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ON THE RANKS WITH RESPECT TO CANONICAL OR REDUCIBLE CURVES IN $\mathbb{P}^4$

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $X \subset \mathbb{P}^4$ be a canonically embedded general trigonal curve. Here we prove that every points of $\mathbb{P}^4$ has X-rank at most 3. We give similar statements for certain reducible curves in $\mathbb{P}^n$.

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1. Introduction

Fix an integral and non-degenerate variety $X \subseteq \mathbb{P}^n$ defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \rangle$ denotes the linear span. Hence $r_X(P) = 1$ if and only if $P \in X$. Since X is non-degenerate, the X-ranks are defined and $r_X(P) \le n+1$ for all $P \in \mathbb{P}^n$. Set $\rho(X) := \max_{P \in \mathbb{P}^n}\{r_X(P)\}$. Now we drop the assumption that X is irreducible and call $X_1, \dots, X_s$ its irreducible components. We may define the integers $r_X(P)$, $P \in \mathbb{P}^n$ as in the case $s = 1$. For any $P \in \mathbb{P}^n$ set $\tau_X(P) := \min_{i=1}^{s} r_{X_i}(P)$, with the convention $r_{X_i}(P) = +\infty$ if $P \notin \langle X_i \rangle$.

Now assume again that X irreducible. Take $n = g+1$ and assume that X is a smooth and non-hyperelliptic canonically embedded curve of genus $g \ge 4$. Looking at the first integer k such that $\mathbb{P}^{g-1}$ is the k-th secant variety of X,

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we get $r_X(P) = \lceil g/2 \rceil$ for a general $P \in \mathbb{P}^{g-1}$ (see [1], Remark 1.6). Thus $\rho(X) \ge \lceil g/2 \rceil$. By [5], Proposition 5.1 (case char$(\mathbb{K}) = 0$) or by [2] (case char$(\mathbb{K}) > 0$) we have $\rho(X) \le g - 1$. Let $\mathcal{H}_g \subset \mathcal{M}_g$ denote the hyperelliptic locus. For any $Y \in \mathcal{M}_g \backslash \mathcal{H}_g$, let $j_K : Y \to \mathbb{P}^{g-1}$ be the canonical embedding. Set $\eta(Y) := \rho(j_K(Y))$. Set

$$\eta_g := \min_{Y \in \mathcal{M}_g \backslash \mathcal{H}_g} \eta(Y), \quad \widetilde{\eta}_g : \max_{Y \in \mathcal{M}_g \backslash \mathcal{H}_g} \eta(Y).$$

Since the function r_X is constructible, there is a non-empty open subset U of $\mathcal{M}_g$ on which the function $\eta : U \to \mathbb{Z}$ is constant. Set $\rho_g := \eta(Y)$ for any $Y \in U$ (the general value of the function η or the value of η at a general curve of genus g). Notice that if $g = 4$ (canonical space curves), the bounds just given imply $\eta(Y) \in \{2, 3\}$. R. Piene proved that both values are achieved (hence $\eta_4 = 2$ and $\widetilde{\eta}_4 = 3$) and that $\rho_4 = 2$ (see [7], Theorem 2 and first Example at pp. 108–109).

Question 1. Compute the integers η_g, $\widetilde{\eta}_g$ and ρ_g for all g (or at least give very strong restrictions for their values). Study the stratification by the integer $\eta(Y)$ of $\mathcal{M}_g \backslash \mathcal{H}_g$.

Here we look at the case $g = 5$ and prove the following result.

Theorem 1. *Assume* char$(\mathbb{K}) \ne 2, 3, 5$. *Let $X \subset \mathbb{P}^4$ be the canonical embeddeding of a general trigonal curve of genus 5. Then $\rho(X) = 3$.*

As an immediate consequence of Theorem 1 we get $\eta_5 = 3$, because $\eta(X) \in \{3, 4\}$ for every smooth and non-degenerate curve of $\mathbb{P}^4$ (see [5], Proposition 5.1, in characteristic zero, [2] in positive characteristic).

Then we look at reducible curves. Certain binary curves where used to study the canonical embedding of general smooth curve of arbitrary genus (see [6]). This was our motivation to consider nodal unions of two rational normal curves. We prove the following results.

Theorem 2. *Fix an integer s such that $0 \le s \le 6$. Let $X \subset \mathbb{P}^4$ be a general nodal curve with two irreducible components, C_1 and C_2, each of them a rational normal curve, such that $\sharp(C_1 \cap C_2) = s$ (with the convention $C_1 \cap C_2 = \emptyset$ if $s = 0$). Then $r_X(P) \le 3$ for all $P \in \mathbb{P}^4$.*

Theorem 3. *Assume* char$(\mathbb{K}) = 0$. *Fix integers n, s, k such that $n \ge 5$, $0 \le s \le n$ and $(3n + 1)/4 \le k \le n - 1$. Let $X = C_1 \cup C_2$ be the general nodal curve with two irreducible components, C_1, C_2, both of them rational normal curves, and $\sharp(C_1 \cap C_2) = s$. Then for each $P \in \mathbb{P}^n \backslash \langle C_1 \cap C_2 \rangle$ there is $i \in \{1, 2\}$ such that $r_{C_i}(P) \le k$.*

In the set-up of Theorem 3 we obviously have $r_{C_i}(P) \le s$ for all $i \in \{1, 2\}$

and all $P \in \langle C_1 \cap C_2 \rangle$.

2. The Proofs

Proof of Theorem 1. Since X is trigonal, it is the scheme-theoretic intersection of a smooth degree 3 surface $S \subset \mathbb{P}^4$ and a quadric hypersurface. The surface S is isomorphic to the Hirzebruch surface F_1. Let $\pi : S \to \mathbb{P}^1$ denote its ruling. Thus $\pi|X$ is a g^1_3 on X. Thus $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 2}$ and we may take as a basis of $\mathrm{Pic}(S)$ any fiber f of the ruling of S and a section of the ruling with negative self-intersection. We have $h^2 = -1$, $h \cdot f = 1$, $f^2 = 0$, h is a line, every element of $|f|$ is a lineand S contains no other line. It is easy to check using the adjunction formula that $X \in |3h + 5f|$. Notice that $\dim(|h + f|) = 2$ (use the projection formula) and that an element of $|h + f|$ is either a smooth conic or a reducible, but reduced, conic, union of h and an element of $|f|$. To prove Theorem 1 we may assume $P \notin X$.

(a) Here we assume $P \in S \backslash X$. Let F_P be the fiber of π containing P. Notice that $X \cdot (h + f) = 5$. Set $V_P := \{E \in |h + f| : P \in E\}$.

Claim. *V_P is base point free outside P.*

Proof of the Claim. Fix $Q \in S \setminus \{P\}$. If $Q \notin (h \cup F_P)$, then $h + F_P$ is an element of V_P not containing Q. Now assume $Q \in F_P$. Since $\pi_*(\mathcal{O}_S(h)) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, we have $h^1(S, \mathcal{O}_S(h)) = 0$. Thus the restriction to F_P of the linear system $|h + f|$ is the complete degree 1 linear system on $F_P \cong \mathbb{P}^1$. Hence the restriction of V_P to F_P separates P from any other point of F_P. Now assume $Q \in h$. Since $h^1(S, \mathcal{O}_S(f)) = 0$, the restriction to h of the linear system $|h + f|$ is the complete degree 1 linear system on $h \cong \mathbb{P}^1$. Hence the restriction of V_P to h separates P from any other point of h, concluding the proof of the claim.

By the claim V_P induces a base point free linear system W_P on X. First assume $\mathrm{char}(\mathbb{K}) = 0$. Since we are in characteristic zero, a general element $E \in V_P$ intersects X at 5 distinct points. Any 3 of the points of $E \cap X$ spans $\langle E \rangle$. Hence $r_X(P) \leq 3$. If $p := \mathrm{char}(\mathbb{K}) > 0$, we use the separability of a certain map degree ≤ 5 morphism $X \to \mathbb{P}^1$, because $p > 5 = X \cdot (h + f)$.

(b) Here we assume $P \notin S$. Only here we assume that X is a general member of $|3h + 5f|$.

(b1) For a general trigonal curve X its degree 3 covering $X \to \mathbb{P}^1$ has no total ramification point. Hence $\sharp((X \cap F)_{red}) \geq 2$ for all $F \in |f|$ for a general $X \in |3h + 5f|$. Notice that $h \cdot (3h + 5f) = 2$. A general $X \in |3h + 5f|$ intersects transversally h (in arbitrary characteristic), because $h^1(S, \mathcal{O}_S(2h + 5f)) = 0$.

Thus (if X is general) $\sharp((E \cap X)_{red}) \geq 3$ for every singular (i.e. reducible) element E of $|h+f|$. Fix a smooth conic $E \in |h+f|$. Notice that $(h+f)\cdot(3h+5f) = 5$. Since $h^1(S, \mathcal{O}_S(2h+4f)) = 0$ a dimensional count show that the set of all $X \in |3h+5f|$ such that $\sharp((X \cap E)_{red}) \leq 2$ has codimension 3 in the projective space $|3h+5f|$. Since $dim(|h+f|) = 2$, we get that a general trigonal curve $X \in |3h+5f|$ has the following property: $\sharp((X \cap E)_{red}) \geq 3$ for all $E \in |h+f|$.

(b2) The surface $\ell_P(S)$ is a non-normal cubic. Hence it has a double line, which is the image by ℓ_P of the unique element $E \in |h+f|$ such that $P \in \langle E \rangle$. Hence either E is a smooth conic or $E = h+F$ with $F \in |f|$. We have length$(E \cap X) = 5$. In both cases it is easy to check that the set $(E \cap X)_{red}$ spans the plane $\langle E \rangle$ if and only if $\sharp((E \cap X)_{red}) \geq 3$. Hence step (b1) gives $r_X(P) \leq 3$. □

Proof of Theorem 2. Fix $P \in \mathbb{P}^4$ and assume $r_X(P) \geq 4$. Hence $r_{C_i}(P) \geq 4$ for all i. We have $r_{C_i}(P) = 4$ if and only if $r_{C_i}(P) \geq 4$ if and only if there is $Q_i \in C_i$ such that $P \in T_{Q_i}C_i \backslash \{Q_i\}$ (see [3] or [5], Theorem 5.1; for $n = 3, 4$, this is true in arbitrary characteristic). Thus we may assume the existence of $Q_1 \in C_1$ and $Q_2 \in C_2$ such that $P \in T_{Q_1}C_1 \cap T_{Q_2}C_2$ and $P \notin X$. For general X there are only finitely many pairs $(Q_1, Q_2) \in (C_1, C_2)$ such that $T_{Q_1}C_1 \cap T_{Q_2}C_2 \cap (\mathbb{P}^4 \backslash X) \neq \emptyset$. Set $D_i := \ell_P(C_i)$, $Y := \ell_P(Y)$ and $v := \ell_P|X$. If v is not injective, then $r_X(P) \leq 2$. Hence we may assume the injectivity of v. Each D_i is a cuspidal complete intersection of two quadric surfaces, Y has $s+2$ singular points and every singular point of Y as a plane as its Zariski tangent space. Fix a general $A \in D_1$. Let $E \subset \mathbb{P}^4$ be a general rational normal curve containing $C_1 \cap C_2$. Since $s \leq 6$ we may take E passing through a general point of $\mathbb{P}^4$. Hence taking a general E as above instead of C_1 (but keeping C_2 and $C_1 \cap C_2$ fixed) we may assume that D_1 is not contained in the tangent developable Γ of D_2. Fix a general $A \in D_1$. Hence $A \notin D_2$. We just saw that $A \notin \Gamma$. Set $u := \ell_A|D_2$. Since the singular point of D_2 is a planar singularity, we get that $u : D_2 \to \mathbb{P}^2$ is unramified and generically injective. Since $\deg(u(D_2)) = 4$, we have $p_a(u(D_2)) > 1 = p_a(D_2)$. Hence u is not injective. Take $A_1, A_2 \in D_2$ such that $u(A_1) = u(A_2)$ and $A_1 \neq A_2$. Since $u(A_1) = u(A_2)$ and $A \notin D_2$, the points A, A_1, A_2 are distinct and collinear. Take $B \in C_1$ and $B_1, B_2 \in C_2$ such that $v(B) = A$, $v(B_1) = A_1$ and $v(B_2) = A_2$. Since the points A, A_1, A_2 are collinear, the points P, B_1, B_2, B_3 spans at most a plane. Thus to get $r_X(P) \leq 3$ it is sufficient to prove that for general A the points B, B_1 and B_2 are not collinear. Assume that B, B_1 and B_2 are collinear. Since as B we may be take a general point of C_1, while B_1 and B_2 are distinct points of C_2 we get that C_1 is contained in the secant variety of C_2. This is false far a general rational normal curve containg $x \le 6$ general points of C_2. □

For any integral and non-degenerate variety $Y \subset \mathbb{P}^n$ and every integer integer $k \ge 1$ set $E_Y(k) := \{P \in X : r_Y(P) = k\}$ and $F_X(k) := \cup_{y \ge k} E_Y(y)$. Each set $E_Y(k)$ and $F_y(k)$ is constructible. For any constructible subset $A \subseteq \mathbb{P}^n$ let $\dim(A)$ denote the maximal dimension of an irreducible component of $\overline{A}$.

Remark 1. Assume $\mathrm{char}(\mathbb{K}) = 0$. Let $C \subset \mathbb{P}^n$, $n \ge 3$, be a rational normal curve. For any integer k such that $\ge \lceil (n+2)/2 \rceil \le k \le n$ we have a precise description of the sets $E_C(k)$ and $F_C(k)$ (see [3] or [5], Theorem 5.1). This description gives $\dim(E_C(k)) = \dim(F_C(k)) = 2n + 2 - 2k$.

Proof of Theorem 3. It is sufficient to prove $F_{C_1}(k+1) \cap F_{C_2}(k+1) = \emptyset$. Since $2n+2-2(k+1) = 2n-2k \ge (n-1)/2$, Remark 3 gives $\dim(F_{C_i}(P)) \le (n-1)/2$, $i = 1, 2$. Fix C_1 and a general $S \subset C_1$ such that $\sharp(S) = s$. Let G_S be the set of all $h \in \mathrm{Aut}(\mathbb{P}^n)$ such that $h(Q) = Q$ for all $Q \in S$. Since $s \le n$, the set $\mathbb{P}^n \backslash \langle S \rangle$ is an open orbit of G_S. Fix any rational normal curve $D \subset \mathbb{P}^n$ such that $C_1 \cap D = S$ and D intersects transversally C_1. It is sufficient to prove $h(F_D(k+1)) \cap F_{C_1}(k+1) = \emptyset$ for general $h \in G_S$. By Kleiman's form of Bertini's Theorem (see [4], part (i) of Theorem 2; notice that it does not require the completeness of the homogeneous variety) and the inequality $\dim(F_D(k+1)) + \dim(F_{C_1}(k+1)) < n$, we have $h(F_D(k+1)) \cap F_{C_1}(k+1) \cap (\mathbb{P}^n \backslash \langle S \rangle) = \emptyset$. □

We leave to the interested reader to extend Theorem 3 to more components and/or the case in which some of the curves have higher genera, using the results on the X-ranks for curves with positive genera which are appearing in the literature.

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