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GENERATING FUNCTIONS FOR CHEBYSHEV POLYNOMIALS

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Abstract: The binomial expansion of $(1 - te^{i\theta})^{-\mu}$ is used to derive families of generating functions for the four kinds of Chebyshev polynomials.

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1. Introduction

Chebyshev polynomials arise in many fields of mathematics, and are of particular importance in numerical analysis, where they are used in approximation theory and many other applications. A comprehensive account of the uses and properties of Chebyshev polynmials is presented in the book by Mason and Handscomb which has a lengthy bibliography (see [3]).

Apart from their intrinsic interest, generating functions have a number of practical uses in the evaluation of certain integrals and the proof of certain identities.

In what follows,we shall be working with the trigonometric definitions of the Chebyshev polynomials. The Chebyshev polynomials of the first and second kind can be defined by:

$$T_n(\cos\theta) = \cos n\theta, \qquad U_n(\cos\theta) = \frac{\sin[(n+1)\theta)]}{\sin\theta}.$$

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Chebyshev polynomials of the third and fourth kinds were first named by Gautschi [2] and can be defined by:

$$V_n(\cos\theta) = \frac{\cos[(n+1/2)\theta]}{\cos(\theta/2)}, \qquad W_n(\cos\theta) = \frac{\sin[(n+1/2)\theta]}{\sin(\theta/2)}.$$

We shall use the binomial series expansion of $(1-te^{i\theta})^{-\mu}$

$$\frac{1}{(1-te^{i\theta})^{\mu}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\mu)}{n!\Gamma(\mu)} t^n e^{in\theta}. \tag{I1}$$

By taking the real and imaginary parts of (I1) for various values of μ, we shall derive families of generating functions for the Chebyshev polynomials of the first and second kinds. On multiplying by $e^{i\theta/2}$ and then taking the real and imaginary parts, we shall derive families of generating function for the Chebyshev polynomials of the third and fourth kinds. The simplest cases arise when μ takes the values 1, 2 and $\pm 1/2$. We shall consider these cases explicitly. Other more complicated generating functions can be obtained with further values of μ.

We shall also use the integral of (I1) for $\mu = 1$

$$\sum_{n=0}^{\infty} \frac{t^{n+1}e^{i(n+1)\theta}}{n+1} = -\ln(1-te^{i\theta}). \tag{I2}$$

We shall take the real and imaginary parts of (I2) to derive further generating functions for Chebyshev polynomials of the first and second kinds. We shall also multiply this equation by $e^{i\theta/2}$ to obtain generating functions for the third and fourth kinds of the Chebyshev polynomials.

In what follows, we shall denote $x = \cos\theta$ and $R = \sqrt{1-2xt+t^2}$. We shall also suppose that $|t| < 1$ and $|x| \leq 1$.

2. Chebyshev Polynomials of the First Kind

If we take the real part of equation (I1), we have a family of generating functions for the Chebyshev polynomials of the first kind depending on our choice of μ.

For $\mu = 1$, we get

$$\sum_{n=0}^{\infty} t^n T_n(x) = \mathrm{Re}\left(\frac{1}{1-te^{i\theta}}\right) = \frac{1-xt}{1-2xt+t^2}. \tag{$T1$}$$

For $\mu = 2$,

$$\sum_{n=0}^{\infty}(n+1)t^n T_n(x) = \mathrm{Re}\left(\frac{1}{(1-te^{i\theta})^2}\right) = \frac{1-2xt+2x^2t^2-t^2}{(1-2xt+t^2)^2}. \qquad (T2)$$

For $\mu = 1/2$,

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{n!\Gamma(1/2)}t^n T_n(x) = \mathrm{Re}\left(\frac{1}{\sqrt{1-te^{i\theta}}}\right) = \frac{\sqrt{1-xt+R}}{\sqrt{2}R}. \qquad (T3)$$

For $\mu = -1/2$,

$$\sum_{n=0}^{\infty}\frac{\Gamma(n-1/2)}{n!\Gamma(-1/2)}t^n T_n(x) = \mathrm{Re}\sqrt{1-te^{i\theta}} = \frac{\sqrt{1-xt+R}}{\sqrt{2}}. \qquad (T4)$$

If we now take the real part of equation (2), we find

$$\sum_{n=0}^{\infty}\frac{t^{n+1}}{n+1}T_{n+1}(x) = \sum_{n=1}^{\infty}\frac{t^n}{n}T_n(x) = -\frac{1}{2}\ln(1-2xt+t^2). \qquad (T5)$$

For this generating function the term for $T_0(x)$ is missing.

More generating functions can be obtained by choosing different values for μ. Clearly negative integer values of μ lead only to trivial cases.

Three of these generating functions appear in the book by Abramowitz and Stegun [1]. Equation (T1) is the same as their equation (22.9.9) or equivalently equation (22.9.6); equation (T3) is the same as equation (22.9.7) and equation (T5) is equivalent to equation (22.9.8).

As a further test for the generating functions, it can be shown that the operator

$$\left\{(1-x^2)\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}t\frac{\partial}{\partial t}\right\}$$

acting on each of the expressions on the right hand sides of equations (T1) to (T5) gives zero. Substution of the left hand side of each equation and equating coefficients of powers of t to zero confirms that $T_n(x)$ defined by each of these equations satifies the differential equation for the Chebyshev polynomials of the first kind of order n. If we now consider the limit as $x \to 1$ in each of equations (T1) to (T5), we see that for the polynomials defined in these equations, $T_n(1) = 1$ and therefore these polynomials must be the Chebyshev polynomials of the first kind.

3. Chebyshev Polynomials of the Second Kind

If we take the imaginary part of equation (I1), we find a family of generating functions for the Chebyshev polynomials of the second kind. We note first of all that the first term in the expansion of equation (I1) is always 1 which is real. We shall therefore start the expansion at the second term and write the imaginary part of equation (I1) as

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1+\mu)}{(n+1)!\Gamma(\mu)} t^{n+1} \sin[(n+1)\theta] = \operatorname{Im}\left(\frac{1}{(1-te^{i\theta})^{\mu}}\right). \qquad \text{(I1}a\text{)}$$

For $\mu = 1$,

$$\sum_{n=0}^{\infty} t^{n+1} \sin((n+1)\theta = \operatorname{Im}\left(\frac{1}{1-te^{i\theta}}\right) = \frac{t\sin\theta}{R^2}.$$

Therefore

$$\sum_{n=0}^{\infty} t^n U_n(x) = \frac{1}{R^2}. \qquad (U1)$$

For $\mu = 2$,

$$\sum_{n=0}^{\infty} (n+2) t^{n+1} \sin[(n+1)\theta] = \operatorname{Im}\left(\frac{1}{(1-te^{i\theta})^2}\right) = \frac{2t(1-xt)\sin\theta}{R^4}.$$

Therefore

$$\sum_{n=0}^{\infty} (n+1) t^n U_n(x) = 2\frac{1-xt}{R^4}. \qquad (U2)$$

For $\mu = 1/2$,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+3/2)}{(n+1)!\Gamma(1/2)} t^{n+1} \sin[(n+1)\theta] = \operatorname{Im}\left(\frac{1}{\sqrt{1-te^{i\theta}}}\right) = \frac{t\sin\theta}{\sqrt{2}R\sqrt{1-xt+R}}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+3/2)}{(n+1)!\Gamma(1/2)} t^n U_n(x) = \frac{1}{\sqrt{2}R\sqrt{1-xt+R}}. \qquad (U3)$$

For $\mu = -1/2$,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{(n+1)!\Gamma(-1/2)} t^{n+1} \sin[(n+1)\theta] = \operatorname{Im}\left(\sqrt{1-te^{i\theta}}\right) = \frac{-t\sin\theta}{\sqrt{2}\sqrt{1-xt+R}}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{(n+1)!\Gamma(-1/2)} t^n U_n(x) = \frac{-1}{\sqrt{2}\sqrt{1-xt+R}}. \qquad (U4)$$

If we now take the imaginary part of equation (I2) above, we find

$$\sum_{n=0}^{\infty} \frac{t^n}{n+1} U_n(x) = \frac{1}{t\sqrt{1-x^2}} \tan^{-1}\left(\frac{t\sqrt{1-x^2}}{1-xt}\right). \qquad (U5)$$

We can obtain further generating functions by taking different values of μ. Again, negative integer values of μ lead only to trivial identities.

Two of these generating functions are the same as those in Abramowitz and Stegun [1]. Equation (U1) is the same as equation (29.9.10) and equation (U3) is the same as equation (29.9.11) in Abramowitz and Stegun.

As a further test for the generating functions, it can be shown that the operator

$$\left\{(1-x^2)\frac{\partial^2}{\partial x^2} - 3x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\frac{1}{t}\frac{\partial}{\partial t}t^2\right\}$$

acting on each of the expressions on the right hand sides of equations (U1) to (U5) gives zero. Substution of the left hand side of each equation and equating coefficients of powers of t to zero confirms that $U_n(x)$ defined by each of these equations satifies the differential equation for the Chebyshev polynomials of the second kind of order n. If we now consider the limit as $x \to 1$ in each of equations (U1) to (U5), we see that for the polynomials defined in these equations, $U_n(1) = n+1$ and therefore these polynomials must be the Chebyshev polynomials of the second kind.

4. Chebyshev Polynomials of the Third Kind

If we multiply equation (I1) by $e^{i\theta/2}$ and take the real part, we get

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\mu)}{n!\Gamma(\mu)} t^n \cos[(n+1/2)\theta] = \mathrm{Re}\left(\frac{e^{i\theta/2}}{(1-te^{i\theta})^{\mu}}\right).$$

For $\mu = 1$,

$$\sum_{n=0}^{\infty} t^n \cos[(n+1/2)\theta] = \mathrm{Re}\left(\frac{e^{i\theta/2}}{1-te^{i\theta}}\right) = \frac{\cos(\theta/2) - t\cos(\theta/2)}{R^2}.$$

Therefore

$$\sum_{n=0}^{\infty} t^n V_n(x) = \frac{1-t}{R^2}. \tag{V1}$$

For $\mu = 2$,

$$\sum_{n=0}^{\infty} (n+1)t^n \cos[(n+1/2)\theta] = \mathrm{Re}\left(\frac{e^{i\theta/2}}{(1-te^{i\theta})^2}\right)$$

$$= \frac{\cos(\theta/2)[1-2t+t^2(2x-1)]}{R^4}.$$

Therefore

$$\sum_{n=0}^{\infty} (n+1)t^n V_n(x) = \frac{1-2t+t^2(2x-1)}{R^4}. \tag{V2}$$

For $\mu = 1/2$,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!\Gamma(1/2)} t^n \cos[(n+1/2)\theta] = \mathrm{Re}\left(\frac{e^{i\theta/2}}{\sqrt{1-te^{i\theta}}}\right)$$

$$= \mathrm{Re}\left[\frac{e^{i\theta/2}}{\sqrt{2}R}\left(\sqrt{1-xt+R} + \frac{it\sin\theta}{\sqrt{1-xt+R}}\right)\right] = \frac{\cos(\theta/2)(1-t+R)}{\sqrt{2}R\sqrt{(1-xt+R)}}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!\Gamma(1/2)} t^n V_n(x) = \frac{1-t+R}{\sqrt{2}R\sqrt{(1-xt+R)}}. \tag{V3}$$

For $\mu = -1/2$,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-1/2)}{n!\Gamma(-1/2)} t^n \cos[(n+1/2)\theta] = \mathrm{Re}\left(e^{i\theta/2}\sqrt{1-te^{i\theta}}\right)$$

$$= \mathrm{Re}\left[\frac{e^{i\theta/2}}{\sqrt{2}}\left(\sqrt{1-xt+R} - \frac{it\sin\theta}{\sqrt{1-xt+R}}\right)\right] = \frac{\cos(\theta/2)(1-2xt+t+R)}{\sqrt{2(1-xt+R)}}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-1/2)}{n!\Gamma(-1/2)} t^n V_n(x) = \frac{1-2xt+t+R}{\sqrt{2(1-xt+R)}}. \tag{V4}$$

If we multiply equation (I2) above by $e^{i\theta/2}$ and take the real part, we obtain the more complicated generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n+1} V_n(x) = \frac{1}{t}\left\{\sqrt{\frac{1-x}{1+x}}\tan^{-1}\left(\frac{t\sqrt{1-x^2}}{1-xt}\right) - \frac{1}{2}\ln(1-2xt+t^2)\right\}. \tag{V5}$$

We can obtain further generating functions by taking different values of μ. Again, negative integer values of μ lead only to trivial identities.

As a further test for the generating functions, it can be shown that the operator

$$\left\{(1-x^2)\frac{\partial^2}{\partial x^2}+(1-2x)\frac{\partial}{\partial x}+t\frac{\partial^2}{\partial t^2}t\right\}$$

acting on each of the expressions on the right hand sides of equations (V1) to (V5) gives zero. Substution of the left hand side of each equation and equating coefficients of powers of t to zero confirms that $V_n(x)$ defined by each of these equations satifies the differential equation for the Chebyshev polynomials of the third kind of order n. If we now consider the limit as $x \to 1$ in each of equations (V1) to (V5), we see that for the polynomials defined in these equations, $V_n(1) = 1$ and therefore these polynomials must be the Chebyshev polynomials of the third kind.

5. Chebyshev Polynomials of the Fourth Kind

If we multiply equation (I1) by $e^{i\theta/2}$ and take the imaginary part, we get

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+\mu)}{n!\Gamma(\mu)}t^n\sin[(n+1/2)\theta]=\text{Im}\left(\frac{e^{i\theta/2}}{(1-te^{i\theta})^{\mu}}\right).$$

For $\mu = 1$,

$$\sum_{n=0}^{\infty}t^n\sin[(n+1/2)\theta]=\text{Im}\left(\frac{e^{i\theta/2}}{1-te^{i\theta}}\right)=\frac{\sin(\theta/2)+t\sin(\theta/2)}{R^2}.$$

Therefore

$$\sum_{n=0}^{\infty}t^nW_n(x)=\frac{1+t}{R^2}. \qquad (W1)$$

For $\mu = 2$,

$$\sum_{n=0}^{\infty}(n+1)t^n\sin[(n+1/2)\theta]=\text{Im}\left(\frac{e^{i\theta/2}}{(1-te^{i\theta})^2}\right)$$

$$=\frac{\sin(\theta/2)[1+2t-t^2(2x+1)]}{R^4}.$$

Therefore

$$\sum_{n=0}^{\infty}(n+1)t^nW_n(x)=\frac{1+2t-t^2(2x+1)}{R^4}. \qquad (W2)$$

For $\mu = 1/2$,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!\Gamma(1/2)} t^n \sin[(n+1/2)\theta] = \mathrm{Im}\left(\frac{e^{i\theta/2}}{\sqrt{1-te^{i\theta}}}\right)$$

$$= \mathrm{Im}\left[\frac{e^{i\theta/2}}{\sqrt{2}R}\left(\sqrt{1-xt+R} + \frac{it\sin\theta}{\sqrt{1-xt+R}}\right)\right] = \frac{\sin(\theta/2)(1+t+R)}{\sqrt{2}R\sqrt{(1-xt+R)}}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!\Gamma(1/2)} t^n W_n(x) = \frac{1+t+R}{\sqrt{2}R\sqrt{(1-xt+R)}}. \qquad (W3)$$

For $\mu = -1/2$,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-1/2)}{n!\Gamma(-1/2)} t^n \sin[(n+1/2)\theta] = \mathrm{Im}\left(e^{i\theta/2}\sqrt{1-te^{i\theta}}\right)$$

$$= \mathrm{Im}\left[\frac{e^{i\theta/2}}{\sqrt{2}}\left(\sqrt{1-xt+R} - \frac{it\sin\theta}{\sqrt{1-xt+R}}\right)\right] = \frac{\sin(\theta/2)(1-2xt-t+R)}{\sqrt{2(1-xt+R)}}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-1/2)}{n!\Gamma(-1/2)} t^n W_n(x) = \frac{1-2xt-t+R}{\sqrt{2(1-xt+R)}}. \qquad (W4)$$

If we multiply equation (I2) above by $e^{i\theta/2}$ and take the imaginary part, we obtain the more complicated generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n+1} W_n(x)$$

$$= \frac{1}{t}\left\{\sqrt{\frac{1+x}{1-x}}\tan^{-1}\left(\frac{t\sqrt{1-x^2}}{1-xt}\right) + \frac{1}{2}\ln(1-2xt+t^2)\right\}. \quad \text{(W5)}$$

We can obtain further generating functions by taking different values of μ. Again, negative integer values of μ lead only to trivial identities.

As a further test for the generating functions, it can be shown that the operator

$$\left\{(1-x^2)\frac{\partial^2}{\partial x^2} - (1+2x)\frac{\partial}{\partial x} + t\frac{\partial^2}{\partial t^2}t\right\}$$

acting on each of the expressions on the right hand sides of equations (W1) to (W5) gives zero. Substution of the left hand side of each equation and equating coefficients of powers of t to zero confirms that $W_n(x)$ defined by each of these equations satifies the differential equation for the Chebyshev polynomials of the fourth kind of order n. If we now consider the limit as $x \to 1$ in each of equations (W1) to (W5), we see that for the polynomials defined in these equations, $W_n(1) = 2n + 1$ and therefore these polynomials must be the Chebyshev polynomials of the fourth kind.

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