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MODULUS HYPERINVARIANT CLOSED IDEALS FOR QUASINILPOTENT OPERATORS WITH MODULUS ON l_p-SPACES

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Abstract: In this paper, it is proved that every non-zero continuous operator with modulus on an l_p-space whose modulus is quasinilpotent at a non-zero positive vector has a non-trivial modulus hyperinvariant closed ideal.

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In 1954, N. Aronszajn and K.T. Smith [5] showed that every compact operator on a Banach space has a non-trivial invariant closed subspace.

But it was not until 1986 that people solved the invariant closed ideal problem for a special kind of compact operators. To be more precise, in 1986, B. de Pagter [11] proved the long standing conjecture that every positive quasinilpotent compact operator on a Banach lattice has a non-trivial invariant closed ideal.

In 1993, Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw [2] showed the following theorem.

Theorem A. *Let $S : l_p \to l_p (1 \leq p < \infty)$ be a continuous operator with modulus. If there exists a non-zero positive operator $T : l_p \to l_p$ such that:*

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(1) T commutes with the modulus of S, and

(2) T is quasinilpotent at a non-zero positive vector.

Then S has a non-trivial invariant closed subspace.

In this paper, using the Abramovich-Aliprantis-Burkinshaw technique based on the idea from [1], [2], [3] and so on, we obtain an invariant closed ideal theorem for a large class of operators. To be more precise, we show that every non-zero continuous operator on an l_p-space whose modulus is quasinilpotent at a non-zero positive vector has a non-trivial modulus hyperinvariant closed ideal. In particular, all continuous operators on an l_p-space whose modulus commute with a non-zero positive operator T that is quasinilpotent at a non-zero positive vector have a common non-trivial invariant closed ideal (hence they have a common non-trivial invariant closed subspace).

For a convenience of the reader, we first recall some basic notions and facts from [1], [2], [4], [11], and others. For the notation and terminology not explained in the text we refer to [2] and any standard book in this area.

Following [2], a continuous operator T on a Banach space X is said to be quasinilpotent at a vector $x \in X$ if $\lim_{n\to\infty} \|T^n x\|^{1/n} = 0$.

Let E be a Banach lattice. A linear subspace $\mathcal{I}$ of E is said to be an (order) ideal whenever $|x| \leq |y|$ and $y \in \mathcal{I}$ imply that $x \in \mathcal{I}$. The ideal generated by a non-empty subset F of E is defined by $\mathcal{I}_F = \{x \in E;$ there are $x_1, \cdots, x_n \in F$ and $\lambda_1, \cdots, \lambda_n > 0$ with $|x| \leq \sum_{k=1}^n \lambda_k |x_k|\}$. In particular, the ideal generated by a singleton $\{y\}$ is given by

$$\mathcal{I}_y = \{x \in E; \text{there is } \lambda > 0 \text{ such that } |x| \leq \lambda |y|\}.$$

A positive vector $y \in E$ is called a quasi-interior point in E whenever $\mathcal{I}_y$ is norm dense in E, that is, $\overline{\mathcal{I}_y} = E$

If T and B are continuous operators on a Banach lattice E with B positive, then T is said to be dominated by B whenever $|Tx| \leq B(|x|)$ holds for all $x \in E$.

In addition, the proof of our main theorem will use the following lemma which is due to de Pagter [11].

Lemma 1. (see [11], Lemma 1) *Let E be a Banach lattice. If E admits a quasi-interior point and vectors $x, y \in E$ with $0 \leq y \leq |x|$, then there is a sequence $\{T_n\}$ of continuous operators on E such that $\|T_n x - y\| \longrightarrow 0$ as $n \longrightarrow \infty$, and $|T_n z| \leq |z|$ for every $n = 1, 2, \cdots$ and all $z \in E$.*

From now on, we only deal with the classical l_p-spaces over complex numbers, where $1 \leq p < \infty$. A vector $x = (x_1, x_2, \cdots, x_n, \cdots)$ in an l_p-space is said to be positive, in symbols $x \geq 0$, if its components are non-negative real numbers. The absolute value $|x|$ of x is the vector $|x| = (|x_1|, |x_2|, \cdots, |x_n|, \cdots)$. The symbol e_n will denote the vector in an l_p-space whose n-th component is one and every other zero. It is well known there is a sequence $\{f_n\}$ in the dual space of the l_p-space such that $f_n(e_m) = \delta_{nm}$.

It is well known that every positive operator on an l_p-space is automatically continuous ([4], Theorem 12.3).

A continuous operator $T : l_p \longrightarrow l_p$ with matrix $[t_{ij}]$ has modulus whenever the matrix $[|t_{ij}|]$ of absolute values also defines a continuous operator on the l_p-space. In this case, the operator defined by the matrix $[|t_{ij}|]$ is called the modulus of T and is denoted by $|T|$. If a continuous operator T on an l_p-space has a modulus, then we have $|Tx| \leq |T|(|x|)$ for each $x \in l_p$.

If T is a continuous operator with modulus on an l_p-space, then $\{T\}'_{\mathrm{m}}$ denotes the set of all continuous operators C with modulus on the l_p-space such that $|T||C| = |C||T|$.

Moreover, we say that a continuous operator T with modulus on an l_p-space has a non-trivial modulus hyperinvariant closed ideal if there exists a non-trivial closed ideal M of l_p such that M is invariant under $\{T\}'_{\mathrm{m}}$.

It is clear that every invariant closed ideal is necessarily an invariant closed subspace, but the converse is not true.

Now we are in a position to give the main result.

Theorem 1. *Let T be a non-zero continuous operator with modulus on an l_p-space. If the modulus of T is quasinilpotent at a non-zero positive vector $x_0 \in l_p$, then T has a non-trivial modulus hyperinvariant closed ideal.*

Proof. Since T is non-zero, it is clear that

$$\mathcal{I}_{\,\mathrm{ran}T} = \overline{\{x \in l_p;\ \text{there exists}\ \ y \geq 0 \text{ such that } |x| \leq |T|y\}}$$

is a non-zero closed ideal. Let $S \in \{T\}'_{\mathrm{m}}$, that is $|S||T| = |T||S|$. If $x \in \mathcal{I}_{\,\mathrm{ran}T}$, then there exists $x_n \in \{x \in l_p;\ \text{there exists}\ \ y \geq 0 \text{ such that } |x| \leq |T|y\}$ $(n = 1, 2, \ldots)$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Thus there are $y_n \geq 0$ $(n = 1, 2, \ldots)$ such that $|x_n| \leq |T|y_n$. Therefore we have

$$|Sx_n| \leq |S||x_n| \leq |S||T|y_n = |T||S|y_n,$$

and so $Sx_n \in \mathcal{I}_{\,\mathrm{ran}T}$. Hence $Sx \in \mathcal{I}_{\,\mathrm{ran}T}$. Consequently $\mathcal{I}_{\,\mathrm{ran}T}$ is a modulus, hyperinvariant closed ideal for T. If $\mathcal{I}_{\,\mathrm{ran}T} \neq l_p$, then $\mathcal{I}_{\,\mathrm{ran}T}$ is a non-trivial modulus hyperinvariant closed ideal for T. So, assume that $\mathcal{I}_{\,\mathrm{ran}T} = l_p$. Since the l_p-space is separable, there is a countable subset $\{z_1, z_2, \cdots, z_n, \cdots\}$ of the range of T consisting of non-zero vectors that is norm dense in ran$T$. Let

$z = \sum_{n=1}^{\infty} \frac{|z_n|}{2^n \|z_n\|}$. Then $z > 0$ and $\{z_1, z_2, \cdots, z_n, \cdots\} \subset \mathcal{I}_z$, where $\mathcal{I}_z$ denotes the ideal generated by z, and so ran$T \subset \overline{\mathcal{I}_z}$. Consequently we have $l_p = \mathcal{I}_{\text{ ran}T} \subset \overline{\mathcal{I}_z}$. This implies $\overline{\mathcal{I}_z} = l_p$. Thus l_p admits a quasi-interior point.

Since $x_0 > 0$, there is an appropriate scalar $\lambda > 0$ and a positive integer n_0 such that $\lambda x_0 \geq e_{n_0} > 0$. Since T is quasinilpotent at x_0, it is also quasinilpotent at λx_0. Let $\mathcal{A}$ be the algebra of all continuous operators on the l_p-space such that each $A \in \mathcal{A}$ is dominated by someone operator in the form of $\sum_{j=1}^{n} |C_j||T|^j$ with $C_j \in \{T\}_{\rm m}^{'}$.

(1). If $Ae_{n_0} = 0$ for all $A \in \mathcal{A}$, then $\mathcal{N}_A = \{x; A(|x|) = 0 \text{ for all } A \in \mathcal{A}\}$ is a non-zero closed ideal in the l_p-space, and $0 \neq T \in \mathcal{A}$ implies $\mathcal{N}_A \neq l_p$. It only remains to show that $\mathcal{N}_A$ is invariant under all operators in the modulus commutant $\{T\}_{\rm m}^{'}$ of T. To this end, take $x \in \mathcal{N}_A$ and $C \in \{T\}_{\rm m}^{'}$. For any $A \in \mathcal{A}$, it follows from the definition of $\mathcal{A}$ that there are operators $C_1, C_2, \cdots, C_n \in \{T\}_{\rm m}^{'}$ such that $|Ay| \leq \sum_{j=1}^{n} |C_j||T|^j(|y|)$ for all $y \in l_p$. Thus we have $|A(|Cx|)| \leq \sum_{j=1}^{n} |C_j||T|^j(|Cx|) \leq \sum_{j=1}^{n} |C_j||C||T|^j(|x|)$. Since $|C_j||C| \in \{T\}_{\rm m}^{'}$, it follows that $\sum_{j=1}^{n} |C_j||C||T|^j \in \mathcal{A}$. Thus for $x \in \mathcal{N}_A$ we obtain $\sum_{j=1}^{n} |C_j||C||T|^j(|x|) = 0$. Consequently we have $|A(|Cx|)| = 0$. This implies $A(|Cx|) = 0$ for all $A \in \mathcal{A}$, and so $Cx \in \mathcal{N}_A$. Consequently $\mathcal{N}_A$ is a non-trivial modulus hyperinvariant closed ideal for T.

(2). If there is an operator $A_0 \in \mathcal{A}$ such that $A_0 e_{n_0} \neq 0$, then $M = \{Ae_{n_0}; A \in \mathcal{A}\}$ is a non-zero linear subspace in the l_p-space. We now prove that $\overline{M}$ is an ideal in the l_p-space. Let $\mathcal{I}_M$ denote the ideal generated by M, and take $x \in \mathcal{I}_M$. Then there are $A_1, A_2, \cdots, A_n \in \mathcal{A}$ such that $|x| \leq \sum_{k=1}^{n} |A_k e_{n_0}|$, and so $|\text{Re}x| \leq \sum_{k=1}^{n} |A_k e_{n_0}|, |\text{Im}x| \leq \sum_{k=1}^{n} |A_k e_{n_0}|$, where Re$x$ and Imx denote the real part and imaginary part of x respectively. Hence $(\text{Re}x)^+ \leq \sum_{k=1}^{n} |A_k e_{n_0}|$. Thus we can write $(\text{Re}x)^+ = \sum_{k=1}^{n} x_k$ with $0 \leq x_k \leq |A_k e_{n_0}|$ for each k. Since l_p admits a quasi-interior point, it follows from Lemma 1 that for each $k = 1, 2, \cdots, n$, there is a sequence $\{T_{k,m}\}_{m=1}^{\infty}$ of continuous operators on the l_p-space such that $\|T_{k,m} A_k e_{n_0} - x_k\| \longrightarrow 0$ as $m \longrightarrow \infty$, and $|T_{k,m} y| \leq |y|$ for every $m = 1, 2, \cdots$ and all $y \in l_p$. Since $A_k \in \mathcal{A}$, it follows that there is $C_{k_j} \in \{T\}_{\rm m}^{'}$ such that

$$|T_{k,m} A_k y| \leq |A_k y| \leq \sum_{j=1}^{n} |C_{k_j}||T|^j(|y|)$$

for all $y \in l_p$, and so $T_{k,m} A_k \in \mathcal{A}$. It follows from above that $x_k \in \overline{M}$ for each k, and so $(\text{Re}x)^+ \in \overline{M}$. Similarly, $(\text{Re}x)^- \in \overline{M}, (\text{Im}x)^+ \in \overline{M}, (\text{Im}x)^- \in M$ and so $x = (\text{Re}x)^+ - (\text{Re}x)^- + i[(\text{Im}x)^+ - (\text{Im}x)^-] \in \overline{M}$. This shows that

$M \subset \mathcal{I}_M \subset \overline{M}$, which implies that $\overline{M} = \overline{\mathcal{I}_M}$. Since the closure of an ideal is an ideal, we may conclude that $\overline{M}$ is an ideal.

Next we prove that M is invariant under all operators in the modulus commutant $\{T\}'_{\rm m}$ of T. To this end, take $z \in M$ and $C \in \{T\}'_{\rm m}$. Then there is an operator $A \in \mathcal{A}$ such that $z = Ae_{n_0}$. It follows from the definition of $\mathcal{A}$ that there exist operators $C_1, C_2, \cdots, C_n \in \{T\}'_{\rm m}$ such that $|Au| \leq \sum_{j=1}^{n} |C_j||T|^j(|u|)$ for all $u \in l_p$. This implies $|CAu| \leq \sum_{j=1}^{n} |C||C_j||T|^j(|u|)$ for all $u \in l_p$. Since $|C||C_j| \in \{T\}'_{\rm m}$, we have $CA \in \mathcal{A}$, $Cz = CAe_{n_0} \in M$.

We now show that $\overline{M} \neq l_p$. Let P denote the natural projection from the l_p-space onto the linear subspace generated by e_{n_0}. It is clear that $0 \leq Pv \leq v$ holds whenever $0 \leq v \in l_p$. We claim that

$$P|C||T|^j e_{n_0} = 0 \tag{1}$$

for each $j \geq 0$ and each $C \in \{T\}'_{\rm m}$. To this end, we write $P|C||T|^j e_{n_0} = ae_{n_0}$ for some $a \geq 0$. Since P is a positive operator and the composition of positive operators is also a positive operator, it follows that the estimate

$$0 \leq a^k e_{n_0} = (P|C||T|^j)^k e_{n_0} \leq (|C||T|^j)^k e_{n_0} = |C|^k |T|^{jk}(\lambda x_0) \tag{2}$$

holds for every positive integer k. Since f_{n_0} is a positive functional on the l_p-space, it follows from (2) that

$$0 \leq a^k = f_{n_0}(a^k e_{n_0}) \leq f_{n_0}(|C|^k |T|^{jk}(\lambda x_0)). \tag{3}$$

Since the modulus of T is finitely quasinilpotent at λx_0, it follows from (3) that $0 \leq a \leq \|f_{n_0}\|^{1/k} \||C|^k |T|^{jk}(\lambda x_0)\|^{1/k} \leq \|f_{n_0}\|^{1/k} \||C|\| (\||T|^{jk}(\lambda x_0)\|^{1/(jk)})^j \to 0$ as $k \to \infty$, from which it follows that $a = 0$.

For every $w \in M$, the definition of M implies that there is an operator $A \in \mathcal{A}$ such that $w = Ae_{n_0}$. Thus by the definition of $\mathcal{A}$ there are operators $C_1, C_2, \cdots, C_n \in \{T\}'_{\rm m}$ such that $|Ae_{n_0}| \leq \sum_{j=1}^{n} |C_j||T|^j e_{n_0}$. Thus by (1) we obtain $P(w) = P(Ae_{n_0}) \leq \sum_{j=1}^{n} P|C_j||T|^j e_{n_0} = 0$.

Hence it is easy to see that $f_{n_0}(w) = f_{n_0}(Pw) = 0$ for every $w \in M$. Consequently $f_{n_0}(w) = 0$ for every $w \in \overline{M}$. Observing that $f_{n_0}(e_{n_0}) = 1$, we obtain $\overline{M} \neq l_p$.

From above we conclude that $\overline{M}$ is a non-trivial modulus hyperinvariant closed ideal for T, and this completes the proof of Theorem 1. □

Corollary 1. *Let T be a non-zero positive operator on an l_p-space. If T is quasinilpotent at a non-zero positive vector $x_0 \in l_p$, then T has a non-trivial modulus hyperinvariant closed ideal.*

In other words, all continuous operators on an l_p-space whose modulus commute with a non-zero positive operator T that is quasinilpotent at a non-zero positive vector have a common non-trivial invariant closed ideal.

Since every invariant closed ideal is necessarily the invariant closed subspace, we derive Theorem A from Corollary 1.

Remark 1. It is also worth mentioning that C.J. Read [12] presented a quasinilpotent continuous operator T on l_1 without non-trivial invariant closed subspaces.

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