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EXISTENCE AND CONSTRUCTION OF FINITE FRAMES WITH A GIVEN FRAME OPERATOR

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Abstract: Let S be a positive self-adjoint invertible operator on an N-dimensional Hilbert space H_N and let $M \geq N$. We give necessary and sufficient conditions on real sequences $a_1 \geq a_2 \geq \cdots \geq a_M \geq 0$ so that there is a frame $\{\varphi_n\}_{n=1}^M$ for H_N with frame operator S and $\|\varphi_n\| = a_n$, for all $n = 1, 2, \ldots M$. As a consequence, given any *frame operator* S as above, there is a set of equal norm vectors in H_N which have precisely S as their frame operator.

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1. Introduction

A sequence $\{\varphi_n\}_{n=1}^M$ is a *frame* for an N-dimesional Hilbert space H_N if the positive self-adjoint *frame operator*

$$S = \sum_{m=1}^{M} \langle \varphi, \varphi_m \rangle \varphi_m$$

is an invertible operator on H_N. A frame $\{\varphi_n\}_{n=1}^{n=M}$ is a λ-*tight frame* if $S = \lambda I$ and if $\lambda = 1$, it is a *Parseval frame.* If the frame vectors all have the same norm, this is an *equal-norm frame.* The *analysis operator* of the frame is $T : H_N \to \ell_2(M)$ given by: $T(\varphi) = \{\langle \varphi, \varphi_m \rangle\}_{m=1}^M$. The *synthesis operator* is T^* where $T^*(\{a_m\}_{m=1}^M) = \sum_{m=1}^M a_m \varphi_m$. So the frame operator is $S = T^*T$. The

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Gram matrix is the matrix of the operator TT^*. For an introduction to frame theory, see [1, 5, 6]. In [2] there is given necessary and sufficient conditions on real sequences $a_1 \geq a_2 \geq \cdots \geq a_M > 0$ so that there exists a tight frame $\{\varphi_m\}_{m=1}^M$ for H_N with $\|\varphi_m\| = a_m$, for all $n = 1, 2, \ldots M$. The condition for the existence of a λ-tight frame given in [2] is that

$$\lambda = \sum_{m=1}^{M} a_m^2 \geq a_1^2.$$

One interpretation of this result is that it gives necessary and sufficient conditions on $\|\varphi_m\|$ for $\{\varphi_m\}_{m=1}^M$ to form a frame for H_N with frame operator $S = \lambda I$. An alternative constructive proof of this result appears in [4] where an algorithm is given for this construction.

In this paper we generalize these results to the case where λI is replaced by any positive self-adjoint invertible operator S on H_N. That is, for a given S and $M \geq N$, we give necessary and sufficient conditions on $a_1 \geq a_2 \geq \cdots \geq a_M > 0$ so that there is a frame $\{\varphi_m\}_{m=1}^M$ for H_N with frame operator S and satisfying: $\|\varphi_m\| = a_m$, for all $m = 1, 2, \ldots M$. We will then see that every *frame operator* S can be realized as the frame operator of an equal norm frame with M-elements, for any $M \geq N$.

2. Main Result

The main result in this paper is:

Theorem 2.1. *Let S be a positive self-adjoint operator on a N-dimensional Hilbert space H_N. Let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N > 0$ be the eigenvalues of S. Fix $M \geq N$ and real numbers $a_1 \geq a_2 \geq \cdots \geq a_M > 0$. The following are equivalent:*

(1) There is a frame $\{\varphi_j\}_{j=1}^M$ for H_N with frame operator S and $\|\varphi_j\| = a_j$, for all $j = 1, 2, \ldots, M$.

(2) For every $1 \leq k \leq N$,

$$\sum_{i=1}^{k} a_i^2 \leq \sum_{i=1}^{k} \lambda_i, \quad \text{and} \quad \sum_{i=1}^{M} a_i^2 = \sum_{i=1}^{N} \lambda_i. \tag{2.1}$$

Now we proceed to prove Theorem 2.1. To show that (1) implies (2) in the theorem we will actually prove a more general result.

Theorem 2.2. *Let* $\{\varphi_j\}_{j=1}^{j=M}$ *be a frame for* H_N *with frame operator* S *having eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_N$. *If* P *is an orthogonal projection of* H_N *onto a* k*-dimensional subspace of* H_N, $1 \leq k \leq N$, *then*

$$\sum_{j=N-k+1}^{N} \lambda_j \leq \sum_{j=1}^{M} \|P\varphi_j\|^2 \leq \sum_{j=1}^{k} \lambda_j.$$

Proof. Let $\{e_n\}_{n=1}^{n=N}$ be an orthonormal basis for H_N with $Se_n = \lambda_n e_n$, $n = 1, 2, \ldots, N$. Let P be a rank k orthogonal projection on H_N and let $\{\psi_i\}_{i=1}^{k}$ be an orthonormal basis for PH_N. It is known (see e.g. [3, 5]), that:

(1) $\sum_{n=1}^{M} |\langle \varphi_n, e_m \rangle|^2 = \lambda_m$, for all $1 \leq m \leq N$.

(2) $\sum_{n=1}^{M} \langle \varphi_n, e_l \rangle \overline{\langle \varphi_n, e_m \rangle} = 0$ for all $1 \leq l \neq m \leq N$.

Now we compute

$$\sum_{n=1}^{M} \|P\varphi_n\|^2 = \sum_{n=1}^{M}\sum_{i=1}^{k} |\langle \psi_i, P\varphi_n \rangle|^2 = \sum_{n=1}^{M}\sum_{i=1}^{k} |\langle \psi_i, \varphi_n \rangle|^2$$

$$= \sum_{n=1}^{M}\sum_{i=1}^{k} \left| \sum_{m=1}^{N} \langle \psi_i, e_m \rangle \overline{\langle \varphi_n, e_m \rangle} \right|^2$$

$$= \sum_{n=1}^{M}\sum_{i=1}^{k}\sum_{m=1}^{N}\sum_{l=1}^{N} \langle \psi_i, e_m \rangle \overline{\langle \varphi_n, e_m \rangle \langle \psi_i, e_l \rangle} \langle \varphi_n, e_l \rangle$$

$$= \sum_{m=1}^{N}\sum_{n=1}^{M}\sum_{i=1}^{k} |\langle \psi_i, e_m \rangle|^2 |\langle \varphi_n, e_m \rangle|^2 + \sum_{i=1}^{k}\sum_{m \neq l} \langle \psi_i, e_m \rangle \overline{\langle \psi_i, e_l \rangle} \sum_{n=1}^{N} \langle \varphi_n, e_l \rangle \overline{\langle \varphi_n, e_m \rangle}$$

$$= \sum_{m=1}^{N}\sum_{i=1}^{k} |\langle \psi_i, e_m \rangle|^2 \sum_{n=1}^{M} |\langle \varphi_n, e_m \rangle|^2 = \sum_{m=1}^{N}\sum_{i=1}^{k} |\langle \psi_i, e_m \rangle|^2 \lambda_m.$$

Since $\{\psi_i\}_{i=1}^{k}$ is an orthonormal basis for its span, we have that

$$\sum_{i=1}^{k} |\langle \psi_i, e_m \rangle|^2 \leq 1, \quad \text{for all} \quad 1 \leq i \leq k, \quad 1 \leq m \leq N$$

and

$$\sum_{m=1}^{N}\sum_{i=1}^{k} |\langle \psi_i, e_m \rangle|^2 = \sum_{i=1}^{k}\sum_{m=1}^{N} |\langle \psi_i, e_m \rangle|^2 = \sum_{i=1}^{k} \|\psi_i\|^2 = k.$$

Hence (see Lemma 4.1),

$$\sum_{m=1}^{k}\lambda_m \geq \sum_{m=1}^{N}\left(\sum_{i=1}^{k}|\langle\psi_i, e_m\rangle|^2\right)\lambda_m = \sum_{m=1}^{N}\|P\varphi_m\|^2 \geq \sum_{m=N-k}^{N}\lambda_m. \qquad \square$$

We now give two corollaries. The first is the implication (1) $\Rightarrow$ (2) of Theorem 2.1.

Corollary 2.3. *Let $\{\varphi_j\}_{j=1}^M$ be a frame for H_N with frame operator S having eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \lambda_N > 0$. If $\|\varphi_1\| \geq \|\varphi_2\| \geq \cdots \geq \|\varphi_M\|$, then for every $1 \leq k \leq N$,*

$$\sum_{j=1}^{k}\|\varphi_j\|^2 \leq \sum_{j=1}^{k}\lambda_j.$$

Proof. Given k, let P be an orthogonal projection of rank $\leq k$ on H_N so that $\varphi_j \in PH_N$, for all $1 \leq j \leq k$. By Theorem 2.2 we have:

$$\sum_{j=1}^{k}\|\varphi_j\|^2 = \sum_{j=1}^{k}\|P\varphi_j\|^2 \leq \sum_{j=1}^{M}\|P\varphi_j\|^2 \leq \sum_{j=1}^{k}\lambda_j. \qquad \square$$

Corollary 2.4. *Let S be a positive self-adjoint operator on H_N with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_N > 0$. If P is a rank k orthogonal projection on H_N then*

$$Tr(PSP) \leq \sum_{j=1}^{k}\lambda_j.$$

Proof. If $\{e_j\}_{j=1}^N$ is an orthonormal sequence in H_N with $Se_j = \lambda_j e_j$, then $\{\varphi_j = \sqrt{\lambda_j}e_j\}_{j=1}^N$ is a frame for H_N with frame operator S. Hence, PSP is the frame operator for $\{P\varphi_j\}_{j=1}^N$. Applying Theorem 2.2, for every $1 \leq k \leq N$, we have

$$Tr(PSP) = \sum_{j=1}^{N}\|P\varphi_j\|^2 \leq \sum_{j=1}^{k}\lambda_j. \qquad \square$$

3. Equal Norm Frames

We will start with a frame $\{\varphi_j\}_{j=1}^M$ with frame operator S. The vectors used in Corollary 2.4 can be extended to a frame on H_N with frame operator S. More generally, since S is symmetric,

$$S = V\Lambda V^*,$$

where V is a unitary matrix and Λ is a diagonal matrix with $\mathrm{diag}\,(\Lambda) = \lambda_1, \lambda_2, \ldots, \lambda_N$. Let $\Delta_{M\times N}$ be such that its top N rows equal $\Lambda^{\frac{1}{2}}$ and all remaining entries are zero. Let $W_{M\times M}$ be unitary, and let $\varphi_j = j$-th row of $F = W\Delta V^*$. Then

$$F^*F = (W\Delta V^*)^* W\Delta V^* = V\Lambda V^* = S.$$

The Gram operator is given by $G = FF^*$, and

$$\mathrm{diag}\,(G) = (\|\varphi_1\|^2, \ldots, \|\varphi_M\|^2).$$

Then (see e.g. [7]) there is an orthogonal matrix $U_{M\times M}$ and an diagonal matrix Λ such that

$$G = U^*\Lambda U, \quad \text{where} \quad \mathrm{diag}\,(G) = (\lambda_1, \ldots, \lambda_N, 0, \ldots, 0).$$

Let $V_{M\times M}$ be (see Lemma 4.2) an orthogonal matrix such that if

$$T = V\Lambda V^*, \quad \text{then}, \quad \mathrm{diag}\,(T) = (a_1^2, \ldots, a_M^2).$$

Let $\psi_j = j^{th}$ row of $H = VUF$. Then $\{\psi_j\}_{j=1}^M$ is a frame, since rank$(H) =$ rank$(F) = N$. Its frame operator is given by

$$H^*H = (VUF)^*VUF = F^*F = S,$$

and the diagonal of its Gram matrix is

$$\mathrm{diag}\,(VUF(VUF)^*) = \mathrm{diag}\,(VUFF^*U^*V^*) = \mathrm{diag}\,(V\Lambda V^*) = (a_1^2, \ldots, a_M^2).$$

Now we check that the requirements of Theorem 2.1 are always met for equal norm frames.

Corollary 3.1. *Let S be a positive self-adjoint operator on a N-dimensional Hilbert space H_N. For any $M \geq N$ there is an equal norm sequence $\{\varphi_m\}_{m=1}^M$ in H_N which has S as its frame operator.*

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N \geq 0$ be the eigenvalues of S. Let

$$a^2 = \frac{1}{M}\sum_{i=1}^N \lambda_i. \tag{3.1}$$

Now we check the conditions of Theorem 2.1 (2) to see that there is a sequence $\{\varphi_m\}_{m=1}^M$ in H_N with $\|f_m\| = a$ for all $m = 1, 2, \ldots, M$ and having frame operator S. We are letting $a_1 = a_2 = \cdots a_M = a$. For the second equality in Theorem 2.1, by equation 3.1,

$$\sum_{m=1}^M \|\varphi_m\|^2 = \sum_{m=1}^M a_m^2 = Ma^2 = \sum_{i=1}^N \lambda_i. \tag{3.2}$$

For the first inequality in Theorem 2.1, we note that by equation (3.1) we have that

$$a_1^2 = a^2 = \frac{1}{M}\sum_{i=1}^{N}\lambda_i \le \frac{1}{N}\sum_{i=1}^{N}\lambda_i \le \lambda_1.$$

So our inequality holds for $m = 1$. Suppose there is an $1 < m \le N$ for which this inequality fails, and m is the first time this fails, and we will come to a contradiction. So,

$$\sum_{i=1}^{m-1} a_i^2 = (m-1)a^2 \le \sum_{i=1}^{m-1}\lambda_i,$$

while

$$\sum_{i=1}^{m} a_i^2 = ma^2 > \sum_{i=1}^{m}\lambda_i.$$

It follows that

$$a_m^2 = a^2 > \lambda_m \ge \lambda_{m+1} \ge \lambda_N.$$

Hence,

$$\begin{aligned} Ma^2 = \sum_{m=1}^{M} a_m^2 \ge \sum_{i=1}^{m} a_i^2 + \sum_{i=m+1}^{N} a_i^2 > \sum_{i=1}^{m}\lambda_i + \sum_{i=m+1}^{N} a_i^2 \\ \ge \sum_{i=1}^{m}\lambda_i + \sum_{i=m+1}^{N}\lambda_i = \sum_{i=1}^{N}\lambda_i. \end{aligned}$$

But this contradicts equation (3.2). □

4. Proof of the Main Theorem

Now we complete the proof of the main result by showing that (2) implies (1).

Lemma 4.1. *Assume we have two sets of numbers* $\{c_m\}_{m=1}^N$ *and* $\{\lambda_m\}_{m=1}^N$ *satisfying:*

1. $\lambda_1 \ge \lambda_2 \cdots \ge \lambda_N$.
2. *We have* $0 \le c_m \le 1$ *and* $\sum_{m=1}^N c_m = k$.

Then

$$\sum_{m=1}^{k}\lambda_m \ge \sum_{m=1}^{N} c_m\lambda_m \ge \sum_{m=N-k}^{N}\lambda_m.$$

Proof. We will check the first inequality. The second will follow similarly. Choose non-negagtive numbers $\{b_m\}_{m=1}^k$ so that $c_m + b_m = 1$, for all $m = 1, 2, \ldots, k$. We first observe that

$$\sum_{m=1}^{k} c_m + \sum_{m=k+1}^{N} c_m = \sum_{m=1}^{N} c_m = k = \sum_{m=1}^{k} (c_m + b_m) = \sum_{m=1}^{k} c_m + \sum_{m=1}^{k} b_m.$$

That is,

$$\sum_{m=1}^{k} b_m = \sum_{m=k+1}^{N} c_m.$$

Since $\{\lambda_m\}_{m=1}^N \downarrow$, we have

$$\sum_{m=1}^{N} c_m\lambda_m \le \sum_{m=1}^{k} c_m\lambda_m + \left(\sum_{m=k+1}^{N} c_m\right)\lambda_{k+1} = \sum_{m=1}^{k} c_m\lambda_m + \left(\sum_{m=1}^{k} b_m\right)\lambda_{k+1}$$
$$= \sum_{m=1}^{k} (c_m\lambda_m + b_m\lambda_{k+1}) \le \sum_{m=1}^{k} (c_m\lambda_m + b_m\lambda_m) = \sum_{m=1}^{k} (c_m + b_m)\lambda_m$$
$$= \sum_{m=1}^{k} \lambda_m. \quad \square$$

Every matrix in $\boldsymbol{O}(M)$ (the orthogonal group) is obtained as a product of Givens rotations $\theta(t, j, k) \in \boldsymbol{O}(M), j < k$, where

$$\theta(t,j,k) = \begin{pmatrix} I_{j-1,j-1} & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & \sin(t) & 0 \\ 0 & 0 & I_{M-j-k-2,M-j-k-2} & 0 & 0 \\ 0 & -\sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & I_{k-1,k-1} \end{pmatrix}.$$

It is clear that

$$\theta(t, j, k)^{-1} = \theta(-t, j, k).$$

Lemma 4.2. *Let $\lambda_1, \ldots, \lambda_M$ and $a_1, \ldots, a_M$ be real numbers such that $a_1^2 \ge a_2^2 \ge \cdots \ge a_M^2$ and for every $1 \le k \le M$,*

$$\sum_{i=1}^{k} a_i^2 \le \sum_{i=1}^{k} \lambda_i, \quad \text{and} \quad \sum_{i=1}^{M} a_i^2 = \sum_{i=1}^{M} \lambda_i. \tag{4.1}$$

Let Λ be a diagonal matrix with $\operatorname{diag}(\Lambda) = (\lambda_1, \ldots, \lambda_M)$. Then there is a matrix $O \in \boldsymbol{O}(M)$ such that

$$\operatorname{diag}(O\Lambda O^*) = (a_1^2, \ldots, a_M^2).$$

Proof. We will prove the lemma by induction on M. If $M = 2$, let $t = \arcsin(\sqrt{\lambda_2 - a_2^2/\lambda_2 - a_1^2}$ and $O = \theta(t, 1, 2, 2)$, and we are done. Now, assume the result holds for $M - 1$. From the hypothetsis, $\lambda_1 \geq a_1^2$, let k be such that $\lambda_j \geq a_1^2$ for $j = 1, \dots, k-1$ and $a_1^2 \geq \lambda_k$. Let

$$t = \arcsin(\sqrt{\lambda_1 - a_1^2/\lambda_1 - \lambda_k} \quad \text{and} \quad O_1 = \theta(t, 1, k, M).$$

Then

$$O_1 \Lambda O_1^* = \begin{pmatrix} a_1^2 & * \; 0 \dots 0 \\ * & \dots \\ 0 & \dots \\ \vdots & \dots \\ 0 & \dots \end{pmatrix}.$$

Let Λ_1 be the $(M-1) \times (M-1)$ bottom right box of $O_1 \Lambda O_1^*$. Then, Λ_1 is a diagonal matrix and, since $Tr(\Lambda) = Tr(O_1 \Lambda O_1^*)$,

$$\operatorname{diag}(\Lambda_1) = (\lambda_2, \dots, \lambda_{k-1}, \lambda_k + \lambda_1 - a_1^2, \lambda_{k+1}, \dots, \lambda_M).$$

Now we will verify that Λ_1 and $a_2, \dots, a_M$ satisfy the hypotheses of the lemma.

If $m < k$,

$$\lambda_2 + \lambda_3 + \dots + \lambda_m \geq (m-1) * a_1^2 \geq (m-1) * a_2^2 \geq a_2^2 + \dots + a_m^2.$$

If $m \geq k$,

$$\begin{aligned} \lambda_2 + \lambda_3 + \dots + \lambda_m &= \lambda_2 + \lambda_3 + \dots + \lambda_{k-1} + \lambda_k + \lambda_1 - a_1^2 + \lambda_{k+1} + \dots + \lambda_m \\ &= \lambda_1 + \lambda_2 + \dots + \lambda_m - a_1^2 \geq a_2^2 + \dots + a_m^2, \end{aligned}$$

thus giving

$$\lambda_1 + \lambda_2 + \dots + \lambda_m \geq a_1^2 + a_2^2 + \dots + a_m^2.$$

Then, by our induction hypothesis, there is an $O_2 \in \boldsymbol{O}(M-1)$ such that

$$\operatorname{diag}(O_2 \Lambda_2 O_2^*) = (a_2^2, \dots, a_M^2).$$

Then

$$O = \begin{pmatrix} 1 & 0 \\ 0 & O_2 \end{pmatrix} O_1$$

will satisfy the claim $\square$

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