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# SEVERI VARIETIES, VARIETIES WITH AN APPARENT DOUBLE POINT AND THE STRATIFICATION OF $\mathbb{P}^n$ BY THEIR $X$-RANKS

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**Abstract:** Let $X \subset \mathbb{P}^n$ be an integral variety. For any $P \in \mathbb{P}^n$ the $X$-rank $r_X(P)$ is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. Here we study the stratification of $\mathbb{P}^n$ when $X$ is a Severi variety or a smooth variety with only one apparent double point (OADP).

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Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate $m$-dimensional variety defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{chark}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ the *$X$-rank* $r_X(P)$ of $P$ is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle\ \ \rangle$ denotes the linear span (see [3]). For any integer $t$ set $E(X,t) := \{P \in \mathbb{P}^n : r_X(P) = t\}$ and $E(X, \geq t) := \cup_{x \geq t} E(X,x)$. The *$s$-th secant variety* $\sigma_s(X) \subseteq \mathbb{P}^n$ of $X$ is the closure in $\mathbb{P}^n$ of the union of all $(s-1)$-dimensional linear subspaces spanned by $s$ points of $X$. Hence with this convention $\sigma_1(X) = X$ and $\sigma_2(X)$ is the secant variety $\mathrm{Sec}(X)$ of $X$. For any $P \in \mathbb{P}^n$ let $b_X(P)$ denote the *border $X$-rank* of $P$, i.e. the first integer

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$s > 0$ such that $P \in \sigma_s(X)$. Sometimes $b_X(P)$ is called the *secant $X$-rank* of $P$. Set $G(X,t) := \{P \in \mathbb{P}^n : b_X(P) = t\}$ and $G(X, \geq t) := \cup_{x \geq t} G(X,x)$. Notice that $X = E(X,1) = G(X,1)$ for all $X$. Let $\tau(X) \subseteq \mathbb{P}^n$ denote the tangent developable of $X$, i.e. the closure in $\mathbb{P}^n$ of the union of all $m$-dimensional tangent spaces $T_QX$, $Q \in X_{reg}$. Notice that if $X$ is smooth, then $\sigma_2(X) \backslash \tau(X) \subseteq E(X,2)$. If $X$ is smooth let $G(X,2)'$ (resp. $G(X,2)''$) be the set of all $P \in \mathbb{P}^n$ such that there are infinitely many lines (resp. at least two lines) $D$ such that $P \in D$ and length$(D \cap X) \geq 2$. We have $G(X,2)' \subseteq G(X,2)'' \subseteq \sigma_2(X)$. If $\dim(\sigma_2(X)) = 2m+1$, then a dimentional count gives $G(X,2)' \subsetneqq \sigma_2(X)$.

**Theorem 1.** *Let $X \subset \mathbb{P}^n$ be one of the 4 Severi varieties (see [6], [4]). Hence $X$ is smooth, $m \in \{2,4,8,16\}$ and $n = 3m/2 - 1$. Then Sec$(X)$ is a cubic hypersurface with $X$ as its singular locus, $X = E(X,1) = G(X,1)$, Sec$(X) \setminus X = E(X,2) = G(X,2)$, $\mathbb{P}^n \setminus$ Sec$(X) = E(X,3) = G(X,3)$ and $E(X, \geq 4) = G(X, \geq 4) = \emptyset$.*

Assume that $X$ is OADP in the sense of [1]. Since $X$ is smooth, $\sigma_2(X) \setminus \tau(X) \subseteq E(X,2)$. Fix $P \in \tau(X) \subset X$. Since $X$ is OADP, either $r_X(P) \geq 3$ or $P$ lies on infinitely many secant lines to $X$ (see [1], statement i) at p. 480). The latter occurs only on

**Proposition 1.** *Assume that $X \subset \mathbb{P}^n$, $n = 2m+1$, is a smooth $m$-dimensional OADP. Then $\dim(E(X,3)) = n-1$, $E(X,3) \subseteq \tau(X)$ and $E(X,3)$ contains a non-empty open subset of $\tau(X)$.*

All smooth OADP of dimension 2 and 3 are classified (see [1]). Using the classification and a case-by-case analysis we are able to show that if $m = 2$ then $E(X,3) = \tau(X) \setminus X$ (see Examples 1, 2 and 3 below). Then we consider another OADP with $m = 3$: the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (Example 4). Motivated by this computation we raise the following question.

**Question 1.** Fix integers $s \geq 2$ and $m_i \geq 1$, $1 \leq i \leq s$. Set $n := -1 + \prod_{i=1}^{s}(m_1+1)$. Set $X := \prod_{i=1}^{s} \mathbb{P}^{m_i}$ embedded in $\mathbb{P}^n$ by the complete linear system $|\mathcal{O}_X(1,\dots,1)|$. Is $r_X(P) = s$ for all $P \in \tau(X) \setminus X$?

In the set-up of Question 1 the proof of Example 4 gives $r_X(P) \leq s$ and that for every $Q \in X$ there are infinitely many $S \subset X \cap T_QX$ such that $\sharp(S) = s$ and $P \in \langle S \rangle$.

**Example 1.** Here we assume that $X$ is a del Pezzo surface of degree 5, i.e. it is isomorphic to the anticanonical model $X \subset \mathbb{P}^5$ of the blowing up of $\mathbb{P}^2$ at 4 not collinear points. Fix $Q \in X$ and $P \in T_QX \setminus X \cap T_QX$. To prove the inequality $r_X(P) \geq 3$ (and hence that $r_X(P) = 3$ by [3], Proposition

5.1) it is sufficient to prove the non-existence of infinitely many secant lines to $X$ containing $P$ (see [1], i) at p. 480). Assume that this is the case and call $T \subset X$ the one-dimensional part of the entry locus of $P$. Fix a general hyperplane $H \subset \mathbb{P}^5$ containing the line $\langle\{P,Q\}\rangle$. Since $\langle\{P,Q\}\rangle$ is tangent to $X$ at $Q$ and $X$ is cut out by quadrics, we have $(\langle\{P,Q\}\rangle)\cap X)_{red} = \{Q\}$. Hence Bertini's Theorem gives the smoothness of the curve $C := X \cap H$. Thus $C$ is a rational normal curve of $H$. By construction $P \in T_QC\backslash\{Q\}$. Hence $r_C(P) = 4$ (see [2] or [3], Theorem 4.1). Since $r_C(P) > 2$ and any two tangent lines of $C$ are disjoint, we get $(T \cap H)_{red} \subseteq \{Q\}$. Varying $H$ we get that $T$ is a union of lines through $Q$. There are only finitely many lines in $X$ and even if $Q$ is on one of these lines, say $R$, we conclude, because $P$ is not in all $T_AX$, $A \in R$.

**Example 2.** Here we assume that $X$ is the degree 4 rational normal scroll $S(2,2)$, i.e. the embedding of $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^5$ induced by the complete linear system $|\mathcal{O}_X(1,2)|$. For any $P \in X$ let $F_Q$ be the only line contained in $X$ and containing $Q$. Thus $F_Q$ is a fiber of one of the rulings of $X$. Call $C_Q$ the fiber of the other ruling of $X$ containing $Q$. The curve $C_Q$ is embedded in $\mathbb{P}^5$ as a smooth conic. Set $M_Q := \langle F_Q \cup C_Q\rangle$. Notice that $\dim(M_Q) = 3$ and that $T_QX = \langle F_Q \cup T_QC_Q\rangle$ is a hyperplane of it. Fix $Q \in X$ and $P \in T_QX \setminus (F_Q \cup T_QC_Q)$. Let $\ell : M_Q \setminus \{P| \to \langle C_Q\rangle$ denote the linear projection from $P$. Since $P \notin (F_Q \cup T_QC_Q)$, $\ell(F_Q)$ is a line not tangent to $C_Q$ at $Q$. Hence there is $A \in C_Q \cap \ell(F_Q)$ such that $A \neq Q$. Thus there are $A_1 \in F_Q \setminus Q$ and $A_2 \in C_Q \setminus \{Q\}$ such that $\ell(A_1) = A_2$, i.e. such that $P \in \langle\{A_1, A_2\}\rangle$. Thus $r_X(P) \leq 2$. Now take $P \in T_QC_Q \setminus \{Q\}$. Since $T_QC_Q$ is contained in the plane $\langle C_Q\rangle$, there is $S \subset C_Q$ such that $\sharp(S) = 2$ and $P \in \langle S\rangle$. Thus $r_X(P) \leq 2$.

**Example 3.** Here we assume that $X$ is the degree 4 rational normal scroll $S(1,3)$. For any $Q \in X$ let $F_Q$ denote the fiber of the ruling $\pi$ of $X$ containing $Q$. $F_Q$ is a line and $\{F_Q\}$, $Q \in X$, and the section $\underline{h}$ with negative self-intersection of $\pi$ are the only lines of $X$. If $Q \in \underline{h}$, then $T_QX \cap X = \underline{h} \cup F_Q$ spans $T_QX$. Thus $r_X(P) \leq 2$ for all $Q \in \underline{h}$ and all $P \in T_QX$. Fix $Q \in X \setminus \underline{h}$ and $P \in T_QX \setminus F_Q$. To prove the inequality $r_X(P) \geq 3$ (and hence that $r_X(P) = 3$ by [3], Proposition 5.1) it is sufficient to prove the non-existence of infinitely many secant lines to $X$ containing $P$ (see [1], i) at p. 480). Assume that this is the case and call $T \subset X$ the one-dimensional part of the entry locus of $P$. Fix a general hyperplane $H \subset \mathbb{P}^5$ containing the line $\langle\{P,Q\}\rangle$. Since $P \notin F_Q$, $\langle\{P,Q\}\rangle$ is tangent to $X$ at $Q$ and $X$ is cut out by quadrics, we have $(\langle\{P,Q\}\rangle)\cap X)_{red} = \{Q\}$. Hence Bertini's Theorem gives the smoothness of the curve $C := X \cap H$. Thus $C$ is a rational normal curve of $H$. By construction

$P \in T_QC\backslash\{Q\}$. Hence $r_C(P) = 4$ (see [2] or [3], Theorem 4.1). Since $r_C(P) > 2$ and any two tangent lines of $C$ are disjoint, we get $(T \cap H)_{red} \subseteq \{Q\}$. Varying $H$ we get that $T$ is a union of lines through $Q$. Since $Q \notin \underline{h}$ and $P \notin F_Q$, this is absurd.

**Example 4.** Here we take $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ embedded by the complete linear system $|\mathcal{O}_X(1,1,1)|$. $X$ is a smooth OADP (see [1], Proposition 2.3). Thus $\sigma_2(X) = \mathbb{P}^7$. Fix $Q \in X$. Notice that $T_XQ \cup X$ is the union of 3 lines $D_1, D_2, D_3$ containing $Q$ and spanning $T_QX$. For all $i, j \in \{1, 2, 3\}$ such that $i < j$ set $D_{ij} := \langle D_i \cup D_j \rangle$. Each point of $D_{ij} \setminus (D_i \cup D_j)$ has $X$-rank 2 anf this rank is computed by infinitely many set $S \subset D_i \cup D_j$. Fix $P \in T_QX \setminus (D_{12} \cup D_{13} \cup D_{23})$. A general plane $M \subseteq T_QX$ containing $P$ is spanned by the 3 points $M \cap (D_1 \cup D_2 \cup D_3)$. Thus there are infinitely many $S \subset D_1 \cup D_2 \cup D_3 \setminus \{Q\}$ such that $\sharp(S \cap D_i) = 1$ for all $i$ and $P \in \langle S \rangle$. Hence $r_X(P) \leq 3$. Since $X$ is a variety with OADP, [1], i) at p. 480, gives $r_X(P) \geq 3$ for all $P \in \tau(X) \setminus G(X, 2)'$ and in particular for a general $P \in \tau(X)$. Since all pairs $(Q, P)$ with $P \in T_QX$ and $P$ not on any plane $D_{ij}$ are projectively equivalent, we get $r_X(P) = 3$ for all $P \in T_QX \setminus (D_{12} \cup D_{13} \cup D_{23})$.

*Proof of Theorem 1.* In all cases $X$ is homogeneous, say $X = G/P$ with $G$ a connected algebraic group and $P$a parabolic subgroup of $G$, and the embedding $X \hookrightarrow \mathbb{P}^n$ is stable by the $G$-action (see [6], Chapter III). In all cases $X$, $\mathrm{Sec}(X) \setminus X$ and $\mathbb{P}^n \setminus \mathrm{Sec}(X)$ are the $G$-orbits for this linear action of $G$ on $\mathbb{P}^n$. Hence all the results on the border rank are obvious, as well the results on $E(X, 2)$. Thus to conclude the proof it is sufficient find $P \in \mathbb{P}^n \setminus \mathrm{Sec}(X)$ such that $r_X(P) \leq 3$ (use the $G$-action). Fix any $P \in \mathbb{P}^n \setminus \mathrm{Sec}(X)$. Since $\mathrm{Sec}(X) \setminus X = E(X, 2)$, it is sufficient to find $Q \in X$ such that $\langle\{P, Q\}\rangle \cap \mathrm{Sec}(X) \setminus X \neq \emptyset$. This is obvious at least for some $P \in \mathbb{P}^n \setminus \mathrm{Sec}(X)$, because $\mathrm{Sec}(X)$ is not a cone with vertex containing $X$. $\square$

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