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THE SUM OF SUBTRACTION OF THE EIGENVALUES OF TWO SELF ADJOINT DIFFERENTIAL OPERATORS WITH UNBOUNDED OPERATOR COEFFICIENT

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Abstract: In this work, a formula for the sum of the eigenvalues of two second order differential operators with unbounded operator coefficient is found.

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1. Introduction

Let H be a separable Hilbert space. Let us consider the following differential expression in the space $H_1 = L_2(0, \pi; H)$

$$\ell_0(y) = -y^{''}(x) + Ay(x).$$

Here, the operator $A : D(A) \to H$ in the space H satisfies the conditions

$$A = A^* \geq I \quad A^{-1} \in \sigma_\infty(H).$$

Let $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_n \leq ...$ be the eigenvalues of the operator A and $\varphi_1, \varphi_2, ...\varphi_n, ...$ be the orthonormal eigenvectors corresponding to these eigenvalues. Here, each eigenvalue is added according to its own algebraic multiplicity number.

Let $D(L_0')$ denote the set of the functions $y(x)$ in the space H_1 satisfying the conditions:

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1. $y(x)$ has a continuous derivative of second order with respect to the norm in the space H in the interval $[0, \pi]$.

2. $y(x)$ and $Ay(x)$ are continuous with respect to the norm in the space H for every $x \in [0, \pi]$.

3. $y^{'}(0) = y(\pi) = 0$.

The manifold $D(L_0')$ is dense in the space H_1 and the operator L_0 defined by $L_0'y = \ell_0(y)$ from $D(L_0')$ to H_1 is symmetric. The eigenvalues of L_0' are

$$(k + \frac{1}{2})^2 + \gamma_j \quad (k = 0, 1, 2, ... \quad ; j = 1, 2, ...)$$

and the orthonormal eigenvectors corresponding to these eigenvalues are

$$\sqrt{\frac{2}{\pi}} \cos(k + \frac{1}{2}) x \varphi_j \quad (k = 0, 1, 2, ... \quad ; j = 1, 2, ...)$$

respectively. Let us denote the closure of L_0' by L_0. Since the eigenvectors system $\left\{ \cos(k + \frac{1}{2}) x \varphi_j \right\}_{k=0, j=1}^{\infty \quad \infty}$ of the operator L_0 is symmetric and the set consisting of eigenvectors of the symmetric operator L_0 is complete in the space H_1, then the operator $L_0 : D(L_0) \to H_1$ is self adjoint.

We will denote the inner product in H and H_1 by $(.,.)_H$ and $(.,.)$, respectively. Moreover the set $\sigma_1(H)$ denotes the Banach space consisting of the kernel operators from H to H, and trT denotes the trace of the kernel operator T [4].

Let $Q(x)$ be an operator function satisfying the following conditions:

a. $Q(x)$ has a weak derivative of second order in the interval $[0, \pi]$. The function $(Q''(x)f, g)$ is continuous for every $f, g \in H$.

b. $Q^{(i)}(x) : H \to H \quad (i = 0, 1, 2)$ are self adjoint kernel operators for every $x \in [0, \pi]$. The functions $\|Q^{(i)}(x)\|_{\sigma_1(H)} \quad (i = 0, 1, 2)$ are bounded and measurable in the interval $[0, \pi]$.

c. $\int_0^\pi (Q(x)f, f)_H dx = 0$ for every $f \in H$.

Since the operator $Q : H_1 \to H_1$ defined by

$$Qy = Q(x)y(x)$$

is self adjoint, the operator $L : D(L_0) \to H_1$ defined by

$$L = L_0 + Q$$

is also self adjoint. We say that the operator L is also self adjoint operator in the space $H_1 = L_2(0, \pi; H)$ formed by the following differential expression

$$\ell(y) = -y''(x) + Ay(x) + Q(x)y(x)$$

with the boundary condition

$$y'(0) = y(\pi) = 0.$$

The operators L_0 and L are half-bounded below and these operators have purely-discrete spectrum. Let the eigenvalues of the operators L_0 and L be $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$, respectively. If $\gamma_j \sim aj^{\alpha}$ $(0 < a, \alpha < \infty)$ as $j \to \infty$ then there is a constant $d > 0$ such that

$$\lambda_n, \mu_n \sim dn^{\frac{2\alpha}{2+\alpha}} \tag{1.1}$$

as $n \to \infty$ [8].

By using the asymptotic formula (1.1), it can be seen that the sequence $\{\mu_n\}_{n=1}^{\infty}$ has a subsequence $\mu_{n_1} < \mu_{n_2} < ... < \mu_{n_m} < ...$ such that

$$\mu_k - \mu_{n_m} > d_0\Big(k^{\frac{2\alpha}{2+\alpha}} - n_m^{\frac{2\alpha}{2+\alpha}}\Big) \quad (k = n_m + 1, n_m + 2, ...), \tag{1.2}$$

where d_0 is a positive constant.

In this work, we find a formula for the limit

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} \Bigg(\lambda_k - \mu_k\Bigg),$$

where $\{\mu_{n_m}\}_{m=1}^{\infty}$ is a subsequence of $\{\mu_n\}_{n=1}^{\infty}$, and it satisfies the inequality (1.2). This formula is said to be regularized trace formula.

The first study about the regularized trace formula for scalar differential operators is started with [7]. And after this work, on regularized trace of various scalar differential operator have been studied in the works [5], [9], [12]. The list of the works on this subject is given in [11] and [6]. The formulas for the regularized traces of differential operators with operator coefficient is investigated in [3] and in many other works.

2. The Finite Sums of the Subtraction of the Eigenvalues and Some Formulas about the Resolvents

Let $R_\lambda^0 = (L_0 - \lambda I)^{-1}$, $R_\lambda = (L - \lambda I)^{-1}$ be the resolvents of the operators L_0 and L, respectively. By the formula (1.1), the series

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{\mu_k - \lambda}$$

are uniform convergent for $\alpha > 2$ and $\lambda \neq \lambda_k, \mu_k \quad (k = 1, 2, ...)$. In this case R_λ^0, R_λ are kernel operators and

$$tr(R_\lambda - R_\lambda^0) = trR_\lambda - trR_\lambda^0 = \sum_{k=1}^{\infty}\Big(\frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda}\Big),$$

see [4]. If the last equality is multiplied by $\frac{\lambda}{2\pi i}$ and integrated on the circle $|\lambda| = b_m = \frac{1}{2}(\mu_{n_m+1} + \mu_{n_m})$, then we obtain

$$\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m} \lambda tr\big(R_\lambda - R_\lambda^0\big)d\lambda$$

$$= \sum_{k=1}^{\infty}\Big(\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m} \frac{\lambda}{\lambda_k - \lambda}d\lambda\Big) - \sum_{k=1}^{\infty}\Big(\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m} \frac{\lambda}{\mu_k - \lambda}d\lambda\Big). \quad (2.1)$$

It easy to see that for the large value of m the following inequality is satisfied:

$$\lambda_{n_m} < b_m < \lambda_{n_m+1}. \quad (2.2)$$

By (2.1) and (2.2) we find

$$\sum_{k=1}^{n_m}\big(\lambda_k - \mu_k\big) = -\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m} \lambda tr\big(R_\lambda - R_\lambda^0\big)d\lambda. \quad (2.3)$$

This is a well known formula for the resolvents of the operators L_0 and L:

$$R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0 \quad (\lambda \in \rho(L) \cap \rho(L_0)).$$

By using the last formula, for every natural number $p \geq 2$ it can be shown that

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{p}(-1)^j R_\lambda^0 (QR_\lambda^0)^j + (-1)^{p+1} R_\lambda (QR_\lambda^0)^{p+1}. \quad (2.4)$$

If we substitute (2.4) in (2.3), then we obtain

$$\sum_{k=1}^{n_m}(\lambda_k - \mu_k) = \sum_{j=1}^{p} D_{mj} + D_m^{(p)},$$

where

$$D_{mj} = \frac{(-1)^{j+1}}{2\pi i}\int\limits_{|\lambda|=b_m} tr\Big[R_\lambda^0(QR_\lambda^0)^j\Big]d\lambda \quad (j = 1, 2, ..),$$

$$D_m^{(p)} = \frac{(-1)^p}{2\pi i}\int\limits_{|\lambda|=b_m} \lambda tr\Big[R_\lambda(QR_\lambda^0)^{p+1}\Big]d\lambda. \quad (2.5)$$

We can show that the following formula is satisfied for D_{mj};

$$D_{mj} = \frac{(-1)^j}{2\pi i j} \int\limits_{|\lambda|=b_m} tr\left[(QR_\lambda^0)^j\right] d\lambda \quad (j = 1, 2, ..). \tag{2.6}$$

Let $\{\Psi_q(x)\}_{q=1}^{\infty}$ be the orthonormal eigenvectors corresponding to the eigenvalues $\{\mu_q\}_{q=1}^{\infty}$, respectively.

Since these orthonormal eigenvectors are $\sqrt{\frac{2}{\pi}}\cos(k+\frac{1}{2})x\varphi_j$ $(k = 0, 1, 2, ...;$ $j = 1, 2, ...)$ corresponding to the eigenvalues $(k+\frac{1}{2})^2 + \gamma_j$ $(k = 0, 1, 2, ...; j = 1, 2, ...)$, then we have

$$\Psi_q(x) = \sqrt{\frac{2}{\pi}}\cos(k_q + \frac{1}{2})x\varphi_{j_q} \quad (q = 1, 2, ...). \tag{2.7}$$

Theorem 2.1. *If $\gamma_j \sim a j^\alpha$ $(a > 0, \alpha > 2)$ as $j \to \infty$ and $Q(x)$ satisfies the conditions a, b, and c, then*

$$D_{m1} = \frac{1}{4}(tr(Q(0)) - tr(Q(\pi))).$$

Proof. According to the formula (2.6)

$$D_{m1} = -\frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} tr(QR_\lambda^0) d\lambda. \tag{2.8}$$

Since QR_λ^0 is a kernel operator for every $\lambda \in \rho(L_0)$ and $\{\Psi_q(x)\}_1^\infty$ is a orthonormal basis of the space H_1, then we have

$$tr(QR_\lambda^0) = \sum_{q=1}^{\infty}(QR_\lambda^0\Psi_q, \Psi_q),$$

see [4]. If we substitute this expression in (2.6) and consider the equality

$$R_\lambda^0\Psi_q = (L_0 - \lambda I)^{-1}\Psi_q = (\mu_q - \lambda)^{-1}\Psi_q,$$

then we find

$$D_{m1} = -\frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \left[\sum_{q=1}^{\infty}(QR_\lambda^0\Psi_q, \Psi_q)\right] d\lambda$$

$$= -\frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \left[\sum_{q=1}^{\infty}(Q(\mu_q - \lambda)^{-1}\Psi_q, \Psi_q)\right] d\lambda$$

$$= \left[\sum_{q=1}^{\infty}(Q\Psi_q, \Psi_q)\right] \frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \frac{d\lambda}{\lambda - \mu_q}. \tag{2.9}$$

By (2.7), (2.9) and by the formulas

$$\frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}\frac{d\lambda}{\lambda-\mu_q}=\begin{cases}1, & q\le n_m\\ 0, & q>n_m,\end{cases}$$

we obtain

$$\begin{aligned}D_{m1} &= \sum_{q=1}^{n_m}(Q\Psi_q,\Psi_q)\\ &= \sum_{q=1}^{n_m}\int\limits_0^{\pi}(Q(x)\Psi_q(x),\Psi_q(x))_H dx\\ &= \sum_{q=1}^{n_m}\int\limits_0^{\pi}(Q(x)\sqrt{\frac{2}{\pi}}\cos(k_q+\frac{1}{2})x.\varphi_{j_q},\sqrt{\frac{2}{\pi}}\cos(k_q+\frac{1}{2})x.\varphi_{j_q})_H dx\\ &= \frac{2}{\pi}\sum_{q=1}^{n_m}\int\limits_0^{\pi}\cos^2(k_q+\frac{1}{2})x(Q(x)\varphi_{j_q},\varphi_{j_q})_H dx\\ &= \frac{1}{\pi}\sum_{q=1}^{n_m}\int\limits_0^{\pi}(1+\cos(2k_q+1)x)(Q(x)\varphi_{j_q},\varphi_{j_q})_H dx\\ &= \frac{1}{\pi}\sum_{q=1}^{n_m}\int\limits_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})_H\cos(2k_q+1)xdx.\end{aligned}\tag{2.10}$$

It is easy to see that if $Q(x)$ satisfies the conditions a and b, then the series

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\int\limits_0^{\pi}(Q(x)\varphi_j,\varphi_j)_H\cos(2k+1)xdx\tag{2.11}$$

is absolute convergent. In this case, as known

$$\lim_{m\to\infty}D_{m1}=\sum_{q=1}^{n_m}\int\limits_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})_H\cos(2k_q+1)xdx$$

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\int\limits_0^{\pi}(Q(x)\varphi_j,\varphi_j)_H\cos(2k+1)xdx.$$

If we take the limit of the equation (2.10) as $m\to\infty$, and consider the last relation, then we find

$$\lim_{m\to\infty} D_{m1} = \frac{1}{\pi}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)_H \cos(2k+1)xdx$$

$$= \frac{1}{2\pi}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left[\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)_H \cos kxdx - (-1)^k\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)_H \cos kxdx\right]$$

$$= \frac{1}{4}\sum_{j=1}^{\infty}\left\{\sum_{k=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)_H \cos kxdx\right]\cos(k0)\right.$$

$$\left. -\sum_{k=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)_H \cos kxdx\right]\cos(k\pi)\right\}.$$

The sums according to k on the right hand side of the last equality are the values at the point 0 and π of the Fourier series of the function $(Q(x)\varphi_j,\varphi_j)$ having second order derivative according to the functions $\{Coskx\}_{k=0}^{\infty}$ in the interval $[0,\pi]$ respectively. Therefore we obtain

$$\lim_{m\to\infty} D_{m1} = \frac{1}{4}\sum_{j=1}^{\infty}\left[(Q(0)\varphi_j,\varphi_j)_H + (Q(\pi)\varphi_j,\varphi_j)_H\right]$$

or

$$\lim_{m\to\infty} D_{m1} = \frac{1}{4}(tr(Q(0)) - tr(Q(\pi))). \qquad \square$$

3. The Formulation of Regularized Trace

In this section, we find a formula for the regularized trace of the operator L. In Section 2, we obtained that the formula

$$\sum_{k=1}^{n_m}(\lambda_k - \mu_k) = \sum_{j=1}^{p} D_{mj} + D_m^{(p)} \tag{3.1}$$

is satisfied for the sum of subtraction of eigenvalue of the operators L and L_0. Here, it can be shown that

$$D_{mj} = \frac{(-1)^j}{2\pi ij}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_j=1}^{\infty}{}^{*}$$

$$\int\limits_{|\lambda|=b_m} \left(\prod_{q=1}^{j}(\mu_{k_q}-\lambda)^{-1}\right)d\lambda \prod_{q=1}^{j}\left(Q\Psi_{k_q},\Psi_{k_{g(q)+1}}\right). \quad (3.2)$$

The symbol "*" denotes that the numbers among $\mu_{k_1},\mu_{k_2},\ldots,\mu_{k_j}$ can be greater or less than b_m. Moreover,

$$g(q)=\begin{cases} q, & \text{if} \quad q<j,\\ 0, & \text{if} \quad q=j.\end{cases}$$

Let us limit $\|R_\lambda^0\|_{\sigma_1(H_1)}$. Here, we assume that $\gamma_j \sim aj^\alpha$ $(a>0,\alpha>2)$ as $j\to\infty$. Since R_λ^0 is a normal operator for every $\lambda \notin \{\mu_k\}_{k=1}^\infty$, we have

$$\|R_\lambda^0\|_{\sigma_1(H_1)}=\sum_{k=1}^{\infty}\frac{1}{|\mu_k-\lambda|},$$

see [4]. On the circle $|\lambda|=b_m=2^{-1}(\mu_{n_m+1}+\mu_{n_m})$, we have

$$\begin{aligned}\|R_\lambda^0\|_{\sigma_1(H_1)} \le \sum_{k=1}^{\infty}\frac{1}{||\lambda|-\mu_k|} &= \sum_{k=1}^{n_m}\frac{2}{\mu_{n_m}+\mu_{n_m+1}-2\mu_k}\\ &+\sum_{k=n_m+1}^{\infty}\frac{2}{2\mu_k-\mu_{n_m}-\mu_{n_m+1}}\\ &<\sum_{k=1}^{n_m}\frac{2}{\mu_{n_m+1}-\mu_k}+\sum_{k=n_m+1}^{\infty}\frac{2}{\mu_k-\mu_{n_m}}. \end{aligned} \quad (3.3)$$

By using (2.1) and (3.3), we find

$$\begin{aligned}\sum_{k=1}^{n_m}\frac{1}{\mu_{n_m+1}-\mu_k} &< \frac{n_m}{\mu_{n_m+1}-\mu_{n_m}} < \frac{n_m}{d_0[(n_m+1)^{1+\delta}-{n_m}^{1+\delta}]}\\ &< {d_0}^{-1}{n_m}^{1-\delta}, \end{aligned} \quad (3.4)$$

where $\delta=\frac{2\alpha}{2+\alpha}-1$.

From the inequality (1.2) again, we obtain

$$\begin{aligned}\sum_{k=n_m+1}^{\infty}\frac{1}{\mu_k-\mu_{n_m}} &< \sum_{k=n_m+1}^{\infty}\frac{1}{d_0(k^{1+\delta}-{n_m}^{1+\delta})}\\ &=\frac{1}{d_0((n_m+1)^{1+\delta}-{n_m}^{1+\delta})}+\frac{1}{d_0}\sum_{k=n_m+2}^{\infty}\frac{1}{k^{1+\delta}-{n_m}^{1+\delta}}. \end{aligned} \quad (3.5)$$

Moreover,

$$\sum_{k=n_m+2}^{\infty} \frac{1}{k^{1+\delta}-{n_m}^{1+\delta}} = \sum_{i=1}^{\infty} \int\limits_{n_m+i}^{n_m+i+1} \frac{dx}{(n_m+i+1)^{1+\delta}-{n_m}^{1+\delta}}$$
$$< \sum_{i=1}^{\infty} \int\limits_{n_m+i}^{n_m+i+1} \frac{dx}{x^{1+\delta}-{n_m}^{1+\delta}}$$
$$= \int\limits_{n_m+1}^{\infty} \frac{dx}{x^{1+\delta}-{n_m}^{1+\delta}}. \tag{3.6}$$

Let us evaluate the integral at the end of the relation (3.6) using the substitution $x^{1+\delta}-{n_m}^{1+\delta}=t$, we find

$$\int\limits_{n_m+1}^{\infty} \frac{dx}{x^{1+\delta}-{n_m}^{1+\delta}} = \int\limits_{\alpha_m}^{\infty} \frac{1}{(1+\delta)t}(t+{n_m}^{1+\delta})^{\frac{-\delta}{1+\delta}}dt$$
$$< \frac{1}{1+\delta}\int\limits_{\alpha_m}^{\infty} t^{-1-\frac{\delta}{1+\delta}}dt = \delta^{-1}[(n_m+1)^{1+\delta}-{n_m}^{1+\delta}]^{\frac{-\delta}{1+\delta}}, \tag{3.7}$$

where $\alpha_m=(n_m+1)^{1+\delta}-{n_m}^{1+\delta}$.

From the inequalities (3.5) and (3.7), we obtain

$$\sum_{k=n_m+1}^{\infty} \frac{1}{\mu_k-\mu_{n_m}} < \frac{1+\delta}{d_0\delta}{n_m}^{\frac{-\delta^2}{1+\delta}}. \tag{3.8}$$

By using (3.3), (3.4) and (3.8), we have

$$\|R_\lambda^0\|_{\sigma_1(H_1)} < d_1{n_m}^{1-\delta} \qquad (|\lambda|=b_m, \alpha>2), \tag{3.9}$$

where $d_1>0$ is a constant.

Let us limit the norm of the operator R_λ on the circle $|\lambda|=b_m$. For $|\lambda|=b_m$, we have

$$|\,|\lambda_k|-|\lambda|\,|=|\,|\lambda_k|-\frac{1}{2}(\mu_{n_m+1}+\mu_{n_m})\,|=\frac{1}{2}\,|\,\mu_{n_m+1}+\mu_{n_m}-2|\lambda_k|\,|\,. \tag{3.10}$$

$k\le n_m$ and for the large values of m , we have

$$\begin{aligned}\mu_{n_m}+\mu_{n_m+1}-2|\lambda_k| &\ge \mu_{n_m}+\mu_{n_m+1}-2|\lambda_{n_m}|\\ &= \mu_{n_m+1}-\mu_{n_m}+2(\mu_{n_m}-\lambda_{n_m})\\ &\ge \mu_{n_m+1}-\mu_{n_m}-2\,|\,\mu_{n_m}-\lambda_{n_m}\,|\\ &\ge \mu_{n_m+1}-\mu_{n_m}-2\,\|\,Q\,\|\,.\end{aligned} \tag{3.11}$$

We also have $k \geq n_m + 1$ and for the large values of m

$$\begin{aligned} 2|\lambda_k| - \mu_{n_m} - \mu_{n_m+1} &\geq 2\lambda_{n_m+1} - \mu_{n_m} - \mu_{n_m+1} \\ &= 2(\lambda_{n_m+1} - \mu_{n_m+1}) + \mu_{n_m+1} - \mu_{n_m} \\ &\geq \mu_{n_m+1} - \mu_{n_m} - 2 \mid \lambda_{n_m+1} - \mu_{n_m+1} \mid \\ &\geq \mu_{n_m+1} - \mu_{n_m} - 2 \parallel Q \parallel . \end{aligned} \tag{3.12}$$

If we consider $\lim\limits_{m\to\infty}(\mu_{n_m+1} - \mu_{n_m}) = \infty$, then from (3.10), (3.11) and (3.12), we obtain

$$\mid |\lambda_k| - |\lambda| \mid > \frac{1}{4}(\mu_{n_m+1} - \mu_{n_m}) \qquad (|\lambda| = b_m). \tag{3.13}$$

By (1.2) and (3.13), we find

$$\mid \lambda_k - \lambda \mid > \frac{d_0}{4} {n_m}^{\delta} \tag{3.14}$$

for the large values of m. On the other hand, since the $s - numbers$ of the operator R_λ are $\{\mid \lambda_k - \lambda \mid^{-1}\}_{k=1}^{\infty}$, we have

$$\parallel R_\lambda \parallel = \max_k \{|\lambda_k - \lambda|^{-1}\}_{k=1}^{\infty}, \tag{3.15}$$

see [4]. From (3.14) and (3.15), for the large values of m, we obtain

$$\parallel R_\lambda \parallel < \frac{4}{d_0} {n_m}^{-\delta} \qquad (|\lambda| = b_m; \alpha > 2). \tag{3.16}$$

Theorem 3.1. *If the operator function $Q(x)$ satisfies the conditions a, b and $\gamma_j \sim a j^{\alpha}$ $(a > 0, \alpha > 2)$ as $j \to \infty$ then we have*

$$\lim_{m\to\infty} D_{mj} = 0 \qquad (j = 2, 3, 4, \ldots).$$

Proof. If $T \in \sigma_1(H)$ and the operator $S : H \to H$ is linear bounded, then $ST \in \sigma_1(H)$ and

$$\mid trT \mid \leq \parallel T \parallel_{\sigma_1(H)}, \tag{3.17}$$

$$\parallel ST \parallel_{\sigma_1(H)} \leq \parallel S \parallel_H \parallel T \parallel_{\sigma_1(H)}, \tag{3.18}$$

see [4]. By using (2.6), (3.17) and (3.18), we obtain

$$\begin{aligned} \mid D_{mj} \mid &\leq \frac{1}{2\pi j} \int\limits_{|\lambda|=b_m} \mid tr\big(Q{R_\lambda}^0\big)^j \mid |d\lambda| \\ &\leq \int\limits_{|\lambda|=b_m} \parallel (Q{R_\lambda}^0)^j \parallel_{\sigma_1(H_1)} |d\lambda| \\ &\leq \int\limits_{|\lambda|=b_m} \parallel Q{R_\lambda}^0 \parallel_{\sigma_1(H_1)} \parallel (Q{R_\lambda}^0)^{j-1} \parallel |d\lambda| \end{aligned}$$

$$\leq \int\limits_{|\lambda|=b_m} \| Q \| \| R_\lambda{}^0 \|_{\sigma_1(H_1)} \| (QR_\lambda{}^0)^{j-1} \| |d\lambda|$$
$$\leq \| Q \| \int\limits_{|\lambda|=b_m} \| R_\lambda{}^0 \|_{\sigma_1(H_1)} \| QR_\lambda{}^0 \|^{j-1} |d\lambda|$$
$$\leq \| Q \|^j \int\limits_{|\lambda|=b_m} \| R_\lambda{}^0 \|_{\sigma_1(H_1)} \| R_\lambda{}^0 \|^{j-1} |d\lambda|. \quad (3.19)$$

Since $R_\lambda = R_\lambda{}^0$ for $Q(x) \equiv 0$ according to (3.16), we obtain

$$\| R_\lambda{}^0 \| < \frac{4}{d_0} n_m{}^{-\delta} \qquad (|\lambda| = b_m; \quad \delta = \frac{2\alpha}{2+\alpha} - 1). \quad (3.20)$$

From (3.9), (3.19) and (3.20), we obtain

$$| D_{mj} | < \text{const.} \int\limits_{|\lambda|=b_m} n_m{}^{1-\delta} n_m{}^{-\delta(j-1)} |d\lambda|$$
$$< \text{const.} b_m n_m{}^{1-\delta j} < \text{const.} n_m{}^{2-\delta(j-1)}.$$

From here, it seems

$$\lim_{m\to\infty} D_{mj} = 0 \qquad (j > 1 + 2\delta^{-1}).$$

It is necessary to show that the last equality is satisfied for $2 \leq j \leq 1+2\delta^{-1}$ to complete the proof. By using the formula (3.2), we have the following equality for D_{m2};

$$D_{m2} = \frac{1}{4\pi i} \sum_{j=1}^{n_m} \sum_{k=n_m+1}^{\infty} \left[\int\limits_{|\lambda|=b_m} \frac{1}{(\lambda-\mu_j)(\lambda-\mu_k)} d\lambda \right] (Q\Psi_j, \Psi_k)(Q\Psi_k, \Psi_j)$$
$$+ \frac{1}{4\pi i} \sum_{j=n_m+1}^{\infty} \sum_{k=1}^{n_m} \left[\int\limits_{|\lambda|=b_m} \frac{1}{(\lambda-\mu_j)(\lambda-\mu_k)} d\lambda \right] (Q\Psi_j, \Psi_k)(Q\Psi_k, \Psi_j). \quad (3.21)$$

For $\mu_j < b_m$ and $\mu_k > b_m$ ($j \leq n_m$ and $k \geq n_m + 1$), we have

$$\frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \frac{1}{(\lambda-\mu_j)(\lambda-\mu_k)} d\lambda = \frac{1}{2\pi i} \int\limits_{|\lambda|=b_m} \frac{1}{(\mu_j-\mu_k)} \left[\frac{1}{\lambda-\mu_j} - \frac{1}{\lambda-\mu_k} \right] d\lambda$$
$$= \frac{1}{\mu_j - \mu_k}. \quad (3.22)$$

By (3.21) and (3.22), we obtain

$$\begin{aligned}
\mid D_{m2} \mid &= \mid \sum_{j=1}^{n_m} \sum_{k=n_m+1}^{\infty} (\mu_j - \mu_k)^{-1} (\Psi_j, Q\Psi_k) \overline{(\Psi_j, Q\Psi_k)} \mid \\
&\leq \sum_{j=1}^{n_m} \sum_{k=n_m+1}^{\infty} (\mu_k - \mu_{n_m})^{-1} \mid (\Psi_j, Q\Psi_k) \mid^2 \\
&\leq \sum_{k=n_m+1}^{\infty} \left[(\mu_k - \mu_{n_m})^{-1} \sum_{j=1}^{\infty} \mid (Q\Psi_k, \Psi_j) \mid^2 \right] \\
&= \sum_{k=n_m+1}^{\infty} (\mu_k - \mu_{n_m})^{-1} \parallel Q\Psi_k \parallel^2 \\
&\leq \parallel Q \parallel^2 \sum_{k=n_m+1}^{\infty} (\mu_k - \mu_{n_m})^{-1}. \qquad (3.23)
\end{aligned}$$

From (3.8) and (3.23), we find

$$\lim_{m\to\infty} D_{m2} = 0.$$

Similarly, we can show that

$$\lim_{m\to\infty} D_{mj} = 0$$

for $3 \leq j \leq 1 + 2\delta^{-1}$. □

Theorem 3.2. *If $Q(x)$ satisfies the conditions a, b, c and $\gamma_j \sim aj^{\alpha}$ $(a > 0, \alpha > 2)$ as $j \to \infty$, then the formula*

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4}\Big[trQ(0) - trQ(\pi)\Big]$$

is satisfied.

Proof. By the relations (2.5), (3.17) and (3.18), we obtain

$$\begin{aligned}
\mid D_m{}^{(p)} \mid &\leq \frac{1}{2\pi} \int_{|\lambda|=b_m} \mid \lambda \mid . \mid tr\Big(R_\lambda (QR_\lambda{}^0)^{p+1}\Big) \mid . \mid d\lambda \mid \\
&\leq b_m \int_{|\lambda|=b_m} \parallel R_\lambda (QR_\lambda{}^0)^{p+1} \parallel_{\sigma_1(H_1)} | \, d\lambda \mid \\
&\leq b_m \int_{|\lambda|=b_m} \parallel R_\lambda \parallel \parallel (QR_\lambda{}^0)^{p+1} \parallel_{\sigma_1(H_1)} | \, d\lambda \mid
\end{aligned}$$

$$\leq \quad b_m \int\limits_{|\lambda|=b_m} \| R_\lambda \| \| (QR_\lambda{}^0)^p \| \| QR_\lambda{}^0 \|_{\sigma_1(H_1)} | d\lambda |$$
$$\leq \quad b_m \int\limits_{|\lambda|=b_m} \| R_\lambda \| \| QR_\lambda{}^0 \|^p \| Q \| \| R_\lambda{}^0 \|_{\sigma_1(H_1)} | d\lambda |$$
$$\leq \quad b_m \int\limits_{|\lambda|=b_m} \| R_\lambda \| \| R_\lambda{}^0 \|^p \| Q \|^{p+1} \| R_\lambda{}^0 \|_{\sigma_1(H_1)} | d\lambda | . \tag{3.24}$$

By using (3.9), (3.16), (3.20) and (3.24), we find

$$D_m{}^{(p)} \leq \text{const.} b_m \int\limits_{|\lambda|=b_m} n_m{}^{-(p+1)\delta} n_m{}^{1-\delta} \mid d\lambda \mid s \leq \text{const.} b_m{}^2 n_m{}^{-p\delta-2\delta+1}. \tag{3.25}$$

If we consider that

$$b_m = \frac{1}{2}(\mu_{n_m+1} + \mu_{n_m}) \leq \text{const.} n_m{}^{1+\delta},$$

for the large values of m then from (3.25), we obtain

$$\mid D_m{}^{(p)} \mid \leq \text{const.} n_m{}^{3-p\delta}.$$

From here, it seems that

$$\lim_{m\to\infty} D_m{}^{(p)} = 0 \qquad (p > 3\delta^{-1}). \tag{3.26}$$

By Theorem 2.1, Theorem 3.1 and by the formulas (3.1), (3.26), we find

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4}[trQ(0) - trQ(\pi)].$$

The theorem is proved. □

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