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A MATHEMATICAL MODEL ON DETRITUS IN MANGROVE ESTUARINE ECOSYSTEM

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Abstract: In this paper a mathematical model on litter, detritus and predators in mangrove estuarine ecosystem is formulated by a system of non-linear differential equations. The system of non-linear differential equations is solved by extending the Adomian's decomposition method and conditions for the stability of various equilibria are established and studied.

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1. Introduction

Mathematical equations or models that describe physical or biological phenomena are in most cases non linear differential equations of first order. Living organisms and their non-living environment are inseparable, interrelated and interacts up on each other. It is a well recognized fact that the ecosystem is in

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fact the basic functional unit of ecology. Since it includes both organisms and a biotic environment, each influences the properties of the other. Mathematical symbols provide a useful short hand for describing complex ecological systems and equations permit formal statement of ecosystem components. The process of translating physical or biological concepts about any system in to a set of mathematical relationships and the manipulation of the mathematical system thus derived is called a system analysis. The mathematical system thus derived is called a model and is almost a perfect and an abstract representation of the real problem. It attracted the attention of mathematicians (see [4], [5], [6] and [9]) and ecologists. So it is of great interest to formulate a mathematical model on detritus in mangrove estuarine ecosystems and find their approximate analytical solutions and there by determine the stability region of coexistence of the ecosystem. This paper is organized as follows. In Section 2, we formulate the non-linear mathematical model of the mangrove estuarine ecosystem. Section 3 outlines the decomposition method developed by G. Adomian [1] for matrix first order differential equations as applied to the mathematical model formulated in Section 2. Approximate analytical solutions of the Mangrove estuarine ecosystem is presented in Section 4 by finding interior equilibrium point using the concept of generalized inverse [8]. Section 5 presents the stability analysis of the region of coexistence using different equilibria. Section 6 presents the summary and conclusions.

2. Formulation of the Mathematical Model

It is a well recognized fact that the enormous production of mangrove detritus provides a necessary energy to drive the biological machinery of mangrove estuarine ecosystem besides supporting to certain extent the coastal population. In the process of litter decomposition, if the food material is not ingested by other animals, the decomposition process of litter completes and this liberates nutrients into water and soil. Part of the detritus produced in mangrove estuarine ecosystem may be carried to the adjacent coastal waters through tidal fluxes. In fact here detritus serves a couple of purposes: (i) Regenerates nutrients by undergoing complete decomposition process and (ii) Serves as a food source to estuarine and coastal organisms like polychaeta, crabs, prawn, fish and zooplankton. Here after we call all these consumers as predators. Now we are in a position to formulate the mathematical model. If $L(t)$, $D(t)$ and $W(t)$ denote the amount of litter, detritus and predators in the sea at time t in the mangrove area respectively then the following assumptions are reasonable.

(1) If there is no litter from the mangroves the detritus formation decreases in a way proportional to litter and if there is no detritus formation, the amount of litter increases exponentially within a limited time.

(2) If there are no bacteria, fungi and protozoa, the formation of detritus decreases and the amount of litter increases in a limited time, and

(3) In the absence of consumers like lower Carnivores and higher Carnivores, fish etc. The amount of detritus increases and in the absence of detritus in the sea, the growth rate of above predators decreases.

These assumptions give rise to the following system of non-linear first order differential equations,

$$\dot{L} = \alpha_1 L - \beta_1 LD, \dot{D} = \alpha_2 D + \beta_2 LD - \gamma_2 DW, \dot{W} = -\alpha_3 W + \gamma_3 DW, \quad (1)$$

where $\alpha_i (i = 1, 2, 3), \beta_j (j = 1, 2)$ and $\gamma_k \, (k = 2, 3)$ are all positive constants. The presence of both litter and detritus is beneficial to the growth of predators like fish and carnivores in the sea. More specifically the predator species increases and the detritus decreases at rates proportional to the product of the two. If they also satisfy the condition,

$$\alpha_1 \gamma_3 - \alpha_3 \beta_1 = 0. \quad (2)$$

It amounts to saying that the growth of predators is directly proportional to the formation of the detritus.

3. Decomposition Method for Matrix Differential System

The decomposition method was developed by G. Adomian [1] to linear and non-linear differential and partial differential equations. The decomposition method is also an approximation, but it does not change the original problem. With the assumptions like weak non-linearity and small perturbation, the solution of the simpler mathematical problem may not be a good approximation to the solution of the original problem. The advantage of the decomposition method is that it provides analytical approximation to a wide class of non-linear equations without linearization as in perturbation, closure approximation or discritization methods. In this section, we describe the Adomain's decomposition method for non-linear matrix differential equation. Consider a matrix differential equation $FX(t) = G(t)$, where F represents a general non-linear ordinary differential operator involving both linear and non-linear terms, $X(t)$ and $G(t)$ are square matrices. Decompose the nonlinear operator into $M + N + R$, where M is easily invertible operator and R is the remainder of the linear operator and N

represents non-linear operator. Thus the equation

$$FX(t) = G(t) \tag{3}$$

may be written as

$$MX(t) + RX(t) + NX(t) = G(t). \tag{4}$$

Solving (see 4) for MX,

$$\begin{aligned} MX &= G - RX - NX \\ M^{-1}MX &= M^{-1}G - M^{-1}RX - M^{-1}NX, \end{aligned} \tag{5}$$

since M is easily invertible operator. If this corresponds to an initial value problem, the integral operator M^{-1} can be a definite integral from t_0 to t. Solving (5) for X yields,

$$X = E_1 + E_2 t + M^{-1}G - M^{-1}RX - M^{-1}NX. \tag{6}$$

The non-linear term NX will be equated to $\sum_{n=0}^{\infty} A_n$, where the A_n's are special polynomials, which can be defined further by (9), and X will be decomposed into $\sum_{n=0}^{\infty} X_n$, with X_0 the initial approximate matrix identified as $E_1 + E_2 + M^{-1}G$. Thus (6) can be expressed as

$$\sum_{n=0}^{\infty} X_n = X_0 - M^{-1}R\sum_{n=0}^{\infty} X_n - M^{-1}\sum_{n=0}^{\infty} A_n. \tag{7}$$

Consequently we can write, the matrix approximations,

$$\begin{aligned} X_1 &= -\,M^{-1}RX_0 - M^{-1}A_0, \\ X_2 &= -\,M^{-1}RX_1 - M^{-1}A_1, \\ &\vdots \\ X_{n+1} &= -\,M^{-1}RX_n - M^{-1}A_n. \end{aligned} \tag{8}$$

The matrix polynomials A_n are generated for each non-linearity so that A_0 depends only on X_0, A_1 depends only on X_o and X_1, A_2 depends on X_0, X_1 and X_2 and so on. All of the matrix components X_n are calculable, and thus $X = \sum_{n=0}^{\infty} X_n$. If the series converges, then the n-th partial sum $\Phi_n = \sum_{i=0}^{n-1} X_i$ will be the solution since $\Phi_n \longrightarrow \sum_{i=0}^{n-1} X_i = X$. To calculate the matrix polynomials A_n's consider an equation for which $X(t)$ is the solution, containing a non-linear term $NX = Q(X) = \sum_{n=0}^{\infty} A_n$. These A_n matrix polynomials are defined as

$$\begin{aligned} A_0 &= Q(X_0), \\ A_1 &= (e^{X_1 d/dx_0} - 1)Q(X_0), \\ A_2 &= (e^{X_2 d/dx_0} - 1)Q(X_0) + (e^{X_1 d/dx_0} - 1)(e^{X_2 d/dx_0} - 1)Q(X_0). \end{aligned} \tag{9}$$

Thus $\sum_{i=0}^{n} A_i = Q(X_0) + (X - X_0)dQ/dX_0 + ...$

That is, the partial sums consist of the essential terms of a Taylor expansion about the function $X_0(t)$ rather than about a point. Thus, the sum of the first (n+1) terms of the A_0, A_1,...,A_n approaches $(e^{Xd/dx_0} - 1)Q(X_0)$. With the product terms, products of $n!$ occur in the denominator which also can be ignored after some n. Thus A_n can be written as

$$A_n = B_n Q(X_0), \quad \text{where} \quad B_n = C_n \sum_{j=0}^{n} B_j, \; n \geq 1, \quad B_0 = 1,$$
$$C_n = (e^{X_n d/dx_0} - 1). \tag{10}$$

4. Approximate Analytical Solution

The interior equilibrium point for the system (1) is calculated by pseudo inverse concept [8] and on solving the system of equations

$$\beta_1 D = \alpha_1, \quad -\beta_2 L + \gamma_2 W = \alpha_2, \quad \gamma_3 D = \alpha_3. \tag{11}$$

The possible equilibria are: (i) Trivial equilibrium point $L = 0$, $D = 0$ and $W = 0$; (ii) Equilibrium in the absence of predators $L_w = -\alpha_2/\beta_2$, $D_W = \alpha_1/\beta_1$; (iii) Equilibrium in the absence of detritus $L_D = 0$, $W_D = 0$; (iv) Equilibrium in the absence of litter $D_L = \alpha_3/\gamma_3$, $W_L = \alpha_2/\gamma_2$, and (v) The interior equilibrium

$$L = -(\alpha_2\beta_2)/(\beta_2^2 + \gamma_2^2), \quad D = (\alpha_1\beta_1 + \alpha_3\gamma_3)/(\beta_1^2 + \gamma_3^2),$$
$$W = (\alpha_2\gamma_2)/(\beta_2^2 + \gamma_2^2). \tag{12}$$

Equilibrium (ii) is the case when the entire litter is decomposed in to detritus which is usual prey predator equilibrium, whereas (iii) is the equilibrium with no detritus and equilibrium (iv) is the one with no predators. Neither all these three equilibria (ii), (iii) and (iv) nor the trivial equilibrium are of much interest to us. Define densities of the three species deviating from the interior equilibrium values given in (12) as

$$L_1 = L - a, \quad D_1 = D - b, \quad W_1 = W - c, \tag{13}$$

where $a = -(\alpha_2\beta_2)/(\beta_2^2+\gamma_2^2)$, $b = (\alpha_1\beta_1+\alpha_3\gamma_3)/(\beta_1^2+\gamma_3^2)$ and $c = (\alpha_2\gamma_2)/(\beta_2^2+\gamma_2^2)$. From equation (1) we get

$$\begin{aligned} \dot{L}_1 &= -a\beta_1 D_1 - \beta_1 L_1 D_1, \\ \dot{D}_1 &= b\beta_2 L_1 - b\gamma_2 W_1 + \beta_2 L_1 D_1 - \gamma_2 D_1 W_1, \\ \dot{W}_1 &= c\gamma_3 D_1 + \gamma_3 D_1 W_1. \end{aligned} \tag{14}$$

The above equations can be represented by the system

$$\dot{X} = E_1 X B + E_2 X C + E_1 X F_1 X + E_2 X F_2 X, \tag{15}$$

where $X = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & W_1 \end{bmatrix}$, $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,
$B = \begin{bmatrix} 0 & 0 & 0 \\ -a\beta_1 & 0 & 0 \\ 0 & -b\gamma_2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & b\beta_2 & 0 \\ 0 & 0 & c\gamma_3 \\ 0 & 0 & 0 \end{bmatrix}$, $F_1 = \begin{bmatrix} 0 & 0 & 0 \\ -\beta_1 & 0 & 0 \\ 0 & -\gamma_2 & 0 \end{bmatrix}$,
$F_2 = \begin{bmatrix} 0 & \beta_2 & 0 \\ 0 & 0 & \gamma_3 \\ 0 & 0 & 0 \end{bmatrix}$.

The initial conditions $L(0) = p_1$, $D(0) = p_2$, $W(0) = p_3$ now become $L_1(0) = p_1 - a$, $D_1(0) = p_2 - b$, $W_1(0) = p_3 - c$ and these can be written in the matrix notation as

$$\begin{bmatrix} L_1(0) & 0 & 0 \\ 0 & D_1(0) & 0 \\ 0 & 0 & W_1(0) \end{bmatrix} = \begin{bmatrix} p_1 - a & 0 & 0 \\ 0 & p_2 - b & 0 \\ 0 & 0 & p_3 - c \end{bmatrix}. \tag{16}$$

Now, applying Adomain's decomposition method to the system (15), we get $MX = E_1XB + E_2XC + E_1XF_1X + E_2XF_2X = RX + NX$, where $RX = E_1XB + E_2XC$, the linear term and $NX = E_1XF_1X + E_2XF_2X$ is the non-linear term of the system, and M denotes the differential operator. Therefore,

$$X = M^{-1}[RX] + M^{-1}[NX], \tag{17}$$

where $X = \sum_{n=0}^{\infty} X_n$ and $NX = \sum_{n=0}^{\infty} A_n$. Thus (17) will become

$$\sum_{n=0}^{\infty} X_n = M^{-1}\{E_1 \sum_{n=0}^{\infty} X_n B + E_2 \sum_{n=0}^{\infty} X_n C\} + M^{-1} \sum_{n=0}^{\infty} A_n.$$

Now the components of X_n are therefore easily identified as,

$$\begin{aligned} X_0 &= \text{diag}\left[\ p_1 - a, \quad p_2 - b, \quad p_3 - c,\ \right], \qquad (18) \\ X_1 &= M^{-1}[E_1X_0B + E_2X_0C] + M^{-1}A_0, \\ X_2 &= M^{-1}[E_1X_1B + E_2X_1C] + M^{-1}A_1, \\ &\vdots \\ X_n &= M^{-1}[E_1X_{n-1}B + E_2X_{n-1}C] + M^{-1}A_{n-1}. \end{aligned}$$

Now the polynomials A_n for the above non-linear equations are calculated by using (9) and are given by

$$\begin{aligned} A_0 =&E_1X_0F_1X_0 + E_2X_0F_2X_0, \\ A_1 =&X_1[E_1F_1X_0 + E_1X_0F_1 + E_2F_2X_0 + E_2X_0F_2] \\ &+ (X_1^2/2!)[2E_1F_1 + 2E_2F_2], \\ A_2 =&X_2[E_1F_1X_0 + E_1X_0F_1 + E_2F_2X_0 + E_2X_0F_2] \\ &+ (X_2^2/2!)[2E_1F_1 + 2E_2F_2] + X_1X_2[2E_1F_1 + 2E_2F_2]. \end{aligned}$$

Thus, the initial approximation matrix X_0 is given by (18) and special polynomial A_0 is given by $A_0 = diag \left[\begin{array}{ccc} Q_1, & Q_2, & Q_3 \end{array} \right]$, where

$$\begin{aligned} Q_1 =& - \beta_1(p_1 - a)(p_2 - b), \\ Q_2 =&(p_2 - b)[\beta_2(p_1 - a) - \gamma_2(p_3 - c)], \\ Q_3 =&\gamma_3(p_2 - b)(p_3 - c). \end{aligned}$$

The second approximation X_1 is given by

$$\begin{aligned} X_1 =&M^{-1}[E_1X_0B + E_2X_0C] + M^{-1}A_0 \\ =&\text{diag} \left[(Q_1 + Q_4)t, \quad (Q_2 + Q_5)t, \quad (Q_3 + Q_6)t \right], \end{aligned}$$

$$\begin{aligned} \text{where } Q_4 =& - a\beta_1(p_2 - b), \\ Q_5 =&b[\beta_2(p_1 - a) - \gamma_2(p_3 - c)], \\ Q_6 =&c\gamma_3(p_2 - b). \end{aligned}$$

The third approximation X_2 is given by

$$X_2 = M^{-1}[E_1X_1B + E_2X_1C] + M^{-1}A_1.$$

The special polynomial A_1 is given by

$$A_1 = \text{diag} \begin{bmatrix} (Q_1 + Q_4)Q_7t & (Q_2 + Q_5)Q_8t + & (Q_3 + Q_6)Q_9t + \\ -\beta_1(Q_1 + Q_4)^2t^2, & ((\beta_2 - \gamma_2)(Q_2 + Q_5)^2t^2), & \gamma_3(Q_3 + Q_5)^2t^2 \end{bmatrix},$$

$$\begin{aligned} \text{where } Q_7 =& -\beta_1(p_1 - a + p_2 - b), \\ Q_8 =& (p_2 - b)(\beta_2 - \gamma_2) - \gamma_2(p_3 - c) - \beta_2(p_1 - a), \\ Q_9 =& \gamma_3(p_2 - b + p_3 - c)). \end{aligned}$$

Therefore

$$X_2 = \text{diag} \left[(Q_{10}t^2 + Q_{13}t^3), \quad (Q_{11}t^2 + Q_{14}t^3), \quad (Q_{12}t^2 + Q_{15}t^3) \right],$$

$$\begin{aligned} \text{where } Q_{10} =& [(Q_1 + Q_4)Q_7 - a\beta_1(Q_2 + Q_5)]/2, \\ Q_{11} =& [(Q_2 + Q_5)Q_8 + b(\beta_2(Q_1 + Q_4) - \gamma_2(Q_3 + Q_6)]/2, \\ Q_{12} =& [(Q_3 + Q_6)Q_9 + \gamma_3c(Q_2 + Q_5)]/2, \\ Q_{13} =& [-\beta_1(Q_1 + Q_4)^2]/3, \\ Q_{14} =& [(\beta_2 - \gamma_2)(Q_2 + Q_5)^2]/3, \end{aligned}$$

$$Q_{15} = [\gamma_3(Q_3 + Q_6)^2]/3.$$

The fourth approximation X_3 is given by

$$X_3 = M^{-1}[E_1X_2B + E_2X_2C] + M^{-1}A_2,$$

where the special polynomial A_2 is given by $A_2 = \text{diag}\left[q_1, \quad q_2, \quad q_3\right]$, and

$$\begin{aligned} q_1 =& Q_7Q_{13}t^2 + [Q_7Q_{13} - 2\beta_2(Q_1 + Q_4)Q_{10}]t^3 \\ & - [\beta_2Q_{10}^2 + 2\beta_2(Q_1 + Q_4)Q_{13})]t^4 - 2\beta_2Q_{10}Q_{13}t^5 - \beta_2Q_{13}^2t^6, \\ q_2 =& Q_8Q_{11}t^2 + [Q_8Q_{14} + 2(\beta_2 - \gamma_2)(Q_2 + Q_5)Q_{11}]t^3 \\ & + 2[(\beta_2 - \gamma_2)Q_{11}^2 + (\beta_2 - \gamma_2)(Q_2 + Q_5)Q_{14}]t^4 \\ & + 2(\beta_2 - \gamma_2)(Q_2 + Q_5)Q_{11}Q_{14}t^5 + (\beta_2 - \gamma_2)Q_{14}^2t^6, \\ q_3 =& Q_9Q_{12}t^2 + [Q_9Q_{15} + 2\gamma_3(Q_3 + Q_6)Q_{12}]t^3 \\ & + [\gamma_3Q_{12}^2 + 2\gamma_3Q_{15}(Q_3 + Q_6)]t^4 + 2\gamma_3Q_{12}Q_{15}t^5 + \gamma_3Q_{15}^2t^6. \end{aligned}$$

Then the fourth approximation X_3 is calculated and is given by

$$X_3 = \text{diag}\begin{bmatrix} Q_{16}t^3 + Q_{17}t^4 & Q_{21}t^3 + Q_{22}t^4 & Q_{26}t^3 + Q_{27}t^4 \\ +Q_{18}t^5 + Q_{19}t^6, & +Q_{23}t^5 + Q_{24}t^6, & +Q_{28}t^5 + Q_{29}t^6 \\ +Q_{20}t^7 & +Q_{25}t^7 & +Q_{30}t^7 \end{bmatrix},$$

where

$$\begin{aligned} Q_{16} =& [Q_7Q_{10} - a\beta_1Q_{11}]/3, \\ Q_{17} =& [Q_7Q_{13} - 2\beta_2(Q_1 + Q_4)Q_{10} - a\beta_1Q_{14}]/4, \\ Q_{18} =& [-\beta_2Q_{10}^2 - 2\beta_2(Q_1 + Q_4)Q_{13}]/5, \\ Q_{19} =& [-2\beta_2Q_{10}Q_{13}]/6, \\ Q_{20} =& [-\beta_2Q_{13}^2]/7, \\ Q_{21} =& [Q_8Q_{14} + b\beta_2Q_{10} - b\gamma_2Q_{12}]/3, \\ Q_{22} =& [Q_8Q_{14} + 2(\beta_2 - \gamma_2)(Q_2 + Q_5)Q_{11} + b\beta_2Q_{13} + b(\beta_2Q_{13} - \gamma_2Q_{15})]/4, \\ Q_{23} =& [(\beta_2 - \gamma_2)Q_{11}^2 + 2(\beta_2 - \gamma_2)(Q_2 + Q_5)Q_{14}]/5, \\ Q_{24} =& [2(\beta_2 - \gamma_2)Q_{11}Q_{14}]/6, \\ Q_{25} =& [(\beta_2 - \gamma_2)Q_{14}^2]/7, \\ Q_{26} =& [\gamma_3cQ_{11} + Q_9Q_{12}]/3, \\ Q_{27} =& [Q_9Q_{15} + 2\gamma_3Q_{12}(Q_3 + Q_6) + \gamma_3cQ_{14}]/4, \\ Q_{28} =& [\gamma_3Q_{12}^2 + 2\gamma_3Q_{12}(Q_3 + Q_6)]/5, \\ Q_{29} =& [2\gamma_3Q_{12}Q_{15}]/6, \\ Q_{30} =& [\gamma_3Q_{15}^2]/7. \end{aligned}$$

Thus the approximate analytical solution of (15) is given by $X \approx X_0+X_1+X_2+X_3$, i.e.

$$\begin{aligned} L_1 &= (p_1-a)+(Q_1+Q_4)t+Q_{10}t^2+(Q_{13}+Q_{16})t^3 \\ &\quad + Q_{17}t^4+Q_{18}t^5+Q_{19}t^6+Q_{20}t^7, \\ D_1 &= (p_2-b)+(Q_2+Q_5)t+Q_{11}t^2+(Q_{14}+Q_{21})t^3 \\ &\quad + Q_{22}t^4+Q_{23}t^5+Q_{24}t^6+Q_{25}t^7, \\ W_1 &= (p_3-c)+(Q_3+Q_6)t+Q_{12}t^2+(Q_{15}+Q_{26})t^3 \\ &\quad + Q_{27}t^4+Q_{28}t^5+Q_{29}t^6+Q_{30}t^7. \end{aligned}$$

In terms of original densities of the three species $L(t)$, $D(t)$ and $W(t)$ the approximate solutions are given by

$$\begin{aligned} L(t) &= p_1+(Q_1+Q_4)t+Q_{10}t^2+(Q_{13}+Q_{16})t^3 \\ &\quad + Q_{17}t^4+Q_{18}t^5+Q_{19}t^6+Q_{20}t^7, \\ D(t) &= p_2+(Q_2+Q_5)t+Q_{11}t^2+(Q_{14}+Q_{21})t^3 \\ &\quad + Q_{22}t^4+Q_{23}t^5+Q_{24}t^6+Q_{25}t^7, \\ W(t) &= p_3+(Q_3+Q_6)t+Q_{12}t^2+(Q_{15}+Q_{26})t^3 \\ &\quad + Q_{27}t^4+Q_{28}t^5+Q_{29}t^6+Q_{30}t^7. \end{aligned}$$

The accuracy of the solution is increased by finding the higher iterations.

5. Stability Analysis

Theorem 1. *In the absence of predators system (1) is asymptotically stable if there exists positive constants d_1 and d_2 such that $[\beta_1 d_1 - \beta_2 d_2] \le 0$.*

Proof. In the absence of predators, the transformation

$$L = L_1 - \alpha_2/\beta_2 \quad \text{and} \quad D = D_1 + \alpha_1/\beta_1 \tag{19}$$

in (1) yields the following system of equations

$$\begin{aligned} \dot{L}_1 &= \alpha_1(L_1-\alpha_2/\beta_2) - \beta_1(L_1-\alpha_2/\beta_2)(D_1+\alpha_1/\beta_1), \\ \dot{D}_1 &= \alpha_2(D_1+\alpha_1/\beta_1) + \beta_2(L_1-\alpha_2/\beta_2)(D_1+\alpha_1/\beta_1). \end{aligned}$$

Define a positive definite function $V_1(t)$ as

$$\begin{aligned} V_1(t) &= d_1[(-\alpha_2/\beta_2)\log(1-\beta_2 L_1/\alpha_2) - L_1] \\ &\quad + d_2[(\alpha_1/\beta_1)\log(1+\beta_1 D_1/\alpha_1) - D_1]. \end{aligned}$$

The time derivative of $V_1(t)$ is

$$\begin{aligned}\dot{V}_1(t) =& - d_1[\dot{L}_1 L_1/(L_1 - \alpha_2/\beta_2)] - d_2[\dot{D}_1 D_1/(D_1 + \alpha_1/\beta_1)] \\ =& L_1 D_1(\beta_1 d_1 - \beta_2 d_2) \leq 0.\end{aligned}$$

□

Theorem 2. *In the absence of litter, system (1) is asymptotically stable if there exists positive constants d_2 and d_3 such that $[\gamma_2 d_2 - \gamma_3 d_3] \leq 0$.*

Proof. In the absence of litter, the transformation

$$D = D_1 + \alpha_3/\gamma_3 \quad \text{and} \quad W = W_1 - \alpha_2/\gamma_2 \tag{20}$$

in (1) yields the following system of equations

$$\begin{aligned}\dot{D}_1 =& \alpha_2(D_1 + \alpha_3/\gamma_3) - \gamma_2(D_1 + \alpha_3/\gamma_3)(W_1 + \alpha_2/\gamma_2), \\ \dot{W}_1 =& - \alpha_3(W_1 + \alpha_2/\gamma_2) + \gamma_3(D_1 + \alpha_3/\gamma_3)(W_1 + \alpha_2/\gamma_2).\end{aligned}$$

Define a positive definite function $V_2(t)$ as

$$\begin{aligned}V_2(t) =& d_2[(\alpha_3/\gamma_3)\log(1 + \gamma_3 D_1/\alpha_3) - D_1] \\ &+ d_3[(\alpha_2/\gamma_2)\log(1 + \gamma_2 D_1/\alpha_2) - W_1].\end{aligned}$$

The time derivative of $V_2(t)$ is

$$\begin{aligned}\dot{V}_2(t) =& - d_2[\dot{D}_1 D_1/(D_1 + \alpha_3/\gamma_3)] - d_3[\dot{W}_1 W_1/(W_1 + \alpha_2/\gamma_2)] \\ =& D_1 W_1(d_2\gamma_2 - d_3\gamma_3) \leq 0.\end{aligned}$$

□

Theorem 3. *The system (1) is asymptotically stable if there exists positive constants d_1, d_2 and d_3 such that $[\beta_1 d_1 - \beta_2 d_2] \leq 0$ and $[\gamma_2 d_2 - \gamma_3 d_3] \leq 0$.*

Proof. By taking the transformation

$$L = L_1 + a,\ D = D_1 + b \quad \text{and} \quad W = W_1 + c \tag{21}$$

in (1) where a, b and c are the equilibrium points the following system of equations are obtained

$$\begin{aligned}\dot{L}_1 &= (L_1 + a)(-\beta_1 D_1), \\ \dot{D}_1 &= (D_1 + b)(\beta_2 L_1 - \gamma_2 W_1), \\ \dot{W}_1 &= (W_1 + c)(\gamma_3 D_1).\end{aligned} \tag{22}$$

Define a positive definite function $V_3(t)$ as

$$\begin{aligned}V_3(t) =& d_1[a\log(1 + L_1/a) - L_1] + d_2[b\log(1 + D_1/b) - D_1] \\ &+ d_3[c\log(1 + W_1/c) - W_1].\end{aligned}$$

By realizing (22)the time derivative of $V_3(t)$ is

$$\begin{aligned}\dot{V}_3(t) =& - d_1[\dot{L}_1 L_1/(a + L_1)] - d_2[\dot{D}_1 D_1/(b + D_1)] \\ &- d_3[\dot{W}_1 W_1/(c + W_1)]\end{aligned}$$

$$= - d_1(-\beta_1 L_1 D_1) - d_2(\beta_2 L_1 D_1 - \gamma_2 D_1 W_1) - d_3(\gamma_3 D_1 W_1)$$
$$= L_1 D_1(\beta_1 d_1 - \beta_2 d_2) + D_1 W_1(d_2\gamma_2 - d_3\gamma_3) \leq 0. \qquad \square$$

6. Summary and Conclusions

In this article we study the dynamics of litter, detritus and predators in the mangrove areas. This work can also be interpreted in the context of yield management problem where the litter is debris, detritus is the organic matter and predator is the yield. In the ecological management of the mangrove areas the government may regulate the flow of biodegradable as well as non biodegradable waste in the interest of existence of coastal organisms like Polychaete, Crabs, Prawn, Fish and Zooplankton. The criteria for predator extinction would be important in which case there is some imbalance between litter and detritus. The criteria for litter extinction would be important in which case there is some imbalance between predators and detritus. The asymptotic stability results of Section 5 help to present necessary conditions for simultaneous coexistence of litter, detritus and predators in the mangrove areas. Owing to the generality, the considered model exhibits very rich dynamics. The key results are illustrated using numerical simulation through graphs.

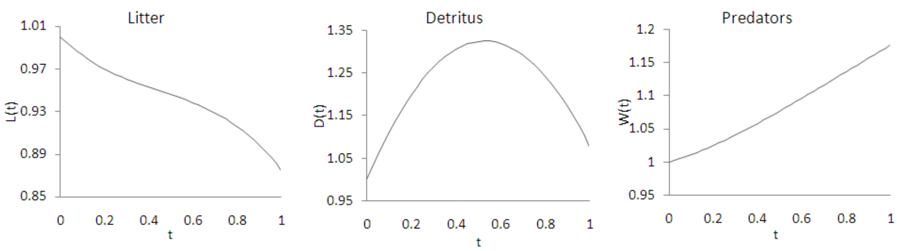

Figure 1: Growth curves of litter, detritus and predators

For the choice of the constants satisfying the necessary conditions for asymptotic stabilities of various equilibria that $\alpha_1 = 0.4, \alpha_2 = 1, \alpha_3 = 0.2, \beta_1 = 0.6, \beta_2 = 1, \gamma_2 = 1, \gamma_3 = 0.3, p_1 = 1, p_2 = 1, p_3 = 1, d_1 = 1, d_2 = 1$ and $d_3 = 4$, we can observe from the Figure 1 that $L(t)$ decreases with the time, detritus increases up to the middle of the time interval and from that it decreases because detritus acts as a food source to the predators and there by contributes to the multiplication of the predators and predators increase with the time because of the continuous decomposition of litter in to detritus in an infinitesimal time interval [0,1].

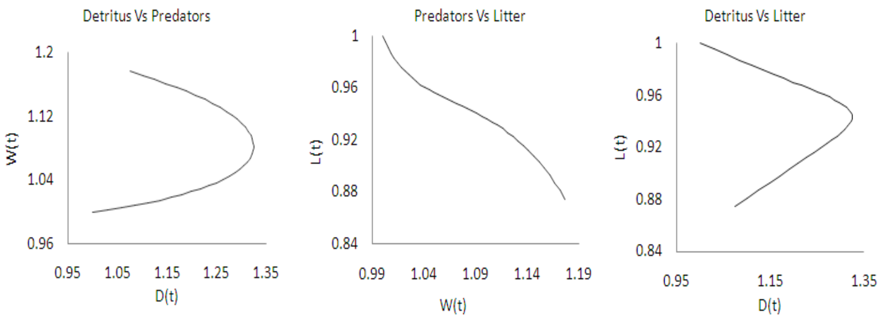

Figure 2: Interactions between litter, detritus and predators

The relational dynamics of litter, detritus and predators represented by the Figure 2 is concurring with the assumptions taken in Section 2 for the formulation of the model in an infinitesimal time interval [0,1].

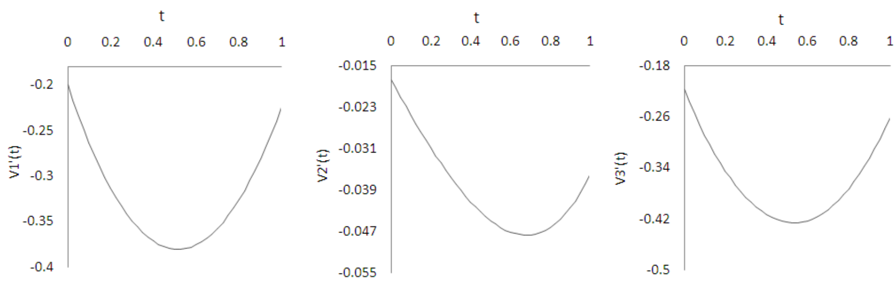

Figure 3: Asymptotic stability of equilibria

The graphs presented in Figure 3 represent the time derivatives of the positive definite functions $V_1(t), V_2(t)$ and $V_3(t)$ constructed for the equilibrium point in the absence of predators, equilibrium point in the absence of litter and interior equilibrium point respectively and all the three equilibria are asymptotically stable as the time derivative of all $V_i(t)$ $(i = 1, 2, 3)$ are negative in an infinitesimal time interval [0,1].

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