathbb{P}^n$ be a general union of Y and y lines. Since $h^0(U, \mathcal{O}_U(2)) = h^0(Y, \mathcal{O}_Y(2)) + 3y$ and U is general, [3], Theorem 1, gives $h^1(\mathcal{I}_U(2)) = 0$ and $h^0(\mathcal{I}_U(2)) = e$. Moreover, $h^1(\mathcal{I}_Y(t)) = 0$ for all $t \geq 2$ (use Lemma 5 and the cohomology of a line). Since $h^1(Y, \mathcal{O}_Y(2)) = 0$, we have $h^1(U, \mathcal{O}_U(2)) = 0$. We get $h^1(\mathcal{I}_Y(t)) = 0$ for all $t \geq 3$. Hence U has maximal rank. Thus Theorem 2 is true for X if $\sigma \leq y$. Hence from now on we assume $\sigma > y$. For each integer $t \geq 0$ set $a_t := h^0(U, \mathcal{O}_U(t))$. For each integer $t \geq 3$ define the integers b_t and v_t by the relations

$$a_t + (t+1)b_t - v_t = \binom{n+t}{n}, \quad 0 \leq v_t \leq t. \tag{3}$$

Taking the difference of the equation in (3) for two consecutive integers we get

$$a_t - a_{t-1} + b_{t-1} + (t+1)(b_t - b_{t-1}) + b_{t-1}) - v_t + v_{t-1} = \binom{n+t-1}{n-1} \tag{4}$$

(any $t \geq 3$, with the convention $b_2 := 0$ and $v_2 := -e$ if $t = 3$). Recall that $a_2 = \binom{n+2}{2} - e$.

Lemma 6. *We have $h^0(\mathcal{I}_U(1)) = 0$ and $h^0(H, \mathcal{I}_{U\cap H}(2)) \geq h^0(\mathcal{I}_U(2))$.*

Proof. We assumed the last inequality. Since $e := h^0(\mathcal{I}_U(2)) \leq 2 \leq n$, we have $h^0(\mathcal{I}_U(1)) = 0$. Take $t = 2$ in the exact sequence

$$0 \to \mathcal{I}_U(t-1) \to \mathcal{I}_U(t) \to \mathcal{I}_{H\cap U,H}(t) \to 0. \tag{5}$$

Since $h^0(\mathcal{I}_U(1)) = 0$, (5) gives $h^0(H, \mathcal{I}_{U\cap H}(2)) \geq h^0(\mathcal{I}_U(2))$. □

Lemma 7. *Fix integers $n \geq 3$ and $t \geq 3$. Then $\binom{n+t-1}{n-1} - (a_t - a_{t-1}) = h^0(H, \mathcal{I}_{U\cap H}(t)) \geq \binom{n+t-3}{n-1}$.*

Proof. Use the exact sequence (5) and that $h^1(\mathcal{I}_U(t-1)) = h^1(\mathcal{I}_U(t)) = 0$ (Lemma 5). Since $h^1(\mathcal{I}_U(x)) = 0$ for all $x \geq 2$ (see [3] and Castelnuovo-Mumford's Lemma), $h^1(U, \mathcal{O}_U(y)) = 0$ for all $y \geq$ (Lemma 5) and $t \geq 3$, the exact sequence (5) gives $a_t - a_{t-1} = \binom{n+t}{n} - h^0(\mathcal{I}_U(t)) - \binom{n+t-1}{n} + h^0(\mathcal{I}_U(t-1)) = \binom{n+t-1}{n-1} - h^0(H, \mathcal{I}_{U\cap H}(t))$. Hence we get the equality. The inequality follows, because $h^0(H, \mathcal{I}_{U\cap H}(2)) > 0$ (Lemma 6). □

Lemma 8. *Fix an integer $t \geq 4$. Let M be a general hyperplane of H. We have $a_t - a_{t-1} - (a_{t-1} - a_{t-2}) = h^0(U \cap M, \mathcal{O}_{U\cap M}(t))$ and $a_t - a_{t-1} - (a_{t-1} - a_{t-2}) = \binom{n+t-2}{n-2} - h^0(M, \mathcal{I}_{U\cap M}(t))$. Moreover, $a_t - a_{t-1} - (a_{t-1} - a_{t-2}) \leq \binom{n+t-2}{n-2} - \binom{n+t-4}{n-2}$.*

Proof. Use the exact sequences of coherent sheaves on H:

$$0 \to \mathcal{I}_{U\cap M}(y-1) \to \mathcal{I}_{U\cap H}(y) \to \mathcal{I}_{U\cap M, M}(y) \to 0 \tag{6}$$

as in the proof of Lemma 7 to get the first part. The second part follows from the first one and the non-vanishing of $h^0(M, \mathcal{I}_{U\cap M}(2))$ (the latter is obvious, because $h^0(H, \mathcal{I}_{U\cap H}(2)) > 0$). □

Lemma 9. *Let x be the minimal dimension of one of the irreducible components of U. If $e > 0$, then $h^0(\mathcal{I}_U(3)) \geq 2n - x + 1$.*

Proof. Look at the multiplication map

$$\mu : H^0(\mathcal{I}_U(2)) \otimes H^0(\mathcal{O}_{\mathbf{P^n}}(1)) \to H^0(\mathcal{I}_U(3)).$$

Since $e > 0$, the bilinear lemma gives $\dim(\mathrm{Im}(\mu)) \geq e + n$. Lemma 5 and Castelnuovo-Mumford's Lemma give that the homogeneous ideal of U is generated by forms of degree ≤ 3. Hence μ is not surjective and $\mathrm{Coker}(\mu)$ has dimension at least $n - x - e$. Since U is general, it is not a complete intersection of e quadrics and $n - x - e$ cubics (proof: it would be equidimensional and connected; a look at the list in [3], Theorem 4.2, shows that it is absurd). □

Lemma 10. *We have $b_t \geq 2v_t$ for all pairs (n, t) such that $n \geq 4$, $t \geq 3$ and one of the following conditions is satisfied:*

$$\binom{n+t}{n} - a_t + t \geq (2t+1)v_t; \tag{7}$$

$$\binom{n+t}{n} - a_t \geq 2t^2; \tag{8}$$

$$\binom{n+t-2}{n} + t \geq (2t+1)v_t; \tag{9}$$

$$\binom{n+t-2}{n} \geq 2t^2. \tag{10}$$

Proof. Since $0 \leq v_t \leq t$, condition (7) is weaker than condition (8) and condition (9) is weaker than condition (10). Since $h^0(\mathcal{I}_U(2)) > 0$, we have $h^0(\mathcal{I}_U(t)) \geq \binom{n+t-2}{n}$. Since $\binom{n+t}{n} - a_t = h^0(\mathcal{I}_U(t))$, condition (7) is weaker than (9). Assume $b_t \leq 2v_t - 1$. From (3) we get that (7) fails. □

Lemma 11. *We have $b_t \geq 2v_t$ for all pairs (n,t) such that either $n = 4$ and $t \geq 7$ or $n = 5$ and $t \geq 5$ or $n \geq 6$ and $t \geq 4$.*

Proof. Use (10). □

Steps of the Proof of Theorem 2. (a) Here we assume $n = 3$. Hence X is a general union of lines. Hence Theorem 2 is true by [5].

(b) From now on we assume $n \geq 4$. We fix a general hyperplane $H \subset \mathbb{P}^n$. For general U we may assume that $U \cap H$ is the union of some general linear subspaces of prescribed dimension and y general point. Since $h^1(\mathcal{I}_Y(2)) = 0$, we easily get $h^0(H, \mathcal{I}_{Y\cap H}(2)) > 0$ even in the only non-trivial case $e = 0$, because $y > 0$ if $e = 0$ and hence $h^1(\mathcal{I}_U(1)) > 0$. Since $Y \cap H$ is a general union of linear subspaces of H, [3], Theorem 1, gives $h^1(H, \mathcal{I}_{H\cap Y}(2)) = 0$. We also get $h^1(H, \mathcal{I}_{U\cap H}(2)) = 0$. From now on we fix any general U. For each integer $t \geq 3$ we define the following assertion $A_{n,t}$:

Assertion $\boldsymbol{A_{n,t}}$. *We say that $A_{n,t}$ is defined if $b_t - 2v_t \geq 0$. We say that $A_{n,t}$ is true if there is a disjoint union $U \sqcup B \sqcup D$, with B a disjoint union of $b_t - 2v_t$ lines, D a disjoint union of v_t reducible conics with their singular point in H, $U \cap B = U \cap D = B \cap D = \emptyset$ and $h^1(\mathcal{I}_{U\cup B\cup D}(t)) = 0$.*

Take any disjoint union $U \sqcup B \sqcup D$ of U, $b_t - 2v_t$ disjoint lines and v_t disjoint reducible conics. We have $h^i(U \cup B \cup D, \mathcal{O}_{U\cup B\cup D}(t)) = 0$ for all $i \geq 1$. By (3) we have $h^0(U \cup B \cup D, \mathcal{O}_{U\cup B\cup D}(t)) = \binom{n+t}{n}$. Thus $h^0(\mathcal{I}_{U\cup B\sqcup D}(t)) = h^1(\mathcal{I}_{U\cup B\sqcup D}(t))$.

(c) Assume $n = 4$. Let $\sharp$ be the number of the 2-dimensional components of Y. Since $h^0(\mathcal{I}_Y(2)) > 0$, we have $\sharp \leq 2$. If $\sharp = 0$ (resp. $\sharp = 1$) then Theorem 2 is true by [5] (resp. [4], Theorem 5.1). Now assume $\sharp = 2$. Since $e \leq 2$, U is a general union of two planes and a line. In this case we have $e = 1$, $a_t = 2\binom{t+2}{2} - 1 + t + 1 = t^2 + 4t + 1$ for all $t \geq 2$, $(b_3, v_3) = (4, 3)$, $(b_4, v_4) = (8, 3)$, $(b_5, v_5) = (14, 4)$, $(b_6, v_6) = (22, 5)$.

Lemma 12. *Fix integers $n \ge 4$ and $t \ge 4$ such that $(n,t) \ne (4,4)$. Fix any Y admissible for $\mathbb{P}^n$ and assume $A_{n,t-1}$ for that Y. Then $b_t \ge b_{t-1}$. If $v_t > v_{t-1}$, then $b_t - b_{t-1} \ge v_t - v_{t-1}$.*

Proof. Take a general solution $Y := U \sqcup B \sqcup D$ of A_{t-1}. Hence $h^i(\mathcal{I}_Y(t-1)) = 0$, $i = 0,1$, and no irreducible component of Y is contained in H. Since $h^2(\mathcal{I}_Y(t-2)) = h^1(Y, \mathcal{I}_U(t-2)) = 0$ (Lemma 5), the exact sequence (5) gives $h^1(H, \mathcal{I}_{Y\cap H}(t-1)) = 0$. Thus $h^0(Y, \mathcal{O}_{Y\cap H}(t-1)) \le \binom{n+t-1}{n-1}$, i.e. $a_{t-1} - a_{t-2} + b_{t-1} \le \binom{n+t-2}{n-1}$. The second part of Lemma 8 gives $a_t - a_{t-1} - (a_{t-1} - a_{t-2}) \le \binom{n+t-2}{n-2} - \binom{n+t-4}{n-2}$. Thus (4) and the equality $\binom{n+t-2}{n-2} + \binom{n+t-2}{n-1} = \binom{n+t-1}{n-1}$ we get

$$t(b_t - b_{t-1}) + v_{t-1} - v_t \ge \binom{n+t-4}{n-2}. \tag{11}$$

Since $v_t \le t$ and $0 \le v_{t-1} \le t-1$ we conclude at least if $n \ge 5$ and $t \ge 5$, because for fixed t the function $n \mapsto \binom{n+t-4}{n-2}$ is increasing and $\binom{t+1}{3} \ge t(t-1)$ for all $t \ge 5$. We conclude in the same way if $n \ge 6$ and $t = 4$. $\square$

(d) Here we prove $A_{n,3}$ under some assumptions on (n, e, b_3, v_3). For the remaining cases, see step (g). Since H is general, no irreducible component of U is contained in H. Thus $\mathrm{Res}_H(U) = U$. Notice again that $h^0(\mathcal{I}_U(2)) = e$. Since $h^1(\mathcal{I}_U(t)) = 0$ for all $t \ge 3$, we have $a_t = \binom{n+t}{n} - h^0(\mathcal{I}_U(t))$ for all $t \ge 3$. Since $h^0(\mathcal{I}_U(1)) = 0$ (Lemma 5) and $h^1(\mathcal{I}_U(2)) = 0$, from the exact sequence

$$0 \to \mathcal{I}_U(2) \to \mathcal{I}_U(3) \to \mathcal{I}_{H\cap U,H}(3) \to 0.$$

we get $\binom{n+2}{3} - a_3 + a_2 = h^0(H, \mathcal{I}_{U\cap H,H}(3))$. Thus (4) for $t = 3$ with the convention $b_2 = 0$ and $v_2 = -e$ gives

$$4b_3 - v_3 - e = h^0(H, \mathcal{I}_{U\cap H}(3)). \tag{12}$$

Lemma 9 gives that if $e > 0$, then the right hand side of (12) is at least $2n-x+1$, where x is the minimal dimension of one of the irreducible components of U. Thus $4b_3 \ge v_3 + 2n - x + 1$.

(d1) Here we assume $e = 2$. We also assume $b_3 - 2v_3 \ge 4$. Fix a general $(P_1, P_2) \in H \times H$ Let $E_i \subset H$, $i = 1,2$, be a general reducible conic of H with P_i as its singular point. Fix a 3-dimensional linear subspace V_i, $i = 1,2$, of $\mathbb{P}^n$ such that $V_i \cap H = \langle E_i \rangle$. Let $B \subset H$ be a general union of $b_3 - 2v_3 - 4$ lines. Let $D \subset H$ be a general union of v_3 reducible conics. Set $Y := U \cup (E_1 \cup \chi_{V_1}(P_1)) \cup (E_2 \cup \chi_{V_2}(P_2)) \cup B \cup D$. The scheme Y is a flat degeneration inside $\mathbb{P}^n$ of a family of disjoint unions of U, $b_3 - 2v_3 - 4$ lines and $v_3 + 2$ reducible conics. Thus to prove $A_{n,3}$ it is sufficient to prove $h^0(\mathcal{I}_Y(3)) = 0$. We have $\mathrm{Res}_H(Y) =$

$U \cup \{P_1, P_2\}$. Since $h^0(\mathcal{I}_U(2)) = 2$, $h^0(\mathcal{I}_{\mathrm{Res}_H(U)}(1)) = 0$ and (P_1, P_2) is general in $H \times H$, $h^0(\mathcal{I}_{\mathrm{Res}_H(U)}(2)) = 0$. Hence to prove $A_{n,3}$ it is sufficient to prove $h^0(H, \mathcal{I}_{U\cap H}(3)) = 0$ (Remark 2). We have $Y \cap H = (U \cap H) \cup B \cup D$. Assuming for the moment $n \geq 5$ we just use that $y \geq 3$ to apply [2], Remark 3.

(d2) Assume $e = 1$. We also assume $b_3 - 2v_3 \geq 2$; this inequality is satisfied if $8 + 8v_3 \leq v_3 + 2 + 2n - x + 1$, i.e. if $5 + 7v_3 \leq 2n - x$. Fix a general $P \in H$. Let E $\subset$be a reducible conic with P as its singular point. Fix a 3-dimensial linear subspace V of $\mathbb{P}^n$ such that $V \cap H = \langle E \rangle$. Let $B \subset H$ be a general union of $b_3 - 2v_3 - 2$ lines. Let $D \subset H$ be a general union of v_3 reducible conics. Set $Y := U \cup (E \cup \chi_V(P)) \cup B \cup D$. The scheme Y is a flat degeneration inside $\mathbb{P}^n$ of a family of disjoint unions of U, $b_3 - 2v_3 - 2$ lines and $v_3 + 1$ reducible conics. Continu as in (d1).

(d3) Assume $e = 0$. We also assume $b_3 - 2v_3 \geq 0$; this inequality is satisfied if $8v_3 \leq v_3 + 2 + 2n - x + 1$, i.e. if $7v_3 \leq 2n - x + 3$. This case is easier than the ones considered in (d1) and (d2) and may be done using no nilpotent.

Lemma 13. *Fix integers n, t such that $n \geq 6$ and $t \geq 6n - 2$. Then $b_{t-1} \geq (5t - 1)/3 - 2n$.*

Proof. Since $\binom{n+t-1}{n-1} - (a_t - a_{t-1}) \geq \binom{n+t-3}{n-1}$, $v_{t-1} \leq t - 1$, and $b_{t-1} \leq \binom{n+t-1}{n}/t$, (4) shows that is is sufficient to prove

$$\binom{n+t-1}{n}/t + (t+1)((5t-1)/2 - n) + t - 1 \leq \binom{n+t-3}{n-1}. \qquad (13)$$

Call $\Psi(n, t)$ the difference between the right hand side of (13) and the left hand side of (13). Set $\Phi(n, t) = t\Psi(n, t)$ and

$$\begin{aligned} m(n) &:= \Psi(n, 6n-2) \\ &= \binom{7n-5}{n-1} - (4n-1)(19n-11)/2 - 4n + 1 - \binom{7n-3}{n}/(4n-2). \end{aligned}$$

Since $n \geq 6$, we have $m(n) > 0$. Call $\Phi^{(i)}(n, t)$, $1 \leq i \leq 4$, the partial derivative of order i of the funtion $\Phi(n, t)$ with respect to the variable t. We first get $\Phi^{(4)}(n, t) \geq 0$ for all $t \geq 6n - 2$, then $\Phi^{(3)}(n, t) \geq 0$ for all $t \geq 6n - 2$, then $\Phi^{(2)}(n, t) \geq 0$ for all $t \geq 6n - 2$, then $\Phi^{(1)}(n, t) \geq 0$ for all $t \geq 6n - 2$ and then $\Phi(n, t) \geq 0$ for all $t \geq 6n - 2$. □

(e) Fix an integer $t \geq 4$. Assume $A_{n,t-1}$. Let $U \sqcup B \sqcup D$ be a general solution of $A_{n,t-1}$. Since $U \sqcup B \sqcup D$ is general, no irreducible component of $U \cup B \cup D$ is contained in H and, for fixed $U \cap H$, the zero-dimensional scheme $(B \cup D) \cap H$ is a general union of v_{t-1} tangent vectors and $b_{t-1} - 2v_{t-1}$ points. Lemma 11 gives $b_t \geq 2v_t$ (i.e. $A_{n,t}$ is defined), if either $n = 4$ and $t \geq 7$ or $n = 5$ and $t \geq 5$ or $n \geq 6$ and $t \geq 4$. If $n = 4$ and $4 \leq t \leq 6$, then $b_t \geq 2v_t$ (see step (c)). For the case $(n,t) = (5,4)$, see part (f).

(e1) Here we assume $v_t \geq v_{t-1}$. We have $b_t - b_{t-1} \geq v_t - v_{t-1}$ (Lemma 12 and that $v_4 = v_3$ if $(n,t) \neq (4,4)$ by step (c)). Let $A \subset H$ be a general disjoint union of $b_t - b_{t-1}$ lines, with the only restriction that exactly $v_t - v_{t-1}$ contains a different point of $B \cap H$. Thus $Y := U \cup B \cup D \cup A$ is a disjoint union of U, $b_t - 2v_t$ lines and v_t reducible conics whose sngular point is contained in H. Hence to prove $A_{n,t}$ it is sufficient to prove $h^1(\mathcal{I}_Y(t)) = 0$. Since $\mathrm{Res}_H(Y) = U \cup B \cup D$ and $h^1(\mathcal{I}_{U \cup B \cup D}(t-1)) = 0$, it is sufficient to prove $h^1(H, \mathcal{I}_{Y \cap H}(t)) = 0$. Since $B \cap H$ is general in H, the condition that $v_t - v_{t-1}$ lines of H contain a different point of $B \cap H$ is not restrictive, i.e. the scheme $Y \cap H$ is a general union of $U \cap H$, $b_t - b_{t-1} - vt + v_{t-1}$ lines, $b_{t-1} - 2v_{t-1} - (v_t - v_{t-1})$ points and v_{t-1} tangent vectors. Notice that $h^0(Y \cap H, \mathcal{O}_{Y \cap H}(t)) = \binom{n+t-1}{n-1}$. Let Y' be the union of the connected components of $Y \cap H$ with dimension > 0. Since $h^1(H, \mathcal{I}_{H \cap U}(2)) = 0$, $U \cap H$ is a general union inside H of finitely many linear subspaces with prescribed dimension, the theorem in $\mathbb{P}^{n-1}$ gives $h^1(H, \mathcal{I}_{Y'}(t)) = 0$, i.e. $h^0(H, \mathcal{I}_{Y'}(t)) = \binom{n+t-1}{n-1} - h^0(Y', \mathcal{O}_{Y'}(t))$. Since $h^0(Y \cap H, \mathcal{O}_{Y \cap H}(t)) = \binom{n+t-1}{n-1}$, Lemma 1 gives $h^i(H, \mathcal{I}_{Y \cap H}(t)) = 0$, $i = 0, 1$.

(e2) Here we assume $v_t < v_{t-1}$. We have $b_t \geq b_{t-1}$ (Lemma 12 and the fact $(n,t) \neq (4,4)$, because $v_4 = v_3$ if $n = 4$). Fix $S \subseteq \mathrm{Sing}(D)$ such that $\sharp(S) = v_{t-1} - v_t$. For each $P \in S$ fix a general 3-dimensional linear space H_P of $\mathbb{P}^n$ containing the connected component of D containing P. Set $\chi := \cup_{P \in S} \chi_{H_P}(P)$. Let $A \subset H$ be a general union of $b_t - b_{t-1}$ lines. Set $Y := U \cup B \cup D \cup A \cup \chi$. The scheme Y is a flat degeneration of a family of disjoint unions of U, $b_t - v_t$ lines and v_t reducible conics with their singular point contained in H (Remark 1). Since $\mathrm{Res}_H(Y) = U \cup B \cup D$ and $h^0(Y \cap H, \mathcal{O}_{Y \cap H}(t)) = \binom{n+t-1}{n-1}$, it is sufficient to prove $h^1(H, \mathcal{I}_{Y \cap H}(t)) = 0$. The scheme $Y \cap H$ is a general union of $U \cap H$, $v_{t-1} - v_t$ planar length 3 schemes, v_t tangent vector and $b_{t-1} - 2v_{t-1}$ points. Let Y' be the union of the connected components of $Y \cap H$ with dimension > 0. Since $h^1(H, \mathcal{I}_{H \cap U}(2)) = 0$, $U \cap H$ is a general union inside H of finitely many linear subspaces with prescribed dimension, the theorem in $\mathbb{P}^{n-1}$ gives $h^1(H, \mathcal{I}_{Y'}(t)) = 0$, i.e. $h^0(H, \mathcal{I}_{Y'}(t)) = \binom{n+t-1}{n-1} - h^0(Y', \mathcal{O}_{Y'}(t))$. If $b_{t-1} \geq t+3$, then Lemmas 3 and 1 give $h^i(H, \mathcal{I}_{Y \cap H}(t)) = 0$, $i = 0, 1$. Use the inequality $b_{t-1} - 2v_{t-1} + y \geq (t+3)/2$ (Remark 1) and [2], Remark 3.

(f) Here we assume $n = 5$. Let f be the number of the irreducible components $M_1, \dots, M_f$ of Y and $3 \geq m_1 \geq \cdots \geq m_f \geq 2$ their dimension (with $m_i = \dim(M_i)$). Set $y := \deg(U \backslash Y)$. If $f \geq 3$ and $m_3 = 3$, then $h^0(Y, \mathcal{O}_Y(3)) \geq$

$10+10+10-3-3-3=21=\binom{7}{2}$, contradiction. Now assume $m_1=m_2=3$. Since $h^0(M_1\cup M_2,\mathcal{O}_{M_1\cup M_2}(2))=17$, we get $f=2$, $y=1$ and $e=1$; we have $a_3=20+20-4+4=40$; from (3) we get $(b_3,v_3)=(4,0)$; hence in this case we proved $A_{5,3}$; we have $a_4=35+35-5+5=70$ and hence $(b_4,v_4)=(12,4)$. Now assume $m_1=3$ and $f\geq 2$; since $10+(6-1)+(6-1)+(6-1)\geq 21$, we get $f=2$; hence $y=2$ and $e=0$; here $a_3=20+10-1+6=35$ and hence $(b_3,v_3)=(6,3)$; since $e=0$, even in this case we proved $A_{5,3}$; we have $a_4=35+20-1+8=62$ and hence $(b_4,v_4)=(13,1)$. Now assume $f=1$ and $m_1=3$; we get $y=3$ and $e=2$; we have $a_3=32$ and $(b_3,v_3)=(6,0)$; hence $b_3-2v_3\geq 2e$ and $A_{5,3}$ was proved even in this case; we have $a_3=35+3\cdot 4=47$ and $(b_4,v_4)=(16,1)$. Now assume $m_1=2$; we have $1\leq f\leq 3$, $y=7-2f$ and $e=0$; we have $a_3=10f+4(7-2f)=28+2f$; if $f=3$, then $(b_3,v_3)=(6,2)$, $a_4=3\cdot 15+5=50$ and $(b_4,v_4)=(16,4)$; if $f=2$, then $(b_3,v_3)=(6,0)$, $a_4=45$ and $(b_4,v_4)=(17,4)$; if $f=1$, then $(b_3,v_3)=(7,2)$, $a_4=40$ and $(b_4,v_4)=(18,4)$. Hence we always have $b_3-2v_3\geq 0=2e$ and hence we proved $A_{n,3}$. Moreover, $b_4\geq 2v_4+(4+4)/2$ as needed for step (e).

(g) Here we consider the cases of $A_{n,3}$ not covered in part (d), i.e. all cases with $b_3-2v_3<2e$. We use the inequality $4b_3\geq v_3+2n-x+1$. Hence either $e=2$ and $12+7v_3\geq 2n-x$ or $e=1$ and $4+7v_3\geq 2n-x$ or $e=0$ and $7v_3\geq 2n-x-4$. Since parts (a), (c) and (f) cover the cases $n\leq 5$, we may assume $n\geq 6$. Since $h^0(Y,\mathcal{O}_Y(2))<\binom{n+2}{2}-9n$, U has at least $3n$ lines as connected components. In this case if $e=2$ (resp. $e=1$) we may take instead of U the union of Y, $\deg(U\setminus Y)-2e$ general lines and e reducible conics whose singular points in contained in H. Alternatively, we may see that $h^0(\mathcal{I}_U(3))\gg 2n-x+1$.

End of the Proof of Theorem 2. Let $\Sigma(U,x,n)$, $x\in\mathbb{N}$, denote the set of all disjoint unions inside $\mathbb{P}^n$ of U and x lines. Each $\Sigma(U,x,n)$ is irreducible. Set $s:=\deg(Y\setminus U)=\sigma-y$. Let k be the minimal integer ≥ 2 such that $h^0(X,\mathcal{O}_X(k))\leq\binom{n+k}{n}$. It is sufficient to prove $h^0(\mathcal{I}_X(k-1))=0$ and $h^1(\mathcal{I}_Y(k))=0$. The definition of k gives $b_{k-1}<s\leq b_k$ (with the convention $b_1=b_2=0$) and that if $s=b_k$, then $v_k=0$. Since $\Sigma(U,s,n)$ is irreducible, the semicontinuity theorem for cohomology gives that to prove Theorem 2 for the invariants of X it is sufficient to find $X_1,X_2\in\Sigma(U,s,n)$ such that $h^0(\mathcal{I}_{X_1}(k-1))=0$ and $h^1(\mathcal{I}_{X_2}(k))=0$. First assume $k\geq 3$. If $v_k=0$ then we may take as a solution $U\sqcup B$ of $A_{n,k}$ as a solution, because in this case X is a union of Y and disjoint lines and $h^i(\mathcal{I}_{U\cup B}(k))=0$, $i=0,1$. Now assume $v_k>0$. The definition of k gives $s<b_k$. To get $h^1(\mathcal{I}_Y(k))=0$ it is sufficient to prove it when $s=b_k-1$ (just throw away b_k-1-z lines). If $z=b_k-1$ it is sufficient to adapt the proof of $A_{n,k-1}\Longrightarrow A_{n,k}$ given above (just the case

$v_{k-1} > v_k$), inserting in H v_{k-1} planar length 3 subschemes. Similarly, to prove $h^0(\mathcal{I}_X(k-1)) = 0$ it is sufficient to do the case $s = b_{k-1} + 1$. This is for free if $k = 3$ (by [3]) or $k = 2$ (Lemma 5). If $k \geq 5$ we may adapt the proof of $A_{n,k-2} \Longrightarrow A_{n,k-1}$ (the case $v_{k-2} > v_{k-1}$) adding in H v_{k-2} planar length 3 schemes and $b_{k-1} + 1 - b_{k-2}$ lines. If $k = 4$ we adapt the proof of $A_{n,3}$. □

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