International Journal of Pure and Applied Mathematics
Volume 63 No. 1 2010, 75-84

# OSCILLATION CRITERIA FOR RETARDED FUNCTIONAL DIFFERENTIAL SYSTEMS

Ana M. Pedro
Departamento de Matemática
Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa
Quinta da Torre, 2825-114, Monte de Caparica, PORTUGAL

**Abstract:** Several criteria are given for having the retarded functional differential systems $\frac{d}{dt}x\left(t\right)=\int_{-1}^{0}x\left(t-r\left(\theta\right)\right)dq\left(\theta\right)$ oscillatory, depending upon the smoothness of the delay function $r\left(\theta\right)$.

**AMS Subject Classification:** 34K11
**Key Words:** retarded functional differential systems, oscillations

## 1. Introduction

The purpose of this note is to investigate the oscillatory behavior of the solutions of the functional differential system

$$\frac{d}{dt}x\left(t\right)+\int_{-1}^{0}x\left(t-r\left(\theta\right)\right)d\left[q\left(\theta\right)\right]=0, \tag{1}$$

where $x(t)\in\mathbb{R}^{n}$, $r\left(\theta\right)$ is a positive real differentiable function on $[-1,0]$ and $q\left(\theta\right)$ is a real $n\times n$ matrix valued function of bounded variation on $[-1,0]$.

The case of (1) more common in the literature on the oscillatory behavior of retarded functional differential systems (see [1] and [2]), is when the delay function is $r\left(\theta\right)=-r\theta$ for every $\theta\in[-1,0]$. An important new aspect of the system is that it gives the possibility to the delays of playing some role in order to obtain new oscillatory criteria.

Received: June 12, 2010 

The system (1) appears in [3] for continuous delay functions $r(\theta)$. Further we assume that the delays $r(\theta)$ are differentiable functions. For that purpose, we will consider $r(\theta)$ in the set $D^{+}$, of all differentiable and positive functions on $[-1,0]$. This work is a continuation of the paper [7] where some criteria for having systems of this type nonoscillatory were obtained.

Several criteria for having retarded functional difference systems oscillatory or nonocisllatory have been given in [6]. In this work we will use similar methods, to those used in [6], to establish conditions under which the system (1) is oscillatory. However the treatment of systems of type (1) is substantially different comparing with the systems considered in [6].

It will be also considered the relevant class of the differential difference systems

$$\frac{d}{dt}x(t)+\sum_{j=1}^{p}A_{j}x(t-r_{j})=0, \tag{2}$$

where the $A_j$ are nonzero real matrices in $\mathbb{R}^{n\times n}$ and each $r_j$ is a positive real number $(j=1,...,p)$. As it is well-know, this equation can be obtained from (1), under the assumption that $q(\theta)$ is a step function with a number $p$ of jump points. More concretely it can be obtained from (1) with $q(\theta)$ given explicitly, for example, by

$$q(\theta)=\sum_{j=1}^{p}Q(\theta-\theta_{j})A_{j}, \tag{3}$$

where $-1=\theta_1<...<\theta_p<\theta_{p+1}=0$, $Q(\theta-\theta_j)=0$ if $\theta\le\theta_j$, $Q(\theta-\theta_j)=1$ if $\theta>\theta_j$, and the delays, $r_j$, are obtained through any function $r(\theta)\in D^{+}$ satisfying $r(\theta_j)=r_j$ for $j=1,...,p$.

Define

$$\|r\|=\max_{-1\le\theta\le0}r(\theta).$$

By a solution of (1) we mean a continuous function $x:[-\|r\|,\infty[\,\to\mathbb{R}^{n}$, which is differentiable on $[0,+\infty[$ in manner that (1) is satisfied for every $t\ge0$. A solution $x(t)=[x_1(t),...,x_n(t)]$ is said oscillatory whenever each $x_k(t)$ $(k=1,...,n)$ has an infinite number of zeros; otherwhise $x(t)$ will be said nonoscillatory. When all solutions are oscillatory the system (1) is called oscillatory.

According to [4], the analysis of the oscillatory behavior of solutions of the system (1) can be based upon the existence or absence of real zeros of the characteristic equation

$$\det\left[\lambda I+\int_{-1}^{0}\exp\left(-\lambda r\left(\theta\right)\right)d\left[q\left(\theta\right)\right]\right]=0, \tag{4}$$

where by $I$ we mean the $n\times n$ identity matrix. In fact, in this framework, one can conclude that (1) is oscillatory if and only if (4) has no real roots.

Given a function $\phi$ of bounded variation, we define $\phi^{+}$ by the following equality

$$\phi^{+}\left(\alpha,\beta\right)=\phi\left(\beta\right)-\phi\left(\alpha\right). \tag{5}$$

The matrix measures of a matrix (also called logarithm norms) will play an important role in the following section. To each induced norm, $\left\|\cdot\right\|$, in $\mathbb{R}^{n\times n}$, we associate a matrix measure $\mu:\mathbb{R}^{n\times n}\to\mathbb{R}$, which is defined for any $C\in\mathbb{R}^{n\times n}$ as

$$\mu\left(C\right)=\lim_{y\to 0^{+}}\frac{\left\|I+yC\right\|-1}{y}.$$

Well-known matrix measures of a matrix $C=\left[c_{jk}\right]\in\mathbb{R}^{n\times n}$ are, for example,

$$\mu_{1}\left(C\right)=\max\left\{c_{kk}+\sum_{j\neq k}\left|c_{jk}\right|:k=1,...,n\right\},$$

$$\mu_{\infty}\left(C\right)=\max\left\{c_{jj}+\sum_{k\neq j}\left|c_{jk}\right|:j=1,...,n\right\},$$

which correspond, respectively, to the induced norms in $\mathbb{R}^{n\times n}$ given by:

$$\left\|C\right\|_{1}=\max\left\{\sum_{j=1}^{n}\left|c_{jk}\right|:k=1,...,n\right\},$$

$$\left\|C\right\|_{\infty}=\max\left\{\sum_{k=1}^{n}\left|c_{jk}\right|:j=1,...,n\right\}.$$

These matrix measures have some well-known properties which can be found in [5, Lemmas 2.1 and 2.2]. In particular, we have:

*(i)* $\mu\left(C_{1}\right)-\mu\left(-C_{2}\right)\leq\mu\left(C_{1}+C_{2}\right)\leq\mu\left(C_{1}\right)+\mu\left(C_{2}\right)\ \left(C_{1},C_{2}\in\mathbb{R}^{n\times n}\right).$

*(ii)* $\mu\left(\alpha C\right)=\alpha\mu\left(C\right)$, for every $\alpha\geq 0$.

*(iii)* $-\mu\left(-C\right)\leq\operatorname{Re}\lambda_{i}\left(C\right)\leq\mu\left(C\right)$ for all $i=1,...,d$, where $\operatorname{Re}\lambda_{i}\left(C\right)$ denotes the real part of the eigenvalue $\lambda_{i}\left(C\right)$ of $C$.

If $\phi\in BV_{n}$, the Banach space of all real functions of bounded variation, the continuity of $\mu$ on $\mathbb{R}^{n\times n}$ implies that $\mu\circ\phi\in BV$; as a consequence, with $[a,b]\subset[-1,0]$, the following inequalities hold:

$(iv)$ If $\varphi\in C([a,b];\mathbb{R})$ is nonincreasing and positive, then

$$\mu\left(\int_a^b \varphi(\theta)\, d[\phi(\theta)]\right)\leq \int_a^b \varphi(\theta)\, d\left(\mu\left(\phi^+(a,\theta)\right)\right).$$

$(v)$ If $\varphi\in C([a,b];\mathbb{R})$ is nondecreasing and positive, then

$$\mu\left(\int_a^b \varphi(\theta)\, d[\phi(\theta)]\right)\leq -\int_a^b \varphi(\theta)\, d\left(\mu\left(\phi^+(\theta,b)\right)\right).$$

## 2. Oscillation Criteria

Let us define

$$F(\lambda)=\lambda I-\int_{-1}^0 \exp(-\lambda r(\theta))\, d[q(\theta)].$$

**Lemma 1.** *If $\mu(-F(\lambda))<0$ for all $\lambda\in\mathbb{R}$, then the system (1) is oscillatory.*

*Proof.* By property $(iii)$ of the matrix measures 5, we have

$$\mathrm{Re}\lambda_i(-F(\lambda))\ \leq\mu(-F(\lambda))<0 \text{ for all } i=1,...,n,\lambda\in\mathbb{R}.$$

Then $\det(-F(\lambda))\neq 0$ for all $\lambda\in\mathbb{R}$ and the system (1) is oscillatory. □

With $-1\leq a\leq b\leq 0$, let $D^+(a,b)$ be the family of all functions in $D^+$ which are increasing on $[-1,a]$, constant on $[a,b]$ and decreasing on $[b,0]$. In the case of having $a=b=\theta_0$ with $\theta_0\in[-1,0]$ we obtain the family $D^+(\theta_0)$ of all differentiable and positive functions which are increasing on $[-1,\theta_0]$ and decreasing on $[\theta_0,0]$. If $\theta_0=-1$, $D_d^+$ is the class of all positive differentiable and decreasing functions on $[-1,0]$. For $\theta_0=0$ we obtain the family $D_i^+$ of all positive differentiable and increasing funtions on $[-1,0]$. For these families of delays we start by stating the following oscillatory situation.

**Theorem 2.** *Let $q\in BV_n$ be such that*

$$\mu\left(q^+(-1,0)\right)<0 \tag{6}$$

$$\begin{array}{lll}\mu(q^+(-1,\theta))\leq 0, & \mu(q^+(\theta,a))\leq 0 \quad \text{for every} & \theta\in[-1,a],\\ \mu(q^+(b,\theta))\leq 0, & \mu(q^+(\theta,0))\leq 0 \quad \text{for every} & \theta\in[b,0],\\ \multicolumn{3}{l}{\mu(q^+(-1,a))+\mu(q^+(b,0))+\mu(q^+(a,b))<0.}\end{array} \tag{7}$$

*If $r\in D^+(a,b)$ and*

$$\int_b^0 \mu\left(q^+(b,\theta)\right) d\log r(\theta) - \int_{-1}^a \mu\left(q^+(\theta,a)\right) d\log r(\theta)$$
$$< \frac{e}{r(a)}\left[\log\left(r(a)\left|\mu\left(q^+(-1,a)\right) + \mu\left(q^+(b,0)\right) + \mu\left(q^+(a,b)\right)\right|\right) + 1\right] \quad (8)$$

*then (1) is oscillatory.*

*Proof.* With $r \in D^+(a,b)$ and $\lambda = 0$, we have by (6)

$$\mu(-F(0)) = \mu\left(\int_{-1}^0 d[q(\theta)]\right) = \mu(q(0) - q(-1)) = \mu\left(q^+(-1,0)\right) < 0.$$

For $\lambda < 0$ by, using the property $(i)$ of matrix measures, we have

$$\mu(-F(\lambda)) \leq -\lambda + \mu\left(\int_{-1}^0 \exp(-\lambda r(\theta))\, d[q(\theta)]\right).$$

Then, by the properties $(i), (iv)$ and $(v)$ of matrix measures, we obtain

$$-\mu(-F(\lambda)) \geq \lambda - \mu\left(\int_{-1}^a \exp(-\lambda r(\theta))\, d[q(\theta)]\right) -$$
$$-\mu\left(\int_a^b \exp(-\lambda r(\theta))\, d[q(\theta)]\right) - \mu\left(\int_b^0 \exp(-\lambda r(\theta))\, d[q(\theta)]\right)$$
$$\geq \lambda + \int_{-1}^a \exp(-\lambda r(\theta))\, d\mu\left(q^+(\theta,a)\right)$$
$$-\exp(-\lambda r(a))\,\mu\left(q^+(a,b)\right) - \int_b^0 \exp(-\lambda r(\theta))\, d\mu\left(q^+(b,\theta)\right).$$

Integrating by parts each one of the above integrals, using (7) and since

$$\exp(-\lambda r(a)) \geq \exp(-\lambda r(-1)), \quad \exp(-\lambda r(a)) \geq \exp(-\lambda r(0)),$$

we have

$$-\mu(-F(\lambda)) \geq \lambda + \exp(-\lambda r(a))\left|\mu\left(q^+(-1,a)\right) + \mu\left(q^+(b,0)\right) + \mu\left(q^+(a,b)\right)\right|$$
$$-\lambda\left[\int_b^0 \exp(-\lambda r(\theta))\,\mu\left(q^+(b,\theta)\right) dr(\theta) - \int_{-1}^a \exp(-\lambda r(\theta))\,\mu\left(q^+(\theta,a)\right) dr(\theta)\right],$$

We have $\lambda r(\theta)\exp(-\lambda r(\theta)) < e^{-1}$ for every $\lambda \in \mathbb{R}$ and $\theta \in [-1,0]$. Then, using (7)

$$-\mu(-F(\lambda)) \geq \lambda + \exp(-\lambda r(a))\left|\mu\left(q^+(-1,a)\right) + \mu\left(q^+(b,0)\right) + \mu\left(q^+(a,b)\right)\right|$$
$$-e^{-1}\left[\int_b^0 \mu\left(q^+(b,\theta)\right) d\log r(\theta) - \int_{-1}^a \mu\left(q^+(\theta,a)\right) d\log r(\theta)\right].$$

The function

$$\varphi(\lambda) \;=\; \lambda + \exp(-\lambda r(a))\left|\mu\left(q^+(-1,a)\right) + \mu\left(q^+(b,0)\right) + \mu\left(q^+(a,b)\right)\right|$$

$$-e^{-1}\left[\int_b^0 \mu\left(q^+(b,\theta)\right)d\log r(\theta)-\int_{-1}^a \mu\left(q^+(\theta,a)\right)d\log r(\theta)\right],$$

has absolute minimum

$$\begin{aligned}\varphi(\lambda_0) &= \frac{1}{r(a)}\left[\log\left(r(a)\left|\mu\left(q^+(-1,a)\right)+\mu\left(q^+(b,0)\right)+\mu\left(q^+(a,b)\right)\right|\right)+1\right]\\ &\quad -e^{-1}\left[\int_b^0 \mu\left(q^+(b,\theta)\right)d\log r(\theta)-\int_{-1}^a \mu\left(q^+(\theta,a)\right)d\log r(\theta)\right],\end{aligned}$$

attained at

$$\lambda_0=\frac{1}{r(a)}\log\left(r(a)\left|\mu\left(q^+(-1,a)\right)+\mu\left(q^+(b,0)\right)+\mu\left(q^+(a,b)\right)\right|\right)$$

and by (8) $\varphi(\lambda_0)>0$, that is, $-\mu(-F(\lambda))>0$. Therefore $\mu(-F(\lambda))<0$.

For $\lambda>0$, we have

$$\begin{aligned}\mu(-F(\lambda)) &\le -\lambda+\int_{-1}^a \exp(-\lambda r(\theta))\,d\mu\left(q^+(-1,\theta)\right)\\ &\quad +\exp(-\lambda r(a))\,\mu\left(q^+(a,b)\right)-\int_b^0 \exp(-\lambda r(\theta))\,d\mu\left(q^+(\theta,0)\right)\end{aligned}$$

Integrating by parts each one of the above integrals we have

$$\begin{aligned}\mu(-F(\lambda)) &\le -\lambda+\exp(-\lambda r(a))\left[\mu\left(q^+(-1,a)\right)+\mu\left(q^+(b,0)\right)+\mu\left(q^+(a,b)\right)\right]\\ &\quad +\lambda\left[\int_{-1}^a \exp(-\lambda r(\theta))\,\mu\left(q^+(-1,\theta)\right)dr(\theta)\right.\\ &\quad \left.-\int_b^0 \exp(-\lambda r(\theta))\,\mu\left(q^+(\theta,0)\right)dr(\theta)\right].\end{aligned}$$

Thus, by (7), $\mu(-F(\lambda))<0$. Therefore by Lemma 1 the system (1) is oscillatory. □

For the case where $a=b=\theta_0$ we have the following corollary

**Corollary 3.** *Let $r(\theta)\in D^+(\theta_0)$ and $q\in BV_d$ such that $\mu\left(q^+(-1,0)\right)<0$*

$\mu\left(q^+(-1,\theta)\right)\le 0,\quad \mu\left(q^+(\theta,\theta_0)\right)\le 0$ *for every* $\theta\in[-1,\theta_0]$,
$\mu\left(q^+(\theta_0,\theta)\right)\le 0,\quad \mu\left(q^+(\theta,0)\right)\le 0$ *for every* $\theta\in[\theta_0,0]$,
$\mu\left(q^+(-1,\theta_0)\right)+\mu\left(q^+(\theta_0,0)\right)<0.$

*If*

$$\begin{aligned}&\int_{-1}^{\theta_0}\mu\left(q^+(\theta,\theta_0)\right)d\log r(\theta)-\int_{\theta_0}^0\mu\left(q^+(\theta_0,\theta)\right)d\log r(\theta)\\ &\qquad<\frac{e}{r(\theta_0)}\left[\log\left(r(\theta_0)\left|\mu\left(q^+(-1,\theta_0)\right)+\mu\left(q^+(\theta_0,0)\right)\right|\right)+1\right],\end{aligned}$$

*then (1) is oscillatory.*

**Corollary 4.** *Let $\theta_0 = 0$ and $r(\theta) \in D_i^+$, if $\mu(q^+(-1,0)) < 0, \mu(q^+(-1,\theta)) \le 0$, $\mu(q^+(\theta,0)) \le 0$ for every $\theta \in [-1,0]$, and*

$$\int_{-1}^{0} \mu\left(q^+(\theta,0)\right) d\log r(\theta) < \frac{e}{r(0)}\left[\log\left(r(0)\left|\mu\left(q^+(-1,0)\right)\right|\right)+1\right],$$

*then (1) is oscillatory.*

**Corollary 5.** *Let $\theta_0 = -1$ and $r(\theta) \in D_d^+$, if $\mu(q^+(-1,0)) < 0$, $\mu(q^+(-1,\theta)) \le 0$, $\mu(q^+(\theta,0)) \le 0$ for every $\theta \in [-1,0]$, and*

$$\int_{-1}^{0} \mu\left(q^+(-1,\theta)\right) d\log r(\theta) > \frac{-e}{r(-1)}\left[\log\left(r(-1)\left|\mu\left(q^+(-1,0)\right)\right|\right)+1\right].$$

*Then (1) is oscillatory.*

**Example 6.** *Let be $r(\theta) = 2\theta + 2$, for $\theta \in [-1,0]$ and*

$$q(\theta) = \begin{bmatrix} -\theta & 0 & 0 \\ 0 & -\theta & 0 \\ 0 & 0 & -\theta-2 \end{bmatrix}.$$

*Thus*

$$\mu_\infty\left(q^+(-1,0)\right) = \mu_\infty \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \max\{-1,-1,-1\} = -1 < 0,$$

$$\begin{aligned} \mu_\infty\left(q^+(-1,\theta)\right) &= \mu_\infty \begin{bmatrix} -\theta-1 & 0 & 0 \\ 0 & -\theta-1 & 0 \\ 0 & 0 & -\theta-1 \end{bmatrix} \\ &= \max_{-1\le\theta\le0}\{-\theta-1,-\theta-1,-\theta-1\} = 0, \end{aligned}$$

$$\mu_\infty\left(q^+(\theta,0)\right) = \mu_\infty \begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{bmatrix} = \max_{-1\le\theta\le0}\{\theta,\theta,\theta\} = 0,$$

*and*

$$\int_{-1}^{0} \mu\left(q^+(\theta,0)\right) d\left(\log(2\theta+2)\right) \approx 0 < \frac{e}{2}\left(\log(2)+1\right) \approx 2,301.$$

*Then, by Corollary 4, the system*

$$\frac{d}{dt}x(t) = \int_{-1}^{0} x(t-2\theta-2)\,dq(\theta)$$

*is oscillatory.*

Corollaries 4 and 5 enable us to conclude the following corollary relative to the system (2), where $q(\theta)$ is given by (3) and $r(\theta_j) = r_j$ for $j = 1,2,...,p$.

**Corollary 7.** *If* $\mu\left(\sum\limits_{k=1}^{p} A_k\right) < 0$, $\mu\left(\sum\limits_{k=1}^{j} A_k\right) \leq 0$, $\mu\left(\sum\limits_{k=j}^{p} A_k\right) \leq 0$, *for* $j = 1, ..., p$ *and either*

$$\sum_{j=2}^{p} \mu\left(\sum_{k=j}^{p} A_k\right) \log \frac{r_j}{r_{j-1}} < \frac{e}{r_p}\left[\log\left(r_p \left|\mu\left(\sum_{k=1}^{p} A_k\right)\right|\right) + 1\right], \text{ for } r_1 < ... < r_p,$$

*or*

$$\sum_{j=2}^{p} \mu\left(\sum_{k=1}^{j} A_k\right) \log \frac{r_{j-1}}{r_j} < \frac{e}{r_1}\left[\log\left(r_1 \left|\mu\left(\sum_{k=1}^{p} A_k\right)\right|\right) + 1\right], \text{ for } r_1 > ... > r_p,$$

*then (2) is oscillatory.*

**Example 8.** *Let us consider the system*

$$\frac{d}{dt}x(t) = A_1 x(t-1) + A_2 x(t-2) + A_3 x(t-3), \tag{9}$$

*where*

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

*Here*

$$\mu_\infty(A_1 + A_2 + A_3) = \mu_\infty \begin{bmatrix} -3 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & 1 & -6 \end{bmatrix} = \max\{-2, -3, -5\} = -2 < 0,$$

$$\mu_\infty(A_1) = -1 \leq 0; \quad \mu_\infty(A_2 + A_3) = \mu_\infty \begin{bmatrix} -2 & -1 & 0 \\ 0 & -3 & -1 \\ 0 & 1 & -5 \end{bmatrix}$$

$$= \max\{-1, -2, -4\} = -1 \leq 0,$$

$$\mu_\infty(A_3) = -2 \leq 0;$$

$$\text{and} \quad \mu_\infty(A_1 + A_2) = \mu_\infty \begin{bmatrix} -1 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & -4 \end{bmatrix} = 0$$

*since*

$$\mu_\infty(A_2 + A_3) \log \frac{r_2}{r_1} + \mu_\infty(A_3) \log \frac{r_3}{r_2} = -\log 2 - 2\log \frac{3}{2} = \log 2 - 2\log 3$$

$$\begin{aligned} &< \frac{e}{3}\left[\log\left[3\left|\mu_{\infty}\left(A_1+A_2+A_3\right)\right|\right]+1\right] \\ &= \frac{e}{3}\left[\log 9+1\right]. \end{aligned}$$

*Then, by Corollary 7, the system (9) is oscillatory*

**Example 9.** *Let us consider the system*

$$\frac{d}{dt}x(t) = A_1 x(t-4) + A_2 x(t-3) + A_3 x(t-2), \tag{10}$$

*where*

$$A_1 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 0 & -3 \\ -3 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

*Here*

$$\mu_1(A_1+A_2+A_3) = \mu_1\begin{bmatrix} -7 & 0 & -3 \\ -3 & -6 & -1 \\ 1 & 0 & -11 \end{bmatrix} = \max\{-3,-6,-7\} = -3 < 0,$$

$$\begin{aligned} \mu_1(A_1) &= -3 \le 0; \quad \mu_1(A_2+A_3) = \mu_1\begin{bmatrix} -4 & 0 & -3 \\ -3 & -3 & -1 \\ 1 & 0 & -5 \end{bmatrix} \\ &= \max\{0,-3,-1\} = 0, \\ \mu_1(A_3) &= -2 \le 0; \\ \text{and} \quad \mu_1(A_1+A_2) &= \mu_1\begin{bmatrix} -5 & 0 & -3 \\ -3 & -3 & -1 \\ 1 & 0 & -6 \end{bmatrix} = -1 \le 0 \end{aligned}$$

*Since*

$$\mu_1(A_1+A_2)\log\frac{r_1}{r_2} + \mu_1(A_1+A_2+A_3)\log\frac{r_2}{r_3} = -\log\frac{2}{3} - 3\log\frac{3}{2}$$
$$= \log 2 - 2\log 3 < \frac{e}{4}\left[\log\left[4\left|\mu_1\left(A_1+A_2+A_3\right)\right|\right]+1\right] = \frac{e}{4}\left[\log 12+1\right].$$

*Then, by Corollary 7, the system (10) is oscillatory.*

## References

[1] J.M. Ferreira, I. Györi, Oscillatory behavior in linear retarded functional

differential equations, *J. Math. Anal. Appl.*, **128** (1987), 332-346.

[2] Q. Kong, Oscillation for systems of functional differential equations, *J. Math. Anal. Appl.*, **198** (1996), 608-619.

[3] J.M. Ferreira, A.M. Pedro, Oscillatory behaviour in functional differential systems of neutral type, *J. Math. Anal. Appl.*, **269** (2002), 533-555.

[4] T. Krisztyn, Oscillations in linear functional differential systems, *Differential Equations Dynam. Systems*, **2** (1994), 99-112.

[5] J. Kirchner, Uwe Stroinski, Explicit oscillation criteria for systems of neutral differential equations with distributed delay, *Differential Equations Dynam. Systems*, **3** (1995), 101-120.

[6] J.M. Ferreira, Sandra Pinelas, Oscillatory retarded functional systems, *J. Math. Anal. Appl.*, **285** (2003), 506-527.

[7] A.M. Pedro, Nonoscillation criteria for retarded functional differential systems, *International Journal of Pure and Applied Mathematics*, **1** (2006), 47-72.