THE SUM OF SUBTRACTION OF THE EIGENVALUES OF TWO SELF ADJOINT DIFFERENTIAL OPERATORS WITH UNBOUNDED OPERATOR COEFFICIENT

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Abstract: In this work, a formula for the sum of the eigenvalues of two second order differential operators with unbounded operator coefficient is found.

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1. Introduction

Let $H$ be a separable Hilbert space. Let us consider the following differential expression in the space $H_1 = L_2(0, π; H)$

$$\ell_0(y) = -y''(x) + Ay(x).$$

Here, the operator $A : D(A) \to H$ in the space $H$ satisfies the conditions

$$A = A^* \geq I \quad A^{-1} \in \sigma_\infty(H).$$

Let $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_n \leq ...$ be the eigenvalues of the operator $A$ and $\varphi_1, \varphi_2, ... \varphi_n, ...$ be the orthonormal eigenvectors corresponding to these eigenvalues. Here, each eigenvalue is added according to its own algebraic multiplicity number.

Let $D(L_0)$ denote the set of the functions $y(x)$ in the space $H_1$ satisfying the conditions:

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1. \( y(x) \) has a continuous derivative of second order with respect to the norm in the space \( H \) in the interval \([0, \pi]\).

2. \( y(x) \) and \( Ay(x) \) are continuous with respect to the norm in the space \( H \) for every \( x \in [0, \pi] \).

3. \( y'(0) = y(\pi) = 0 \).

The manifold \( D(L_0') \) is dense in the space \( H_1 \) and the operator \( L_0 \) defined by \( L_0' y = \ell_0(y) \) from \( D(L_0') \) to \( H_1 \) is symmetric. The eigenvalues of \( L_0' \) are
\[
(k + \frac{1}{2})^2 + \gamma_j \quad (k = 0, 1, 2, \ldots; j = 1, 2, \ldots)
\]
and the orthonormal eigenvectors corresponding to these eigenvalues are
\[
\sqrt{\frac{2}{\pi}} \cos(k + \frac{1}{2})x \varphi_j \quad (k = 0, 1, 2, \ldots; j = 1, 2, \ldots)
\]
respectively. Let us denote the closure of \( L_0' \) by \( L_0 \). Since the eigenvectors system \( \{ \cos(k + \frac{1}{2})x \varphi_j \}_{k=0, j=1}^{\infty, \infty} \) of the operator \( L_0 \) is symmetric and the set consisting of eigenvectors of the symmetric operator \( L_0 \) is complete in the space \( H_1 \), then the operator \( L_0 : D(L_0) \rightarrow H_1 \) is self adjoint.

We will denote the inner product in \( H \) and \( H_1 \) by \((., .)_H\) and \((., .)_{H_1}\), respectively. Moreover the set \( \sigma_1(H) \) denotes the Banach space consisting of the kernel operators from \( H \) to \( H \), and \( \text{tr}T \) denotes the trace of the kernel operator \( T \) \[^4\].

Let \( Q(x) \) be an operator function satisfying the following conditions:

a. \( Q(x) \) has a weak derivative of second order in the interval \([0, \pi]\). The function \( Q''(x) f, g \) is continuous for every \( f, g \in H \).

b. \( Q^{(i)}(x) : H \rightarrow H \) \( (i = 0, 1, 2) \) are self adjoint kernel operators for every \( x \in [0, \pi] \). The functions \( \|Q^{(i)}(x)\|_{\sigma_1(H)} \) \( (i = 0, 1, 2) \) are bounded and measurable in the interval \([0, \pi]\).

c. \( \int_0^\pi (Q(x)f, f)_H dx = 0 \) for every \( f \in H \).

Since the operator \( Q : H_1 \rightarrow H_1 \) defined by
\[
Q y = Q(x) y(x)
\]
is self adjoint, the operator \( L : D(L_0) \rightarrow H_1 \) defined by
\[
L = L_0 + Q
\]
is also self adjoint. We say that the operator \( L \) is also self adjoint operator in the space \( H_1 = L_2(0, \pi; H) \) formed by the following differential expression
\[
\ell(y) = -y''(x) + Ay(x) + Q(x) y(x)
\]
with the boundary condition
\[ y'(0) = y(\pi) = 0. \]

The operators \( L_0 \) and \( L \) are half-bounded below and these operators have purely-discrete spectrum. Let the eigenvalues of the operators \( L_0 \) and \( L \) be \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots \) and \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \), respectively. If \( \gamma_j \sim a j^n \) \((0 < a, \alpha < \infty)\) as \( j \to \infty \) then there is a constant \( d > 0 \) such that
\[ \lambda_n, \mu_n \sim d n^{\frac{2\alpha}{\alpha+1}} \]
as \( n \to \infty \) [8].

By using the asymptotic formula (1.1), it can be seen that the sequence \( \{\mu_n\}_{n=1}^{\infty} \) has a subsequence \( \mu_{n_1} < \mu_{n_2} < \cdots < \mu_{n_m} \leq \cdots \) such that
\[ \mu_k - \mu_{n_m} > d_0 \left( k^{\frac{2\alpha}{\alpha+1}} - n_m^{\frac{2\alpha}{\alpha+1}} \right) \quad (k = n_m + 1, n_m + 2, \ldots), \]
where \( d_0 \) is a positive constant.

In this work, we find a formula for the limit
\[ \lim_{m \to \infty} \sum_{k=1}^{n_m} \left( \lambda_k - \mu_k \right), \]
where \( \{\mu_{n_m}\}_{m=1}^{\infty} \) is a subsequence of \( \{\mu_n\}_{n=1}^{\infty} \), and it satisfies the inequality (1.2). This formula is said to be regularized trace formula.

The first study about the regularized trace formula for scalar differential operators is started with [7]. And after this work, on regularized trace of various scalar differential operator have been studied in the works [5], [9], [12]. The list of the works on this subject is given in [11] and [6]. The formulas for the regularized traces of differential operators with operator coefficient is investigated in [3] and in many other works.

2. The Finite Sums of the Subtraction of the Eigenvalues and Some Formulas about the Resolvents

Let \( R_0^\lambda = (L_0 - \lambda I)^{-1}, \) \( R_\lambda = (L - \lambda I)^{-1} \) be the resolvents of the operators \( L_0 \) and \( L \), respectively. By the formula (1.1), the series
\[ \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{\mu_k - \lambda} \]
are uniform convergent for \( \alpha > 2 \) and \( \lambda \neq \lambda_k, \mu_k \) \( (k = 1, 2, \ldots) \). In this case \( R^0_\lambda, R_\lambda \) are kernel operators and

\[
tr(R_\lambda - R^0_\lambda) = trR_\lambda - trR^0_\lambda = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda} \right),
\]

see [4]. If the last equality is multiplied by \( \frac{1}{2\pi i} \) and integrated on the circle \( |\lambda| = b_m = \frac{1}{2}(\mu_{n_m+1} + \mu_{n_m}) \), then we obtain

\[
\frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda tr(R_\lambda - R^0_\lambda) d\lambda = \sum_{k=1}^{\infty} \left( \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{\lambda}{\lambda_k - \lambda} d\lambda \right) - \sum_{k=1}^{\infty} \left( \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{\lambda}{\mu_k - \lambda} d\lambda \right).
\] (2.1)

It easy to see that for the large value of \( m \) the following inequality is satisfied:

\[
\lambda_{n_m} < b_m < \lambda_{n_m+1}.
\] (2.2)

By (2.1) and (2.2) we find

\[
\sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda tr(R_\lambda - R^0_\lambda) d\lambda.
\] (2.3)

This is a well known formula for the resolvents of the operators \( L_0 \) and \( L \):

\[
R_\lambda = R^0_\lambda - R_\lambda QR^0_\lambda \quad (\lambda \in \rho(L) \cap \rho(L_0)).
\]

By using the last formula, for every natural number \( p \geq 2 \) it can be shown that

\[
R_\lambda - R^0_\lambda = \sum_{j=1}^{p} (-1)^j R^0_\lambda (QR^0_\lambda)^j + (-1)^{p+1} R_\lambda (QR^0_\lambda)^{p+1}.
\] (2.4)

If we substitute (2.4) in (2.3), then we obtain

\[
\sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \sum_{j=1}^{p} D_{m,j} + D^{(p)}_m,
\]

where

\[
D_{m,j} = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_m} tr \left[ R^0_\lambda (QR^0_\lambda)^j \right] d\lambda \quad (j = 1, 2, \ldots),
\]

\[
D^{(p)}_m = \frac{(-1)^p}{2\pi i} \int_{|\lambda|=b_m} \lambda tr \left[ R_\lambda (QR^0_\lambda)^{p+1} \right] d\lambda.
\] (2.5)
We can show that the following formula is satisfied for $D_{mj}$:

$$D_{mj} = \frac{(-1)^j}{2\pi i j} \int_{|\lambda|=b_m} \text{tr} \left[ (Q R_\lambda^0)^j \right] d\lambda \quad (j = 1, 2, \ldots). \quad (2.6)$$

Let $\{\Psi_q(x)\}_{q=1}^\infty$ be the orthonormal eigenvectors corresponding to the eigenvalues $\{\mu_q\}_{q=1}^\infty$, respectively.

Since these orthonormal eigenvectors are $\sqrt{2} \pi \cos(k + \frac{1}{2}) x \varphi_j \quad (k = 0, 1, 2, \ldots; \quad j = 1, 2, \ldots)$ corresponding to the eigenvalues $\left(k + \frac{1}{2}\right)^2 + \gamma_j \quad (k = 0, 1, 2, \ldots; \quad j = 1, 2, \ldots)$, then we have

$$\Psi_q(x) = \sqrt{2} \pi \cos(k_q + \frac{1}{2}) x \varphi_j \quad (q = 1, 2, \ldots). \quad (2.7)$$

**Theorem 2.1.** If $\gamma_j \sim a j^\alpha \quad (a > 0, \alpha > 2)$ as $j \to \infty$ and $Q(x)$ satisfies the conditions $a$, $b$, and $c$, then

$$D_{m1} = \frac{1}{4} (\text{tr}(Q(0)) - \text{tr}(Q(\pi))).$$

**Proof.** According to the formula (2.6)

$$D_{m1} = -\frac{1}{2\pi i} \int_{|\lambda|=b_m} \text{tr}(Q R_\lambda^0) d\lambda. \quad (2.8)$$

Since $Q R_\lambda^0$ is a kernel operator for every $\lambda \in \rho(L_0)$ and $\{\Psi_q(x)\}_{1}^\infty$ is a orthonormal basis of the space $H_1$, then we have

$$\text{tr}(Q R_\lambda^0) = \sum_{q=1}^\infty (Q R_\lambda^0 \Psi_q, \Psi_q),$$

see [4]. If we substitute this expression in (2.6) and consider the equality

$$R_\lambda^0 \Psi_q = (L_0 - \lambda I)^{-1} \Psi_q = (\mu_q - \lambda)^{-1} \Psi_q,$$

then we find

$$D_{m1} = -\frac{1}{2\pi i} \int_{|\lambda|=b_m} \left[ \sum_{q=1}^\infty (Q R_\lambda^0 \Psi_q, \Psi_q) \right] d\lambda$$

$$= -\frac{1}{2\pi i} \int_{|\lambda|=b_m} \left[ \sum_{q=1}^\infty (Q (\mu_q - \lambda)^{-1} \Psi_q, \Psi_q) \right] d\lambda$$

$$= \left[ \sum_{q=1}^\infty (Q \Psi_q, \Psi_q) \right] \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{d\lambda}{\lambda - \mu_q}. \quad (2.9)$$
By (2.7), (2.9) and by the formulas
\[
\frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{d\lambda}{\lambda - \mu_q} = \begin{cases} 1, & q \leq n_m \\ 0, & q > n_m \end{cases},
\]
we obtain
\[
D_{m1} = \sum_{q=1}^{n_m} (Q\Psi_q, \Psi_q)
\]
\[
= \sum_{q=1}^{n_m} \int_0^\pi (Q(x)\Psi_q(x), \Psi_q(x))_H dx
\]
\[
= \sum_{q=1}^{n_m} \int_0^\pi (Q(x)\sqrt{\frac{2}{\pi}} \cos(k_q + 1) x, \varphi_{j_q})_H dx
\]
\[
= \frac{2}{\pi} \sum_{q=1}^{n_m} \int_0^\pi \cos^2(k_q + 1)x(Q(x)\varphi_{j_q}, \varphi_{j_q})_H dx
\]
\[
= \frac{1}{\pi} \sum_{q=1}^{n_m} \int_0^\pi (Q(x)\varphi_{j_q}, \varphi_{j_q})_H (1 + \cos(2k_q + 1) x) dx.
\] (2.10)

It is easy to see that if \(Q(x)\) satisfies the conditions a and b, then the series
\[
\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \int_0^\pi (Q(x)\varphi_j, \varphi_j)_H \cos(2k + 1)x dx
\] (2.11)
is absolute convergent. In this case, as known
\[
\lim_{m \to \infty} D_{m1} = \sum_{q=1}^{n_m} \int_0^\pi (Q(x)\varphi_{j_q}, \varphi_{j_q})_H \cos(2k_q + 1)x dx
\]
\[
\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \int_0^\pi (Q(x)\varphi_j, \varphi_j)_H \cos(2k + 1)x dx.
\]

If we take the limit of the equation (2.10) as \(m \to \infty\), and consider the last
relation, then we find
\[
\lim_{m \to \infty} D_{m1} = \frac{1}{\pi} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} (Q(x)\varphi_j, \varphi_j)_H \cos(2k+1)x \, dx
\]
\[
= \frac{1}{2\pi} \sum_{j=1}^{\infty} \left\{ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \ldots \sum_{k_l=1}^{\infty} \pi \int_{0}^{\pi} (Q(x)\varphi_j, \varphi_j)_H \cos k_1 x dx \cos(k_2 \pi) \ldots \cos(k_l \pi) \right\}
\]

The sums according to \( k \) on the right hand side of the last equality are the values at the point 0 and \( \pi \) of the Fourier series of the function \( (Q(x)\varphi_j, \varphi_j) \) having second order derivative according to the functions \{Cos\, \varphi\}_k \infty in the interval \([0, \pi]\) respectively. Therefore we obtain
\[
\lim_{m \to \infty} D_{m1} = \frac{1}{4} \sum_{j=1}^{\infty} \left[ (Q(0)\varphi_j, \varphi_j)_H + (Q(\pi)\varphi_j, \varphi_j)_H \right]
\]
or
\[
\lim_{m \to \infty} D_{m1} = \frac{1}{4} (\text{tr}(Q(0)) - \text{tr}(Q(\pi))).
\]

3. The Formulation of Regularized Trace

In this section, we find a formula for the regularized trace of the operator \( L \). In Section 2, we obtained that the formula
\[
\sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \sum_{j=1}^{p} D_{mj} + D_{mjq} \quad (3.1)
\]
is satisfied for the sum of subtraction of eigenvalue of the operators \( L \) and \( L_0 \). Here, it can be shown that
\[
D_{mj} = \frac{(-1)^j}{2\pi ij} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_j=1}^{\infty} *
\]
\[
\int_{|\lambda|=b_m} \left( \prod_{q=1}^{j} (\mu_{k_q} - \lambda)^{-1} \right) d\lambda \prod_{q=1}^{j} \left( Q\Psi_{k_q}, \Psi_{k_{(q)}+1} \right). \tag{3.2}
\]

The symbol "\(*\)" denotes that the numbers among \(\mu_{k_1}, \mu_{k_2}, \ldots, \mu_{k_j}\) can be greater or less than \(b_m\). Moreover,

\[
g(q) = \begin{cases} 
q, & \text{if } q < j, \\
0, & \text{if } q = j.
\end{cases}
\]

Let us limit \(\|R^0_\lambda\|_{\sigma_1(H_1)}\). Here, we assume that \(\gamma_j \sim aj^\alpha (a > 0, \alpha > 2)\) as \(j \to \infty\). Since \(R^0_\lambda\) is a normal operator for every \(\lambda \notin \{\mu_k\}_{k=1}^\infty\), we have

\[
\|R^0_\lambda\|_{\sigma_1(H_1)} = \sum_{k=1}^{\infty} \frac{1}{|\mu_k - \lambda|},
\]

see [4]. On the circle \(|\lambda| = b_m = 2^{-1}(\mu_{n_m+1} + \mu_{n_m})\), we have

\[
\|R^0_\lambda\|_{\sigma_1(H_1)} \leq \sum_{k=1}^{\infty} \frac{1}{|\lambda - \mu_k|} = \sum_{k=1}^{n_m} \frac{1}{\mu_{n_m} + \mu_{n_m+1} - 2\mu_k} + \sum_{k=n_m+1}^{\infty} \frac{2}{2\mu_k - \mu_{n_m} - \mu_{n_m+1}} < \sum_{k=1}^{n_m} \frac{2}{\mu_{n_m+1} - \mu_k} + \sum_{k=n_m+1}^{\infty} \frac{2}{\mu_k - \mu_{n_m}}. \tag{3.3}
\]

By using (2.1) and (3.3), we find

\[
\sum_{k=1}^{n_m} \frac{1}{\mu_{n_m+1} - \mu_k} < \sum_{k=n_m+1}^{\infty} \frac{1}{\mu_{n_m+1} - \mu_{n_m}} < \frac{n_m}{d_0[(n_m + 1)^{1+\delta} - n_m^{1+\delta}]} < d_0^{-1} n_m^{1-\delta}, \tag{3.4}
\]

where \(\delta = \frac{2\alpha}{2+\alpha} - 1\).

From the inequality (1.2) again, we obtain

\[
\sum_{k=n_m+1}^{\infty} \frac{1}{\mu_k - \mu_{n_m}} < \sum_{k=n_m+1}^{\infty} \frac{1}{d_0(k^{1+\delta} - n_m^{1+\delta})} = \frac{1}{d_0((n_m + 1)^{1+\delta} - n_m^{1+\delta})} + \frac{1}{d_0} \sum_{k=n_m+2}^{\infty} \frac{1}{k^{1+\delta} - n_m^{1+\delta}}. \tag{3.5}
\]
Moreover,
\[
\sum_{k=n_m+2}^{\infty} \frac{1}{k^{1+\delta} - n_m^{1+\delta}} = \sum_{i=1}^{n_m+1} \int_{n_m+i}^{\infty} \frac{dx}{(n_m+i+1)^{1+\delta} - n_m^{1+\delta}} \\
< \sum_{i=1}^{n_m+1} \int_{n_m+i}^{\infty} \frac{dx}{x^{1+\delta} - n_m^{1+\delta}} \\
= \int_{n_m+1}^{\infty} \frac{dx}{x^{1+\delta} - n_m^{1+\delta}}.
\]

(3.6)

Let us evaluate the integral at the end of the relation (3.6) using the substitution \(x^{1+\delta} - n_m^{1+\delta} = t\), we find
\[
\int_{n_m+1}^{\infty} \frac{dx}{x^{1+\delta} - n_m^{1+\delta}} = \int_{\alpha_m}^{\infty} \frac{1}{(1+\delta)t} (t + n_m^{1+\delta})^{\frac{\delta}{1+\delta}} dt \\
< \frac{1}{1+\delta} \int_{\alpha_m}^{\infty} t^{-1} \frac{dt}{t^{\frac{\delta}{1+\delta}}} = \delta^{-1}[n_m^{1+\delta} - n_m^{1+\delta}]^{\frac{\delta}{1+\delta}},
\]

(3.7)

where \(\alpha_m = (n_m+1)^{1+\delta} - n_m^{1+\delta}\).

From the inequalities (3.5) and (3.7), we obtain
\[
\sum_{k=n_m+1}^{\infty} \frac{1}{\mu_k - \mu_{n_m}} < \frac{1+\delta}{d_0\delta} n_m^{-\frac{\delta^2}{1+\delta}}.
\]

(3.8)

By using (3.3), (3.4) and (3.8), we have
\[
\| R_\lambda^\delta \|_{\ell_1(H)} < d_1 n_m^{1-\delta} \quad (|\lambda| = b_m, \alpha > 2),
\]

(3.9)

where \(d_1 > 0\) is a constant.

Let us limit the norm of the operator \(R_\lambda\) on the circle \(|\lambda| = b_m\). For \(|\lambda| = b_m\), we have
\[
|\lambda_k| - |\lambda| = |\lambda_k| - \frac{1}{2}(\mu_{n_m+1} + \mu_{n_m}) = \frac{1}{2}|\mu_{n_m+1} + \mu_{n_m} - 2|\lambda_k| |.
\]

(3.10)

For the large values of \(m\), we have
\[
\mu_{n_m} + \mu_{n_m+1} - 2|\lambda_k| \geq \mu_{n_m} + \mu_{n_m+1} - 2|\lambda_{n_m}| \\
= \mu_{n_m+1} - \mu_{n_m} + 2(\mu_{n_m} - \lambda_{n_m}) \\
\geq \mu_{n_m+1} - \mu_{n_m} - 2|\mu_{n_m} - \lambda_{n_m}| \\
\geq \mu_{n_m+1} - \mu_{n_m} - 2\| Q \|.
\]

(3.11)
We also have $k \geq n_m + 1$ and for the large values of $m$

\[2|\lambda_k| - \mu_{n_m} - \mu_{n_m+1} \geq 2\lambda_{n_m+1} - \mu_{n_m} - \mu_{n_{m+1}}
= 2(\lambda_{n_m+1} - \mu_{n_m+1}) + \mu_{n_m+1} - \mu_{n_m}
\geq \mu_{n_m+1} - \mu_{n_m} - 2|\lambda_{n_m+1} - \mu_{n_m+1}|
\geq \mu_{n_m+1} - \mu_{n_m} - 2\|Q\|.
\] (3.12)

If we consider $\lim_{m \to \infty} (\mu_{n_m+1} - \mu_{n_m}) = \infty$, then from (3.10), (3.11) and (3.12), we obtain

\[|\lambda_k| - |\lambda| > \frac{1}{4}(\mu_{n_m+1} - \mu_{n_m}) \quad (|\lambda| = b_m).
\] (3.13)

By (1.2) and (3.13), we find

\[|\lambda_k - \lambda| > \frac{d_0}{4n_m}\delta
\] (3.14)

for the large values of $m$. On the other hand, since the $s$ numbers of the operator $R_\lambda$ are $\{|\lambda_k - \lambda|^{-1}\}_{k=1}^\infty$, we have

\[\|R_\lambda\| = \max_k\{|\lambda_k - \lambda|^{-1}\}_{k=1}^\infty,
\] (3.15)

see [4]. From (3.14) and (3.15), for the large values of $m$, we obtain

\[\|R_\lambda\| < \frac{4}{d_0n_m}\delta \quad (|\lambda| = b_m; \alpha > 2).
\] (3.16)

**Theorem 3.1.** If the operator function $Q(x)$ satisfies the conditions a, b and $\gamma_j \sim aj^\alpha$ ($a > 0, \alpha > 2$) as $j \to \infty$ then we have

\[\lim_{m \to \infty} D_{mj} = 0 \quad (j = 2, 3, 4, \ldots).
\]

**Proof.** If $T \in \sigma_1(H)$ and the operator $S : H \to H$ is linear bounded, then $ST \in \sigma_1(H)$ and

\[|trT| \leq \|T\|_{\sigma_1(H)},
\] (3.17)

\[\|ST\|_{\sigma_1(H)} \leq \|S\|_{H}\|T\|_{\sigma_1(H)},
\] (3.18)

see [4]. By using (2.6), (3.17) and (3.18), we obtain

\[|D_{mj}| \leq \frac{1}{2\pi j} \int_{|\lambda|=b_m} |tr(QR_\lambda^0)^j| |d\lambda|
\leq \int_{|\lambda|=b_m} \|Q_\lambda^0\|_{\sigma_1(H)} |d\lambda|
\leq \int_{|\lambda|=b_m} \|QR_\lambda^0\|_{\sigma_1(H)} \|Q_\lambda^0\|^{-1} |d\lambda|
\]
\[ \leq \int_{|\lambda| = b_m} \| Q \| \| R_\lambda^0 \| \sigma_1(\mathcal{H}_1) \| (QR_\lambda^0)^{j-1} \| d\lambda \]
\[ \leq \| Q \| \int_{|\lambda| = b_m} \| R_\lambda^0 \| \sigma_1(\mathcal{H}_1) \| QR_\lambda^0 \|^{j-1} d\lambda \]
\[ \leq \| Q \| \int_{|\lambda| = b_m} \| R_\lambda^0 \| \sigma_1(\mathcal{H}_1) \| R_\lambda^0 \|^{j-1} d\lambda. \quad (3.19) \]

Since \( R_\lambda = R_\lambda^0 \) for \( Q(x) = 0 \) according to (3.16), we obtain
\[ \| R_\lambda^0 \| < \frac{4}{d_0} n_m^{-\delta} \quad (|\lambda| = b_m; \quad \delta = \frac{2\alpha}{2 + \alpha} - 1). \quad (3.20) \]

From (3.9), (3.19) and (3.20), we obtain
\[ |D_{mj}| < \text{const.} \int_{|\lambda| = b_m} n_m^{1-\delta} n_m^{-\delta(j-1)} d\lambda \]
\[ < \text{const.} b_m n_m^{1-\delta j} < \text{const.} n_m^{2-\delta(j-1)}. \]

From here, it seems
\[ \lim_{m \to \infty} D_{mj} = 0 \quad (j > 1 + 2\delta^{-1}). \]

It is necessary to show that the last equality is satisfied for \( 2 \leq j \leq 1 + 2\delta^{-1} \) to complete the proof. By using the formula (3.2), we have the following equality for \( D_{m2} \):
\[ D_{m2} = \frac{1}{4\pi i} \sum_{j=1}^{n_m} \sum_{k=n_m+1}^{\infty} \left[ \int_{|\lambda| = b_m} \frac{1}{(\lambda - \mu_j)(\lambda - \mu_k)} d\lambda \right] (Q\Psi_j, \Psi_k)(Q\Psi_k, \Psi_j) \]
\[ + \frac{1}{4\pi i} \sum_{j=n_m+1}^{\infty} \sum_{k=1}^{n_m} \left[ \int_{|\lambda| = b_m} \frac{1}{(\lambda - \mu_j)(\lambda - \mu_k)} d\lambda \right] (Q\Psi_j, \Psi_k)(Q\Psi_k, \Psi_j). \quad (3.21) \]

For \( \mu_j < b_m \) and \( \mu_k > b_m \) \((j \leq n_m \) and \( k \geq n_m + 1)\), we have
\[ \frac{1}{2\pi i} \int_{|\lambda| = b_m} \frac{1}{(\lambda - \mu_j)(\lambda - \mu_k)} d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = b_m} \frac{1}{(\mu_j - \mu_k)} \left[ \frac{1}{\lambda - \mu_j} - \frac{1}{\lambda - \mu_k} \right] d\lambda \]
\[ = \frac{1}{\mu_j - \mu_k}. \quad (3.22) \]
By (3.21) and (3.22), we obtain

\[
|D_{m2}| = \left| \sum_{j=1}^{n_m} \sum_{k=n_m+1}^\infty (\mu_j - \mu_k)^{-1} (\Psi_j, Q \Psi_k)(\Psi_j, Q \Psi_k) \right|
\]

\[
\leq \sum_{j=1}^{n_m} \sum_{k=n_m+1}^\infty (\mu_k - \mu_{n_m})^{-1} \left| (\Psi_j, Q \Psi_k) \right|^2
\]

\[
\leq \sum_{k=n_m+1}^\infty \left[ (\mu_k - \mu_{n_m})^{-1} \sum_{j=1}^{\infty} \left| (Q \Psi_k, \Psi_j) \right|^2 \right]
\]

\[
= \sum_{k=n_m+1}^\infty (\mu_k - \mu_{n_m})^{-1} \| Q \Psi_k \|^2
\]

\[
\leq \| Q \|^2 \sum_{k=n_m+1}^\infty (\mu_k - \mu_{n_m})^{-1}.
\]  \hspace{1cm} (3.23)

From (3.8) and (3.23), we find

\[
\lim_{m \to \infty} D_{m2} = 0.
\]

Similarly, we can show that

\[
\lim_{m \to \infty} D_{mj} = 0
\]

for \(3 \leq j \leq 1 + 2\delta^{-1}\).

**Theorem 3.2.** If \(Q(x)\) satisfies the conditions a, b, c and \(\gamma_j \sim a_j^\alpha\) \((a > 0, \alpha > 2)\) as \(j \to \infty\), then the formula

\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4} \left[ trQ(0) - trQ(\pi) \right]
\]

is satisfied.

**Proof.** By the relations (2.5), (3.17) and (3.18), we obtain

\[
|D_{m}^{(p)}| \leq \frac{1}{2\pi} \int_{|\lambda|=b_m} |\lambda| \cdot |tr(R_\lambda(QR_\lambda^0)^{p+1})| \cdot |d\lambda|
\]

\[
\leq b_m \int_{|\lambda|=b_m} \| R_\lambda(QR_\lambda^0)^{p+1} \| \sigma_1(H_1) |d\lambda|
\]

\[
\leq b_m \int_{|\lambda|=b_m} \| R_\lambda \| \| (QR_\lambda^0)^{p+1} \| \sigma_1(H_1) |d\lambda|
\]
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\[ \leq b_m \int_{|\lambda|=b_m} \| R_\lambda \| (QR_\lambda^0)^p \| Q R_\lambda^0 \|_{\sigma_1(H_1)} \| d\lambda \|
\leq b_m \int_{|\lambda|=b_m} \| R_\lambda \| QR_\lambda^0 \| Q \| R_\lambda^0 \|_{\sigma_1(H_1)} \| d\lambda \|
\leq b_m \int_{|\lambda|=b_m} \| R_\lambda \| QR_\lambda^0 \| Q \|^{p+1} \| R_\lambda^0 \|_{\sigma_1(H_1)} \| d\lambda \|. \quad (3.24) \]

By using (3.9), (3.16), (3.20) and (3.24), we find

\[ D_m^{(p)} \leq \text{const.} b_m \int_{|\lambda|=b_m} n_m^{-(p+1)\delta} n_m^{1-\delta} | d\lambda | s \leq \text{const.} b_m^2 n_m^{-p\delta-2\delta+1}. \quad (3.25) \]

If we consider that

\[ b_m = \frac{1}{2}(\mu_{n_m+1} + \mu_{n_m}) \leq \text{const.} n_m^{1+\delta}, \]

for the large values of \( m \) then from (3.25), we obtain

\[ | D_m^{(p)} | \leq \text{const.} n_m^{3-p\delta}. \]

From here, it seems that

\[ \lim_{m \to \infty} D_m^{(p)} = 0 \quad (p > 3\delta^{-1}). \quad (3.26) \]

By Theorem 2.1, Theorem 3.1 and by the formulas (3.1), (3.26), we find

\[ \lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4} [\text{tr} Q(0) - \text{tr} Q(\pi)]. \]

The theorem is proved. \( \square \)

References


[3] E.E. Adıgüzelov, P. Kanar, The regularized trace of a second order differ-


