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WEAK AND STRONG CONVERGENCE THEOREMS WITHOUT SOME WIDELY USED CONDITIONS

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Abstract: We establish weak (strong) convergence of Ishikawa iterates of two asymptotically (quasi-)nonexpansive maps without any condition on the rate of convergence associated with the two maps. Moreover, our weak convergence results do not require any of the Opial condition, Kadec-Klee property or Fréchet differentiable norm.

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1. Introduction

Let C be a nonempty subset of a normed space E. Then a selfmap T on C is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \qquad \text{for all } x, y \in C \text{ and for all } n \geq 1.$$

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of the important class of nonexpansive maps was introduced by Goebel and Kirk [5] in 1972 and it has been studied by many authors. It is well known that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space, then every asymptotically nonexpansive self map on C has a fixed point. Schu [15] modified Mann and Ishikawa iterations to approximate fixed points of asymptotically nonexpansive maps in Hilbert spaces as well as in uniformly convex Banach spaces. For more details see, for example, [3], [6], [8]-[10], [12].

A selfmap T on C is known as uniform L-Lipschitz if for some $L > 0$, we have

$$\|T^n x - T^n y\| \leq L\,\|x - y\| \qquad \text{for all } x, y \in C \text{ and for all } n \geq 1.$$

T is uniformly Hölder continuous if there exists positive constants L and α such that

$$\|T^n x - T^n y\| < L\,\|x - y\|^{\alpha} \text{ for all } x, y \in C \text{ and } n \geq 1.$$

T is termed as uniformly equi-continuous if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|T^n x - T^n y\| \leq \epsilon \text{ whenever } \|x - y\| \leq \delta \text{ for all } x, y \in C \text{ and } n \geq 1$$

or, equivalentely, T is uniformly equi-continuous if and only if

$$\|T^n x_n - T^n y_n\| \to 0 \text{ whenever } \|x_n - y_n\| \to 0 \text{ as } n \to \infty.$$

T is said to be quasi-nonexpansive if $F(T) = \{x \in C : Tx = x\} \neq \phi$ and

$$\|Tx - p\| \leq \|x - p\| \qquad \text{for all } x \in C,\ p \in F(T).$$

An asymptotically quasi-nonexpansive map is derived by combining the notions of asymptotically nonexpansive map and quasi-nonexpansive map and has recently been studied by Qihou [11]-[13] as well as the authors [4] and is defined as follows:

A selfmap T on C is asymptotically quasi-nonexpansive if $F(T) \neq \phi$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n\,\|x - p\| \qquad \text{for all } x \in C,\ p \in F(T) \text{ and for all } n \geq 1.$$

From the above definitions, it is clear that:

Asymptotic nonexpansiveness $\Rightarrow$ uniform L-Lipschitz $\Rightarrow$ uniformly Hölder continuous$\Rightarrow$ uniformly equi-continuous.

However, their converses fail in the presence of the following example [20]:

Example. Define $T : [0, 1] \to [0, 1]$ by $Tx = (1 - x^{\frac{3}{2}})^{\frac{2}{3}}$for all $x \in [0, 1]$. A Banach space E is uniformly convex if for each $r \in (0, 2]$, the modulus of convexity of E, given by

$$\delta(r) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge r\right\},$$

satisfies the inequality $\delta(r) > 0$. For a sequence, the symbol $\to$ stands for norm convergence whereas $\rightharpoonup$ for weak convergence. The space E is said to satisfy:

(i) Opial condition [10] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$;

(ii) Kadec-Klee property [7] if for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $x_n \to x$ as $n \to \infty$.

A map $T : C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

A map $T : C \to C$ is semi-compact (completely continuous) if for any bounded sequence $\{x_n\}$ in C with $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in C$ as $i \to \infty$.

Let $S = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The norm of E is:

(iii) Gâteaux differentiable (see [16]) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S, and

(iv) Fréchet differentiable (see [16]) if for each x in S, the above limit is attained uniformly for $y \in S$.

In the case of *Fréchet differentiable norm*, it has been obtained in [16] that

$$\begin{aligned}\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 &\le \frac{1}{2}\|x+h\|^2 \\ &\le \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|)\end{aligned}$$

for all x, h in E, where J is the normalized duality map from E to E^* defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},$$

$\langle ., .\rangle$ is the duality pairing between E and E^* and b is an increasing function defined on $[0, \infty)$ such that $\lim_{t\downarrow 0} \frac{b(t)}{t} = 0$.

In 2001, Khan and Takahashi [8] constructed and studied the following

Ishikawa iteration process:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \qquad \text{for all } n \geq 1, \end{cases} \tag{1.1}$$

where S, T are self asymptotically nonexpansive maps on C with $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ (rate of convergence) and $0 \leq \alpha_n, \beta_n \leq 1$.

Liu [9] and Xu [17] introduced Ishikawa (Mann) iterates with errors independently. Numerous papers have been produced on Ishikawa (Mann) iterates with errors and follow a similar computational techniques as those without errors. It has been noted that the results proved for Ishikawa (Mann) iterates without errors can easily be extended for iterates with errors.

The rate of convergence condition viz $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ has remained in extensive use to prove both weak and strong convergence theorems to approximate fixed points of asymptotically nonexpansive maps. The conditions like Opial condition, Kadec-Klee property or Fréchet differentiable norm have remained key to prove weak convergence theorems.

Tan and Xu [16] proved some weak convergence results for Mann and Ishikawa iterates of asymptotically nonexpansive maps and remarked: we do not know whether our Theorem 3.1 remains valid if k_n(the sequence associated with the asymptotically nonexpansive map T) is allowed to approach 1 slowly enough so that $\sum_{n=1}^{\infty}(k_n - 1)$ diverges.

Keeping in mind the above remarks, in this paper, we neither demand the rate of convergence condition for weak or strong convergence nor any property like Opial condition, Kadec-Klee property or Fréchet differentiable norm for weak convergence. We will prove weak (strong) convergence theorems using Ishikawa iterates for two asymptotically (quasi-)nonexpansive maps without any of the above conditions.

Many authors have approximated fixed points of asymptotically nonexpansive maps through Mann and Ishikawa iterates assuming that domain of the map is unbounded but the map has at least one fixed point. Following the investigations of Goebel and Kirk in [5], the existence of the fixed points of asymptotically nonexpansive maps needs the boundedness of the domain of the map. We will follow them for the purpose.

The following lemmas will be needed in the main section.

Lemma 1.1. (see [19], Theorem 2) *Let $r > 0$ be a fixed real number. Then a Banach space E is uniformly convex iff there is a continuous strictly* *increasing convex map* $g : [0, \infty) \to [0, \infty)$ *such that:*

(i) $g(0) = 0$

(ii) $\|\lambda x + (1-\lambda)y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$, *where* $x, y \in B_r[0] = \{x \in E : \|x\| \leq r\}$ *and* $\lambda \in [0,1]$.

Lemma 1.2. (see [20], Lemma 2.2) *Let* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *be a continuous strictly increasing map. If a sequence* $\{x_n\}$ *in* $[0, \infty)$ *satisfies* $\lim_{n\to\infty} g(x_n) = 0$, *then* $\lim_{n\to\infty} x_n = 0$.

Lemma 1.3. (see [1], Theorem 1) *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* E *and let* T *be asymptotically nonexpansive map of* C *into itself. Then* $I - T$ *is demiclosed at* 0.

2. Fixed Point Approximation

We start with proving the following lemma.

Lemma 2.1. *Let* C *be a nonempty bounded closed convex subset of a normed space* E. *Let* $S, T : C \to C$ *be uniformly equi-continuous maps. Then for the sequence* $\{x_n\}$ *given in* (1.1) *and satisfying*

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0 = \lim_{n\to\infty} \|x_n - T^n x_n\|,$$

we have that

$$\lim_{n\to\infty} \|x_n - S x_n\| = 0 = \lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

Proof. Set $c_n = \|x_n - S^n x_n\|$ and $d_n = \|x_n - T^n x_n\|$.

From

$$\|x_n - y_n\| = \beta_n \|x_n - T^n x_n\| \to 0 \text{ as } n \to \infty,$$

$$\|x_n - x_{n+1}\| = \alpha_n \|x_n - S^n y_n\| \leq \|x_n - S^n x_n\| + \|S^n x_n - S^n y_n\|,$$

and uniform equi-continuity of S, we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.1}$$

Since

$$\begin{aligned} \|x_n - Sx_n\| \leq\ & \|x_n - x_{n+1}\| + \|x_{n+1} - S^{n+1}x_{n+1}\| \\ & + \|S^{n+1}x_{n+1} - S^{n+1}x_n\| + \|S^{n+1}x_n - Sx_n\|, \end{aligned} \tag{2.2}$$

therefore taking $\limsup$ on both the sides of inequality (2.2) and using the definition of uniform equi-continuity of S, we get that $\limsup_{n\to\infty} \|x_n - Sx_n\| \leq$

0 and hence

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Similarly

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

That is,

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|. \qquad \square$$

Remark 2.1. Lemma 2.1 improves and generalizes the corresponding Lemma 3 of [8].

Lemma 2.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to C$ be uniformly equi-continuous and asymptotically quasi-nonexpansive maps with sequences $\{s_n\}, \{t_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} s_n = 1, \lim_{n\to\infty} t_n = 1$, respectively. Let the sequence $\{x_n\}$ be as in* (1.1) *with $\delta \le \alpha_n, \beta_n \le 1 - \delta$ for some $\delta \in (0, \frac{1}{2})$. If $F(S) \cap F(T) \ne \phi$, then*

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|.$$

Proof. Set $k_n = \max\{s_n, t_n\}$. Then $\lim_{n\to\infty} k_n = 1$ if $\lim_{n\to\infty} s_n = 1 = \lim_{n\to\infty} t_n$. Let $p \in F(S) \cap F(T)$. Since C is bounded, there exists $B_r[0]$ such that $C \subset B_r[0]$ for some $r > 0$. Applying Lemma 1.1 for the scheme (1.1), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(S^n y_n - p) + (1-\alpha_n)(x_n - p)\|^2 \\
&\le \alpha_n \|S^n y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n) g\left(\|x_n - S^n y_n\|\right) \\
&\le \alpha_n s_n^2 \|y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n) g\left(\|x_n - S^n y_n\|\right) \\
&= \alpha_n s_n^2 \|\beta_n(T^n x_n - p) + (1-\beta_n)(x_n - p)\|^2 \\
&\quad + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n) g\left(\|x_n - S^n y_n\|\right) \\
&\le \left[\alpha_n\beta_n s_n^2 t_n^2 + \alpha_n t_n^2(1-\beta_n) + (1-\alpha_n)\right]\|x_n - p\|^2 \\
&\quad - \alpha_n s_n^2 \beta_n(1-\beta_n) g\left(\|x_n - T^n x_n\|\right) - \alpha_n(1-\alpha_n) g\left(\|x_n - S^n y_n\|\right) \\
&\le \left[\alpha_n\beta_n k_n^4 + \alpha_n k_n^2(1-\beta_n) + (1-\alpha_n)\right]\|x_n - p\|^2 \\
&\quad - \alpha_n k_n^2 \beta_n(1-\beta_n) g\left(\|x_n - T^n x_n\|\right) - \alpha_n(1-\alpha_n) g\left(\|x_n - S^n y_n\|\right) \\
&\le \left[\alpha_n\beta_n k_n^4 + \alpha_n k_n^4(1-\beta_n) + (1-\alpha_n)k_n^4\right]\|x_n - p\|^2 \\
&\quad - \alpha_n k_n^2 \beta_n(1-\beta_n) g\left(\|x_n - T^n x_n\|\right) - \alpha_n(1-\alpha_n) g\left(\|x_n - S^n y_n\|\right) \\
&\le \|x_n - p\|^2 + r(k_n^4 - 1) - \delta^3 g\left(\|x_n - T^n x_n\|\right) - \delta^2 g\left(\|x_n - S^n y_n\|\right).
\end{aligned}$$

That is,

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2 + r(k_n^4-1) - \delta^3 g\left(\|x_n - T^n x_n\|\right) - \delta^2 g\left(\|x_n - S^n y_n\|\right). \quad (2.3)$$

From (2.3), the following two important inequalities are obtained:

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2 + r(k_n^4-1) - \delta^2 g\left(\|x_n - S^n y_n\|\right) \quad (2.4)$$

and

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2 + r(k_n^4-1) - \delta^3 g\left(\|x_n - T^n x_n\|\right). \quad (2.5)$$

Now, we prove that

$$\lim_{n\to\infty} \|x_n - S^n y_n\| = 0 = \lim_{n\to\infty} \|x_n - T^n x_n\|.$$

First assume $\limsup_{n\to\infty} \|x_n - S^n y_n\| > 0$. Then there exists a subsequence (use the same notation for subsequence as for the sequence) of$\{x_n\}$ and $\mu > 0$ such that $\|x_n - S^n y_n\| \geq \mu > 0$. By definition of g, we have $g\left(\|x_n - S^n y_n\|\right) \geq g(\mu) > 0$.

From (2.4), it follows that

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2 + r(k_n^4-1) - \delta^2 g(\mu) \quad (2.6) \\ &= \|x_n-p\|^2 + r\left[(k_n^4-1) - \frac{\delta^2}{2r} g(\mu)\right] - \frac{\delta^2}{2} g(\mu). \end{aligned}$$

Since $k_n^4 \to 1$ and $\frac{\delta^2}{2r} g(\mu) > 0$, so there exists $n_0 \geq 1$ such that $(k_n^4 - 1) < \frac{\delta^2}{2r} g(\mu)$ for all $n \geq n_0$. Hence (2.6) reduces to

$$\frac{\delta^2}{2} g(\mu) \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2. \quad (2.7)$$

Let $m \geq n_0$ be any positive integer. Then from (2.7), we have

$$\sum_{n=n_0}^{m} \frac{\delta^2}{2} g(\mu) \leq \|x_{n_0}-p\|^2 - \|x_{m+1}-p\|^2 \leq \|x_{n_0}-p\|^2. \quad (2.8)$$

When $m \to \infty$ in (2.8), we get

$$\infty = \|x_{n_0}-p\|^2 < \infty$$

which contradicts the reality. This proves that $\mu = 0$.

Hence

$$\limsup_{n\to\infty} \|x_n - S^n y_n\| \leq 0.$$

Consequently, we have

$$\lim_{n\to\infty} \|x_n - S^n y_n\| = 0. \quad (2.9)$$

Following the similar procedure of proof with (2.5), we conclude

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \quad (2.10)$$

Now (2.10) assures that

$$\|x_n - y_n\| = \beta_n \|x_n - T^n x_n\| \to 0 \text{ as } n \to \infty. \tag{2.11}$$

Since

$$\|x_n - S^n x_n\| \le \|x_n - S^n y_n\| + \|S^n x_n - S^n y_n\|,$$

therefore with the help of (2.9), (2.10) and the uniform equi-continuity of S, we get

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0. \tag{2.12}$$

Finally Lemma 2.1 appeals that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|. \tag{2.13}$$

Now we prove our weak theorem without making use of any of the Opial condition, Kadec-Klee property or Fréchet differentiable norm.

Theorem 2.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to C$ be asymptotically nonexpansive maps with sequences $\{s_n\}, \{t_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} s_n = 1, \lim_{n\to\infty} t_n = 1$, respectively. Let the sequence $\{x_n\}$ be as in (1.1) with $\delta \le \alpha_n, \beta_n \le 1 - \delta$ for some $\delta \in (0, \frac{1}{2})$. If $F(S) \cap F(T) \ne \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $\omega_w(x_n)$ be the weak ω-limit set of $\{x_n\}$ given by:

$$\omega_w(x_n) = \{y \in E : x_{n_k} \rightharpoonup y \text{ for } \{x_{n_k}\} \subseteq \{x_n\}\}.$$

Since C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p \in \omega_w(x_n)$ as $i \to \infty$ and vice versa. This shows that $\omega_w(x_n) \ne \phi$ and by Lemma 2.1, $\lim_{i\to\infty} \|x_{n_i} - Sx_{n_i}\| = 0 = \lim_{i\to\infty} \|x_{n_i} - Tx_{n_i}\|$. Since $I - S$ and $I - T$ are demiclosed at 0 (by Lemma 1.3), therefore $Sp = p = Tp$. That is, $\omega_w(x_n) \subset F(S) \cap F(T)$. Next, we follow the idea of Chang et al [1]. For any $p \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$x_{n_j} \rightharpoonup p \text{ as } j \to \infty. \tag{2.14}$$

Hence from (2.10) and (2.14), it follows that

$$T^{n_i} x_{n_i} = (T^{n_i} x_{n_i} - x_{n_i}) + x_{n_i} \rightharpoonup p. \tag{2.15}$$

Now from (1.1), (2.14) and (2.15), we get that

$$y_{n_i} = (1 - \beta_{n_i}) x_{n_i} + \beta_{n_i} T^{n_i} x_{n_i} \rightharpoonup p. \tag{2.16}$$

Also from (2.9) and (2.14), we have that

$$S^{n_i} y_{n_i} = (S^{n_i} y_{n_i} - x_{n_i}) + x_{n_i} \rightharpoonup p. \tag{2.17}$$

Again from (1.1), (2.14) and (2.17), we conclude that

$$x_{n_i+1} = (1-\alpha_{n_i})x_{n_i} + \alpha_{n_i} S^{n_i} y_{n_i} \rightharpoonup p.$$

Continuing in this way, by induction, we can prove that, for any $m \geq 0$,

$$x_{n_i+m} \rightharpoonup p.$$

By induction, one can prove that $\bigcup_{m=0}^{\infty}\{x_{n_j+m}\}$ converges weakly to p as $j \to \infty$; in fact $\{x_n\}_{n=n_1}^{\infty} = \bigcup_{m=0}^{\infty}\{x_{n_j+m}\}_{j=1}^{\infty}$ gives that $x_n \rightharpoonup p$ as $n \to \infty$. □

Our strong convergence theorem is as follows. We do not use the rate of convergence condition viz $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ in its proof.

Theorem 2.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to C$ be uniformly continuous and asymptotically quasi-nonexpansive maps with sequences $\{s_n\}, \{t_n\} \subset [1, \infty)$ respectively such that $\lim_{n\to\infty} s_n = 1, \lim_{n\to\infty} t_n = 1$. Let the sequence $\{x_n\}$ be as in (1.1) with $\delta \leq \alpha_n, \beta_n \leq 1-\delta$ for some $\delta \in (0, \frac{1}{2})$. If $F(S) \cap F(T) \neq \phi$ and either S or T is semi-compact (completely continuous), then $\{x_n\}$ converges strongly to the same common fixed point of S and T.*

Proof. Let S be semi-compact. As $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \to p.$$

Using $x_n = x_{n_i}$ in (2.13) and the uniform continuity of S and T, we obtain that $p \in F(S) \cap F(T)$. The rest of the proof follows by replacing $\rightharpoonup$ with $\to$ in Theorem 2.1 and we, therefore, omit the details. □

As an asymptotically nonexpansive map with bounded domain is uniformly continuous as well as asymptotically quasi-nonexpansive, the following strong convergence theorem is the immediate consequence of Theorem 2.2.

Theorem 2.3. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to C$ be asymptotically nonexpansive maps with sequences $\{s_n\}, \{t_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} s_n = 1, \lim_{n\to\infty} t_n = 1$, respectively. Let the sequence $\{x_n\}$ be as in (1.1) with $\delta \leq \alpha_n, \beta_n \leq 1-\delta$ for some $\delta \in (0, \frac{1}{2})$. If $F(S) \cap F(T) \neq \phi$ and either S or T is semi-compact (completely continuous), then $\{x_n\}$ converges strongly to the common fixed point of S and T.*

Remark 2.2. Our theorems improve the theorems obtained by Khan and Takahashi [8], Tan and Xu [16], Rhoades [14] and Qihou [11] by dropping any of the conditions: (i) Opial condition or Fréchet differentiable norm (in case of weak convergence) and (ii) the rate of convergence $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ associated with the map T (both for weak and strong convergences). We can also extend

Theorems 2.1-2.3 for Ishikawa iteration process with errors (Liu [9] or Xu [17]) and thus improving the results appeared in [3], [4], [6], [12], [13].

3. Nonself Mappings Case

A nonself asymptotically (quasi-)nonexpansive map is introduced as follows (see [2]):

Let E be a real Banach space. A subset C of E is said to be a retract of E if there exists a continuous map $P : E \to C$ such that $Px = x$ for all $x \in C$. A map $P : E \to E$ is a retraction if $P^2 = P$. Let $P : E \to C$ be the nonexpansive retraction of E onto C. A map $T : C \to E$ is asymptotically (quasi-)nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq k_n \|x - y\|$$

for all $x, y \in C$ ($x \in C, y \in F(T) \neq \phi$) and for all $n \geq 1$.

A nonself version of (1.1) is as under:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P(\alpha_n S(PS)^{n-1} y_n + (1-\alpha_n)x_n), \\ y_n = P(\beta_n T(PT)^{n-1} y_n + (1-\beta_n)x_n), \ n \geq 1, \end{cases} \tag{3.1}$$

where $S, T : C \to E$ are asymptotically nonexpansive maps, P is the nonexpansive retraction as defined above and $0 \leq \alpha_n, \beta_n \leq 1$.

We can prove the results of previous section for nonself asymptotically (quasi-)nonexpansive maps. Following are their statements for nonself maps.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to E$ be asymptotically nonexpansive maps with sequences $\{s_n\}, \{t_n\} \subset [1, \infty)$ respectively such that $\lim_{n\to\infty} s_n = 1, \lim_{n\to\infty} t_n = 1$ and $F(S) \cap F(T) \neq \phi$. Let the sequence $\{x_n\}$ be as in (3.1) with $\delta \leq \alpha_n, \beta_n \leq 1-\delta$ for some $\delta \in (0, \frac{1}{2})$. Then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Theorem 3.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to E$ be uniformly continuous and asymptotically quasi-nonexpansive maps with sequences $\{s_n\}, \{t_n\} \subset [1, \infty)$ respectively such that $\lim_{n\to\infty} s_n = 1, \lim_{n\to\infty} t_n = 1$ and $F(S) \cap F(T) \neq \phi$. Let the sequence $\{x_n\}$ be as in (3.1) with $\delta \leq \alpha_n, \beta_n \leq 1-\delta$ for some $\delta \in (0, \frac{1}{2})$. If either S or T is semi-compact, then $\{x_n\}$ converges strongly to* *the common fixed point of* S *and* T.

Remark 3.1. Above theorem extends Theorems 3.3-3.5 due to Wang [18]. Further, if we take $T = I$ in Theorem 3.1, it extends Theorem 3.7, Theorem 3.10 and Corollary 3.11 of Chidume et al [2].

At the end, we want to draw the attention of the reader towards the following open problem.

Open Problem. Is it possible to prove Lemma 1.3 (see [1], Theorem 1), for asymptotically quasi-nonexpansive mappings so that our Theorem 2.1 remains valid for such mappings?

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