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ON THE X-RANKS OF TANGENT VECTORS OF CURVES AND VERONESE EMBEDDINGS OF ARBITRARY VARIETIES

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Abstract: Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a smooth curve of genus ≥ 2. Let $\tau(X)$ be the tangent developable of X. Fix a general $P \in \tau(X)$. Here we prove the non-existence of $S \subset X$ such that $\sharp(S) = 2$ and $P \in \langle S\rangle$. We prove a similar result for Veronese embeddings of order ≥ 3 of arbitrary varieties.

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Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ such that chark$(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S\rangle$, where $\langle\ \ \rangle$ denotes the linear span. Hence $r_X(P) = 1$ if and only if $P \in X$. Since X is non-degenerate, the X-rank is defined and $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n$. For any integer $r \geq 1$ let $\sigma_r(X)$ denote the closure in $\mathbb{P}^n$ of the union of all $(r-1)$-dimensional linear subspaces of $\mathbb{P}^n$ spanned by r points of X. Often $\sigma_r(X)$ is called the $(r-1)$-secant variety of X. For any $P \in \mathbb{P}^n$ let $b_X(P)$ be the minimal integer r such that $P \in \sigma_r(X)$. Often $b_X(P)$ is called the *border rank* or *secant rank* of P. Obviously $b_X(P) \leq r_X(P)$ and $b_X(P) = 1$ if and only if $r_X(P) = 1$ if and only if $P \in X$. Let $\eta(X)$ (resp. $\rho(X)$) be the minimal integer t such there is $P \in \mathbb{P}^n$ with $b_X(P) \neq r_X(P)$ and $b_X(P) = t$ (resp. $r_X(P) = t$) with the convention $\eta(X) := \lozenge$ (resp. $\rho(X) = \spadesuit$) if no such integer exists. If

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$\rho(X)$ and/or $\eta(X)$ are integers, then $2 \le \eta(X) < \rho(X)$. There are some cases in which $\eta(X)$ and $\rho(X)$ are not finite (e.g. X a plane curve (see Remark 1) or several smooth curves in $\mathbb{P}^3$ (the smooth curves called without a cuspidal projection in [5]) or the order 2 Veronese embedding of $\mathbb{P}^2$. In [2] the authors computed $r_X(P)$ for all $P \in \sigma_2(X)$ and all $P \in \sigma_3(X)$ when X is a d-Veronese embedding of $\mathbb{P}^m$. The results are easier for $P \in \sigma_2(X)$ for the following reason. Let $\tau(X)$ denote tangent developable of X, i.e. the closure in $\mathbb{P}^n$ of the union of all linear subspaces T_QX with $Q \in X_{reg}$. Obviously $r_X(P) = 2$ for all $P \in \sigma_2(X)\backslash\tau(X)$. Hence to compute the X-ranks of all points of $\sigma_2(X)$ it is sufficient to compute $r_X(P)$ for all $P \in \tau(X)\backslash X$. Here we prove the following result for curves in $\mathbb{P}^n$, $n \ge 4$, and ask a related question.

Theorem 1. *Assume $n \ge 4$ and that X is an integral curve such that the normalization map $C \to X$ is unramified and C has genus at least 2. Then $\rho(X) = 2$. Moreover, for a general $Q \in X_{reg}$ there is $P \in T_QX$ such that $r_X(P) \ge 3$.*

Let $X \subset \mathbb{P}^n$ be an integral curve. We recall that the normalization map of X is unramified if X if each singular point is either an ordinary planar singularity (e.g. an ordinary node) or a seminormal singularity (i.e. it is formally equivalent to the union of the axis through the origin in an affine space $\mathbb{A}^r$).

Question 1. Is Theorem 1 true for all integral curves and (in positive characteristic) for all smooth curves?

Theorem 2. *Fix an integer $d \ge 3$ and an integral projective variety $Y \subset \mathbb{P}^r$. Let $j : Y \hookrightarrow \mathbb{P}^N$, $N := \binom{r+d}{r} - 1$ be the composition of the inclusion $Y \hookrightarrow \mathbb{P}^r$ and the order d Veronese embedding $v_{r,d} : \mathbb{P}^r \hookrightarrow \mathbb{P}^N$. Set $X := j(Y)$ and $\mathbb{P}^n := \langle X \rangle$. Fix $Q \in X_{reg}$ and $P \in T_QX\backslash\{Q\}$. Then $r_X(P) \ge 3$. Thus $\eta(X) = 2$ (if X is not a point).*

Remark 1. Take $n = 2$ and as X a non-degenerate integral curve. In characteristic zero, we always have $\eta(X) = \lozenge$, because $r_X(P) = 2$ for all $P \in \mathbb{P}^2 \setminus X$. In positive characteristic p this is false if and only X is a strange curve with invariants $\mu = 0$ and $s = 1$ and hence with degree p^e (see [1]); for instance, if $p = 2$ and X is a smooth conic, then $\eta(X) = 3$ and $\rho(X) = 2$. Now assume $n \ge 3$ and $\dim(X) = n - 1$. At least in characteristic zero we always have $\eta(X) = \lozenge$ by [4], Proposition 5.1.

Proof of Theorem 1. Since $\tau(X) \subset \sigma_2(X)$, it is sufficient to prove the last assertion. Set $d := \deg(X)$. Fix a general $Q \in X_{reg}$. Since $\mathrm{char}(\mathbb{K}) = 0$, the tangent line T_QX has order of contact 2 with X at Q. Since Q is general and the normalization of X is unramified, $(T_QX \cap X)_{red} = \{Q\}$ (see [3], Theorem 3.1 and Remark 3.8). Since $(T_QX \cap X)_{red} = \{Q\} \subset X_{reg}$, the linear projection from the line T_QX induces a morphism $u : X \to \mathbb{P}^{n-2}$ such that $\deg(u) \cdot \deg(u(X)) = d-2$. Assume $r_X(P) \leq 2$ for all $P \in T_QX$. We get $\deg(u) \geq 2$. Since a general tangent line of X intersects infinitely many secant lines and (since X is not strange) a general secant line of X is not a multisecant line. We get $\deg(v) = 2$ for a general Q. Since X has infinitely many degree 2 morphisms onto other curves, the normalization of X has genus at most one. □

Proof of Theorem 2. See T_QX as a linear subspace of the r-dimensional linear subspace $T_Qv_{r,d}(\mathbb{P}^r)$ of $\mathbb{P}^N$. The proof of [2], Theorem 4.3, gives $r_{v_{r,d}(\mathbb{P}^r)}(P) = d$. Since $X \subseteq v_{r,d}(\mathbb{P}^r)$, we get $v_X(P) \geq d$. The last assertion follows from the inequality just proved and the definition of $\eta(X)$. □

In some cases Theorem 1 is true also in the case $d = 2$ (for quadric hypersurfaces uses [2], Lemma 4.6). We do not know what happens in general for the Segre product of the embeddings of two varieties. We only know the following easy result (use [2], Lemma 4.6, for its proof).

Proposition 1. *Fix integers $s \geq 2$, $m_i \geq 1$, $1 \leq i \leq s$, and $d_i \geq 1$, $1 \leq i \leq s$. Set $Y := \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$ and $n := -1 + \prod_{i=1}^{s} \binom{m_i+d_i}{d_i}$. Let $j : Y \hookrightarrow \mathbb{P}^n$ denote the embedding of Y induced by the complete linear system $|\mathcal{O}_Y(d_1, \ldots, d_s)|$. Set $X := j(Y)$. Assume $d_i \geq 3$ for all i. Set $\delta := \min_{i=1}^{s}\{d_i\}$. Then $\eta(X) = 2$ and $\delta = \min_{P \in \tau(X) \setminus X} r_X(P)$.*

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References

[1] E. Ballico, An upper bound for the X-ranks of points of $\mathbb{P}^n$ in positive characteristic, *Preprint*.

[2] A. Bernardi, A. Gimigliano, M. Idà, On the stratification of secant varieties of Veronese varieties via symmetric rank, *ArXiv:* 0908.1651v4 [math.AG].

[3] H. Kaji, On the tangentially degenerate curves, *J. London Math. Soc.*, **33**, No. 2 (1986), 430-440.

[4] J.M. Landsberg, Z. Teitler, On the ranks and border ranks of symmetric tensors, *ArXiv:* 0901.0487v3.

[5] R. Piene, Cuspidal projections of space curves, *Math. Ann.*, **256**, No. 1 (1981), 95-119.