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STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MARKOVIAN SWITCHING AND NON-LIPSCHITZ COEFFICIENTS

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Abstract: In this paper, we study the convergence of numerical method of stochastic functional differential equations with Markovian switching. Under non-Lipschitz condition, we prove that the Euler approximate solutions converge to the exact solutions in the mean-square sense.

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1. Introduction

Recently, the theory of stochastic functional differential equations (SFDEs) has received a great deal of attention. The fundamental theory of existence and uniqueness of the solution and the stability properties of SFDEs have been studied by Mao [4], [3]. Since most of SFDEs are nonlinear and cannot be solved explicitly, there is a strong need for the development of efficient numerical methods and to study the properties of them. There are significant literatures that have been done concerning approximate schemes for SFDEs and stochastic delay differential equations with Markovian switching (SDDEwMSs)[5], [8], [6], [7].

However, there are not any numerical methods established for stochastic functional differential equations with Markovian switching and jumps (SFDEw MSJs) yet. Motivated by the papers mentioned above, we aim to study the

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convergence properties of the numerical methods for SFDEwMSJs. We show that under non-Lipschitz condition, weaker than the linear growth condition and global Lipschitz condition, the Euler approximate solution converges to the exact solution. Our proofs are more general than what they deal with the Markovian switching term or the delay term.

2. Preliminaries and the Approximate Solution

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\{W(t),\, t \geq 0\}$ be an m-dimensional Brownian motion defined on the probability space. Let $\tau > 0$ and $C([-\tau, 0]; R^n)$ denote the family of continuous function φ from $[-\tau, 0]$ to R^n with the norm $||\varphi||=\sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. Denote by $L^2_{\mathcal{F}_0}([-\tau, 0]; R^n)$ the family of $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R^n)$-valued random variables such that $E||\xi||^2 < \infty$. If $X(t)$ is a continuous R^n-valued stochastic process on $t \in [-\tau, \infty)$, we let $X_t = \{X(t + \theta) : -\tau \leq \theta \leq 0\}$ for all $t \geq 0$, which is regarded as a $C([-\tau, 0]; R^n)$-valued stochastic process.

Let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2 \ldots N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by:

$$P(r(t + \Delta) = j|r(t) = i) = \begin{cases} \gamma_{ij}\Delta + \circ(\Delta), & \text{if } \ i \neq j, \\ 1 + \gamma_{ij}\Delta + \circ(\Delta), & \text{if } \ i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that Markov chain $r(.)$ is independent of the Brownian motion $W(.)$. It is known that almost every sample path of $r(.)$ is right-continuous step function with a finite number of simple jumps in any finite sub-interval of R_+.

In this paper, we consider the stochastic functional differential equations with Markovian switching and compensated Poisson random measures:

$$dX(t) = f(t, X_t, r(t))dt + g(t, X_t, r(t))dW(t) + \int_{R^n} h(t, X_t, u)\tilde{N}(dt, du), \qquad 0 \leq t \leq T, \quad (2.1)$$

with initial data $X_0 = \xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; R^n)$. $X(t) \in R^n$ for each t, $X_t = \{X(t+\theta) : -\tau \leq \theta \leq 0\} \in C([-\tau, 0]; R^n)$. Here $f : [0, T] \times C([-\tau, 0]; R^n) \times S \to R^n$ is the drift coefficient, $g : [0, T] \times C([-\tau, 0]; R^n) \times S \to R^{n\times d}$ is the diffusion coefficient and $h : [0, T] \times C([-\tau, 0]; R^n) \times R^n \to R^n$ is the jump coefficient;

$W(t)$ is a d-dimensional Brownian motion and $\tilde{N}(dt,du)$ is the compensated Poisson random measure given by $\tilde{N}(dt,du) = N(dt,du) - \Pi(du)dt$, here $\Pi(du)$ is the Levy measure associated to N.

To define the Euler approximate solution for the SFDEwMSJs (1), we will need the property of embedded discrete Markov chain. The following lemma (see [1]) describes this property.

Lemma 2.1. *For $h > 0$ and $n \geq 0$, then $\{r(nh), n = 0, 1, 2, \cdots\}$ is a discrete Markov chain with the one-step transition probability matrix $P(h) = (P_{ij}(h))_{N\times N} = e^{h\Gamma}$.*

Given a step size $h > 0$, the discrete Markov chain $\{r(nh), n = 0, 1, 2, \cdots\}$ can be simulated as follows: compute the one-step transition probability matrix $P(h) = (P_{ij}(h))_{N\times N} = e^{h\Gamma}$. Let $r(0) = i_0$ and generate a random number ζ_1 which is uniformly distributed in $[0,1]$. If $\zeta_1 = 1$ then let $r(h) = i_1 = N$ or otherwise find the unique integer $i_1 \in S$ for $\sum_{j=1}^{i_1-1} P_{i_0,j}(h) \leq \zeta_1 < \sum_{j=1}^{i_1} P_{i_0,j}(h)$ and let $r(h) = i_1$, where we set $\sum_{i=1}^{0} P_{i_0,j}(h) = 0$ as usual. Generate independently a new random number ζ_2 which is again uniformly distributed in $[0,1]$. If $\zeta_2 = 1$ then let $r(2h) = i_2 = N$ or otherwise find the unique integer $i_2 \in S$ for $\sum_{j=1}^{i_2-1} P_{i_1,j}(h) \leq \zeta_2 < \sum_{j=1}^{i_2} P_{i_1,j}(h)$ and let $r(2h) = i_2$. Repeating this procedure a trajectory of $\{r(nh), n = 0, 1, 2, \cdots\}$ can be generated. This procedure can be carried out independently to obtain more trajectories.

After explaining how to simulate the discrete Markov chain $\{r(nh), n = 0, 1, 2, \cdots\}$, we can now propose the Euler method to obtain strong approximations of the solution to the equation (2.1). Let the time-step size $h \in (0,1)$ be a fraction of τ, that is $h = \frac{T}{N}$ for some integer N. Compute the discrete Euler approximate solution $\bar{Y}(nh) = X(t_n)$ by setting $X_0(t) = \xi(0)$, $r(0) = i_0$

$$\bar{Y}((n+1)h) = \bar{Y}(nh) + f(t, \bar{Y}_{nh}, r(nh))h + g(t, \bar{Y}_{nh}, r(nh))\Delta W_n + \int_{R^n} h(t, \bar{Y}_{nh}, u)\tilde{N}(h, du), \quad (2.2)$$

with $\bar{Y}(nh) = \xi(nh)$ on $-N \leq n \leq 0$. Here $\bar{Y}_{nh} = \{\bar{Y}_{nh}(\theta) : -\tau \leq \theta \leq 0\}$ is a $C([-\tau, 0]; R^n)$-valued random variable defined as follows:

$$\bar{Y}_{nh}(\theta) = \bar{Y}((n+k)h) + \frac{\theta - kh}{h}[\bar{Y}((n+k+1)h) - \bar{Y}((n+k)h)]$$
$$\text{for } \; kh \leq \theta \leq (k+1)h, \quad k = -N, -(N-1), \cdots - 1. \quad (2.3)$$

Let $\bar{Y}_t = \sum_{n=0}^{\infty} \bar{Y}_{nh} I_{[nh,(n+1)h)}(t)$, $\bar{r}(t) = \sum_{n=0}^{\infty} r(nh) I_{[nh,(n+1)h)}(t)$. So the continuous Euler approximate solution is defined by

$$Y(t) = \begin{cases} \xi(0) + \int_0^t f(s, \bar{Y}_s, \bar{r}(s))ds + \int_0^t g(s, \bar{Y}_s, \bar{r}(s))dW(s) \\ \qquad + \int_0^t \int_{R^n} h(s, \bar{Y}_s, u)\tilde{N}(ds, du), \quad 0 \le t \le T, \\ \xi(t), \quad -\tau \le t \le 0. \end{cases} \tag{2.4}$$

Note that $Y(nh) = \bar{Y}(nh)$ for $n \ge -N$, that is $Y(t)$ and $\bar{Y}(t)$ coincide with the discrete approximate solution at the grid points.

3. Convergence with Non-Lipschitz Coefficients

In this section, we will show the strong convergence of the Euler approximate solution to the exact solution under the non-Lipschitz condition.

Theorem 3.1. *For all $x, y \in C([-\tau, 0]; R^n)$, $u \in R^n$ and $i \in S$,*

$$|f(t,x,i) - f(t,y,i)|^2 \vee |g(t,x,i) - g(t,y,i)|^2 \vee \int_{R^n} |h(t,x,u) - h(t,y,u)|^2 \Pi(du) \le \Gamma(t)k(||x-y||^2), \tag{3.1}$$

where $\Gamma(t) \ge 0$ is locally integrable nondecreasing function and $k(.)$ is a concave nondecreasing function from R_+ to R_+ such that $k(0) = 0$, $k(u) > 0$ for $u > 0$ and $\int_0^a du/k(u) = \infty$. Let $E|\xi(t) - \xi(s)|^2 \le \lambda(t-s)$, then

$$\lim_{h \to 0} E(\sup_{0 \le t \le T} |X(t) - Y(t)|^2) = 0. \tag{3.2}$$

Remark 3.1. Now let us give some examples of the function k. Let $\varepsilon > 0$ be sufficiently small. Define

$$k_1(u) = \begin{cases} u \log(u^{-1}), & 0 \le u \le \varepsilon, \\ \varepsilon \log(\varepsilon^{-1}) + k_1'(\varepsilon-)(u-\varepsilon), & u > \varepsilon, \end{cases}$$

$$k_2(u) = \begin{cases} u \log(u^{-1}) \log\log(u^{-1}), & 0 \le u \le \varepsilon, \\ \varepsilon \log(\varepsilon^{-1}) \log\log(\varepsilon^{-1}) + k_2'(\varepsilon-)(u-\varepsilon), & u > \varepsilon. \end{cases}$$

They are all concave nondecreasing functions satisfying $\int_0^a du/k_i(u) = \infty$, $i = 1, 2$.

Lemma 3.1. *Under (3.1),*

$$E \sup_{-\tau \le t \le T} |Y(t)|^2 \le C_1, \tag{3.3}$$

where C_1 is a constant which is independent of h.

Proof. Applying the basic inequality $|a+b+c+d|^2 \leq 4(|a|^2+|b|^2+|c|^2+|d|^2)$, one gets

$$|Y(t)|^2 \leq 4|\xi(0)|^2 + 4|\int_0^t f(s,\bar{Y}_s,\bar{r}(s))ds|^2 + 4|\int_0^t g(s,\bar{Y}_s,\bar{r}(s))dW(s)|^2 + 4|\int_0^t \int_{R^n} h(s,\bar{Y}_s,u)\tilde{N}(ds,du)|^2.$$

Taking the expectation on both sides, and by the the Hölder inequality, we have

$$E(\sup_{0\leq s\leq t}|Y(s)|^2) \leq 4E|\xi(0)|^2 + 4E\sup_{0\leq s\leq t}|\int_0^s f(v,\bar{Y}_v,\bar{r}(v))dv|^2 + 4E\sup_{0\leq s\leq t}|\int_0^s g(v,\bar{Y}_v,\bar{r}(v))dW(v)|^2 + 4E\sup_{0\leq s\leq t}|\int_0^t \int_{R^n} h(v,\bar{Y}_v,u)\tilde{N}(dv,du)|^2. \quad (3.4)$$

Now, using the Doob inequality in the two martingale terms,

$$E(\sup_{0\leq s\leq t}|Y(s)|^2) \leq 4E|\xi(0)|^2 + 4TE\int_0^t |f(s,\bar{Y}_s,\bar{r}(s))ds|^2 ds + 16E|\int_0^t g(s,\bar{Y}_s,\bar{r}(s))dW(s)|^2 + 16E|\int_0^t \int_{R^n} h(s,\bar{Y}_s,u)\tilde{N}(ds,du)|^2. \quad (3.5)$$

Applying the martingale isometries and (3.1), we have

$$\begin{aligned}E(\sup_{0\leq s\leq t}|Y(s)|^2) &\leq 4E|\xi(0)|^2 \\ &\quad + 4TE\int_0^t |f(s,\bar{Y}_s,\bar{r}(s))ds|^2 ds + 16E\int_0^t |g(s,\bar{Y}_s,\bar{r}(s))|^2 ds \\ &\quad + 16E\int_0^t \int_{R^n} |h(s,\bar{Y}_s,u)|^2 \Pi(du)ds \\ &\leq 4E|\xi(0)|^2 + 8TE\int_0^t [|f(s,\bar{Y}_s,\bar{r}(s)) - f(s,0,\bar{r}(s))|^2 + |f(s,0,\bar{r}(s))|^2]ds \\ &\quad + 32E\int_0^t [|g(s,\bar{Y}_s,\bar{r}(s)) - g(s,0,\bar{r}(s))|^2 + |g(s,0,\bar{r}(s))|^2]ds \\ &\quad + 32E\int_0^t \int_{R^n} [|h(s,\bar{Y}_s,u) - h(s,0,u)|^2 + |h(s,0,u)|^2]\Pi(du)ds \\ &\leq 4E|\xi(0)|^2 + 8TE\int_0^t |f(s,0,\bar{r}(s))|^2 ds + 32E\int_0^t |g(s,0,\bar{r}(s))|^2 ds\end{aligned}$$

$$+32E\int_0^t\int_{R^n}|h(s,0,u)|^2]\Pi(du)ds+(8T+64)E\int_0^t\Gamma(s)k(||\bar{Y}_s||^2)ds. \quad (3.6)$$

Let $K=\max\{\max_{i\in S}|f(s,0,i)|^2,\max_{i\in S}|g(s,0,i)|^2,\int_{R^n}|h(s,0,u)|^2]\Pi(du)\}$, we have

$$E(\sup_{0\le s\le t}|Y(s)|^2)\le 4E\sup_{-\tau\le s\le 0}|\xi(s)|^2+(8T+64)TK$$
$$+(8T+64)\int_0^t\Gamma(s)k(E||X_s^{n-1}||^2)ds$$
$$\le 4E\sup_{-\tau\le s\le 0}|\xi(s)|^2+(8T+64)TK+(8T+64)\int_0^t\Gamma(s)k(E(\sup_{-\tau\le v\le s}|Y(v)|^2))ds,$$

which implies that

$$E(\sup_{-\tau\le s\le t}|Y(s)|^2)\le 5E\sup_{-\tau\le s\le 0}|\xi(s)|^2$$
$$+(8T+64)TK+(8T+64)\int_0^t\Gamma(s)k(E(\sup_{-\tau\le v\le s}|Y(v)|^2))ds, \quad (3.7)$$

because $E(\sup_{-\tau\le s\le t}|Y(s)|^2)\le E(\sup_{-\tau\le s\le 0}|Y(s)|^2)+E(\sup_{0\le s\le t}|Y(s)|^2)$. Let $u(t)=[5E\sup_{-\tau\le t\le 0}|\xi(t)|^2+(8T+64)TK]e^{(8T+64)\int_0^t\Gamma(s)ds}$, then u is the solution of $u(t)=5E\sup_{-\tau\le t\le 0}|\xi(t)|^2+(8T+64)TK+(8T+64)\int_0^t\Gamma(s)u(s)ds$. so we have $E(\sup_{-\tau\le s\le t}|Y(s)|^2)\le u(t)$. Thus for any $t\in[-\tau,T]$, it follows that

$$E(\sup_{-\tau\le t\le T}|Y(t)|^2)\le u(T)=C_1.$$

The proof is complete. □

Lemma 3.2. *Let $E|\xi(t)-\xi(s)|^2\le\lambda(t-s)$ and (3.1) hold, then there is a constant C_2, which is independent of h such that*

$$E||Y_s-\bar{Y}_s||^2\le C_2h. \quad (3.8)$$

Proof. Fixed any $s\in[0,T]$ and $\theta\in[-\tau,0]$. Let $s\in[nh,(n+1)h)_{n=0,1,2\ldots[T/h]}$ and $\theta\in[kh,(k+1)h)_{k=-N,-(N-1)\cdots-1}$. Clearly, $0\le s-nh<h$ and $0\le\theta-kh<h$, So $0\le s+\theta-(n+k)h<2h$, that is, $(n+k)h\le s+\theta<(n+k+2)h$. By (2.2), we have

$$Y_s-\bar{Y}_s=Y(s+\theta)-\{\bar{Y}((n+k)h)+\frac{\theta-kh}{h}[\bar{Y}((n+k+1)h)$$
$$-\bar{Y}((n+k)h)]\}. \quad (3.9)$$

Hence

$$E\|Y_s - \bar{Y}_s\|^2 \le 2E|Y(s+\theta) - \bar{Y}((n+k)h)|^2 + 2E|Y((n+k+1)h) - \bar{Y}((n+k)h)|^2. \quad (3.10)$$

To show the desired result, let us consider the following four possible cases:

(1) If $(n+k+2)h \le 0$, then $(n+k)h < 0$ and $s+\theta < 0$. So

$$E\|Y_s - \bar{Y}_s\|^2 \le 2E|\xi(s+\theta) - \xi((n+k)h)|^2 + 2E|\xi((n+k+1)h) - \xi((n+k)h)|^2 \le 6\lambda h. \quad (3.11)$$

(2) If $(n+k)h \ge 0$, then $s+\theta \ge 0$. It follows from (2.5) that

$$Y(s+\theta) - \bar{Y}((n+k)h) = \int_{(n+k)h}^{s+\theta} f(v, \bar{Y}_v, \bar{r}(v))dv + \int_{(n+k)h}^{s+\theta} g(v, \bar{Y}_v, \bar{r}(v))dW(v) + \int_{(n+k)h}^{s+\theta} \int_{R^n} h(v, \bar{Y}_v, u)\tilde{N}(dv, du). \quad (3.12)$$

By Lemma 3.1 and (3.1), we compute that

$$\begin{aligned}
&E|Y(s+\theta) - \bar{Y}((n+k)h)|^2 \\
&\le 3E\Big|\int_{(n+k)h}^{s+\theta} f(v, \bar{Y}_v, \bar{r}(v))dv\Big|^2 + 3E\Big|\int_{(n+k)h}^{s+\theta} g(v, \bar{Y}_v, \bar{r}(v))dW(v)\Big|^2 \\
&\quad + 3E\Big|\int_{(n+k)h}^{s+\theta} \int_{R^n} h(v, \bar{Y}_v, u)\tilde{N}(dv, du)\Big|^2 \\
&\le 3[s+\theta - (n+k)h]E\int_{(n+k)h}^{s+\theta} |f(v, \bar{Y}_v, \bar{r}(v))|^2 dv \\
&\quad + 3E\int_{(n+k)h}^{s+\theta} |g(v, \bar{Y}_v, \bar{r}(v))|^2 dv + 3E\int_{(n+k)h}^{s+\theta} \int_{R^n} |h(v, \bar{Y}_v, u)|^2 \Pi(du)dv \\
&\le 12(h+1)E\int_{(n+k)h}^{s+\theta} \Gamma(v)k(\|\bar{Y}_v\|^2)dv + 12hE\int_{(n+k)h}^{s+\theta} |f(v, 0, \bar{r}(v)|^2 dv \\
&\quad + 6E\int_{(n+k)h}^{s+\theta} |g(v, 0, \bar{r}(v)|^2 dv + 6E\int_{(n+k)h}^{s+\theta} |h(v, 0, u)|^2 \Pi(du)dv \\
&\le 12(h+1)\int_{(n+k)h}^{s+\theta} \Gamma(v)k(E\|\bar{Y}_v\|^2)dv + 24(h+1)Kh \le C'h,
\end{aligned}$$

where $C' = 24(h+1)\Gamma[(n+k+2)h]k(C_1) + 24(h+1)K$.

Similarly, we have $E|Y((n+k+1)h) - \bar{Y}((n+k)h)|^2 \le C'h$. Using this bound in (3.10) gives

$$E||Y_s - \bar{Y}_s||^2 \leq Ch. \tag{3.13}$$

(3) If $(n+k)h < 0$, i.e. $n+k=-1$, and $-h \leq s+\theta < 0$. So

$$\begin{aligned} E||Y_s - \bar{Y}_s||^2 &\leq 2E|\xi(s+\theta) - \xi(-h)|^2 + 2E|\xi(0) - \xi(-h)|^2 \\ &\leq 2\lambda(s+\theta+h) + 2\lambda \leq 4\lambda h. \end{aligned} \tag{3.14}$$

(4) If $(n+k)h < 0$, i.e. $n+k=-1$, and $0 \leq s+\theta < h$. So

$$\begin{aligned} E||Y_s - \bar{Y}_s||^2 &\leq 2E|Y(s+\theta) - \bar{Y}(-h)|^2 + 2E|\xi(0) - \xi(-h)|^2 \\ &= 2E|Y(s+\theta) - \xi(0)|^2 + 4E|\xi(0) - \xi(-h)|^2 \end{aligned}$$

It can be shown in the same way as in case (2) that $E|Y(s+\theta) - \xi(0)|^2 \leq C'h$. We therefore see that

$$E||Y_s - \bar{Y}_s||^2 \leq 2(C' + 2\lambda)h. \tag{3.15}$$

Combining these different cases (3.11), (3.13)-(3.15) together, we can conclude that we have

$$E||Y_s - \bar{Y}_s||^2 \leq C_2 h. \tag{3.16}$$

The proof is complete. □

Lemma 3.3. *Under (3.1),*

$$\begin{aligned} &E\int_0^T |f(s, \bar{Y}_s, r(s) - f(s, \bar{Y}_s, \bar{r}(s)|^2 ds \leq C_3 h, \\ &E\int_0^T |g(s, \bar{Y}_s, r(s) - f(s, \bar{Y}_s, \bar{r}(s)|^2 ds \leq C_4 h, \end{aligned} \tag{3.17}$$

where C_3, C_4 are constants which are independent of h.

Proof. The proof is similar to that of [6], [7] and we omit it. □

We will also need the Bihari inequality (see [2]) which we cite as a lemma.

Lemma 3.4. (Bihari's Inequality) *Let $T > 0$ and $c > 0$. Let $k : R_+ \to R_+$ be a continuous non-decreasing function such that $k(t) > 0$ for all $t > 0$. Let $u(.)$ be a Borel measurable bounded non-negative function on $[0,T]$, and let $v(.)$ be a non-negative integrable function on $[0,T]$. If $u(t) \leq c + \int_0^t v(s)k(u(s))ds$ for all $0 \leq t \leq T$, then $u(t) \leq G^{-1}(G(c) + \int_0^t v(s)ds$ holds for all such $t \in [0,T]$ that $G(c) + \int_0^t v(s)ds \in \mathrm{Dom}(G^{-1})$, where $G(r) = \int_a^r ds/k(s)$ on $r > 0$, and G^{-1} is the inverse function of G.*

Proof of Theorem 3.1. By the Hölder inequality and the Doob martingale inequality, it is easy to show that

$$E(\sup_{0\le s\le t}|X(s)-Y(s)|^2)\le 3TE\int_0^t|f(s,X_s,r(s))-f(s,\bar{Y}_s,\bar{r}(s))|^2ds$$
$$+12E\int_0^t|g(s,X_s,r(s))-g(s,\bar{Y}_s,\bar{r}(s))|^2ds$$
$$+12E\int_0^t\int_{R^n}|h(s,X_s,u)-h(s,\bar{Y}_s,u)|^2\Pi(du)ds. \quad (3.18)$$

Since

$$E\int_0^t|f(s,X_s,r(s))-f(s,\bar{Y}_s,\bar{r}(s))|^2ds$$
$$\le 2E\int_0^t|f(s,X_s,r(s))-f(s,\bar{Y}_s,r(s))|^2ds$$
$$+2E\int_0^t|f(s,\bar{Y}_s,r(s))-f(s,\bar{Y}_s,\bar{r}(s))|^2ds, \quad (3.19)$$

then, by (3.1) and Lemmas 3.2-3.3,

$$E\int_0^t|f(s,X_s,r(s))-f(s,\bar{Y}_s,\bar{r}(s))|^2ds\le 2E\int_0^t\Gamma(s)k(||X_s-\bar{Y}_s||^2)ds+2C_3h$$
$$\le 2\int_0^t\Gamma(s)k(E||X_s-\bar{Y}_s||^2)ds+2C_3h$$
$$\le 2\int_0^t\Gamma(s)k(E||X_s-Y_s+Y_s-\bar{Y}_s||^2)ds+2C_3h$$
$$\le 2\int_0^t\Gamma(s)k(2E||X_s-Y_s||^2+2E||Y_s-\bar{Y}_s||^2)ds+2C_3h$$
$$\le 2\int_0^t\Gamma(s)[2k(E||X_s-Y_s||^2)+2k(C_2h)]ds+2C_3h$$
$$\le 4\int_0^t\Gamma(s)k(E\sup_{0\le v\le s}|X(v)-Y(v)|^2)ds+4C_2\int_0^t\Gamma(s)k(h)ds+2C_3h. \quad (3.20)$$

Similarly,

$$E\int_0^t|g(s,X_s,r(s))-g(s,\bar{Y}_s,\bar{r}(s))|^2ds$$
$$\le 4\int_0^t\Gamma(s)k(E\sup_{0\le v\le s}|X(v)-Y(v)|^2)ds+4C_2k(h)\int_0^t\Gamma(s)ds+2C_4h. \quad (3.21)$$

and

$$E\int_0^t\int_{R^n}|h(s,X_s,u)-h(s,\bar{Y}_s,u)|^2\Pi(du)ds\le E\int_0^t\Gamma(s)k(||X_s-\bar{Y}_s||^2)ds$$

$$\leq 2\int_0^t \Gamma(s)k(E\sup_{0\leq v\leq s}|X(v)-Y(v)|^2)ds + 2C_2k(h)\int_0^t \Gamma(s)ds. \quad (3.22)$$

Taking (3.20)-(3.22) into (3.18), we obtain

$$E(\sup_{0\leq s\leq t}|X(s)-Y(s)|^2) \leq 12(T+6)\int_0^t \Gamma(s)k(E\sup_{0\leq v\leq s}|X(v)-Y(v)|^2)ds$$
$$+ 12(T+6)C_2k(h)\int_0^t \Gamma(s)ds + 6(TC_3+4C_4)h. \quad (3.23)$$

By applying the Bihari inequality, it follows that

$$E(\sup_{0\leq s\leq t}|X(s)-Y(s)|^2) \leq G^{-1}\{G[12(T+6)\int_0^t \Gamma(s)dsC_2k(h)+6(TC_3+4C_4)h]$$
$$+ 12(T+6)\int_0^t \Gamma(s)ds\}. \quad (3.24)$$

Note that when $h \to 0$, then $12(T+6)\int_0^t \Gamma(s)dsC_2k(h) + 6(TC_3+4C_4)h \to 0$. Recalling the condition $\int_0^a du/k(u) = \infty$, we have $G[12(T+6)\int_0^t \Gamma(s)dsC_2k(h)+ 6(TC_3+4C_4)h] \to -\infty$, and $G[12(T+6)\int_0^t \Gamma(s)dsC_2k(h) + 6(TC_3+4C_4)h] + 12(T+6)\int_0^t \Gamma(s)ds \to -\infty$. So we get $G^{-1}\{G[12(T+6)\int_0^t \Gamma(s)dsC_2k(h) + 6(TC_3+4C_4)h] + 12(T+6)\int_0^t \Gamma(s)ds\} \to 0$. We therefore have

$$\lim_{h\to 0} E(\sup_{0\leq s\leq t}|X(s)-Y(s)|^2) \leq \lim_{h\to 0} G^{-1}\{G[12(T+6)\int_0^t \Gamma(s)dsC_2k(h)$$
$$+ 6(TC_3+4C_4)h] + 12(T+6)\int_0^t \Gamma(s)ds\} = 0.$$

The proof of Theorem 3.1 is now complete. □

Remark 3.2. If $\Gamma(t) = L(L > 0)$ and $k(u) = u$, $u \geq 0$, then condition (3.1) implies a global Lipschitz condition.

Corollary 3.1. *For all $x, y \in C([-\tau, 0]; R^n)$, $u \in R^n$ and $i \in S$,*

$$|f(t,x,i)-f(t,y,i)|^2 \vee |g(t,x,i)-g(t,y,i)|^2$$
$$\vee \int_{R^n}|h(t,x,u)-h(t,y,u)|^2\Pi(du) \leq L||x-y||^2, \quad (3.25)$$

let $E|\xi(t)-\xi(s)|^2 \leq \lambda(t-s)$. Then we have

$$E(\sup_{0\leq t\leq T}|X(t)-Y(t)|^2) \leq Ch^2. \quad (3.26)$$

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