International Journal of Pure and Applied Mathematics
Volume 63 No. 3 2010, 301-309

NONOSCILLATION OF A SECOND ORDER SUBLINEAR DIFFERENTIAL EQUATION WITH "MAXIMA"

D. Kolev[1], N. Markova[2 §], S. Nenov[3]
[1,3]Department of Mathematics
University of Chemical Technology and Metallurgy
8, Kliment Ohridski Blvd., Sofia, 1756, BULGARIA
[1]e-mail: mkolev@math.uctm.edu, kolev@uctm.edu
[1]url: http://math.uctm.edu/ mkolev
[3]e-mail: svety@math.uctm.edu
[2]Department of Mathematics, Physics and Chemistry
Technical University
Sliven, 8800, BULGARIA
e-mail: n_markova54@abv.bg

Abstract: In this paper we consider a second order functional differential equation

$$(r(t)x')' + q(t)f\left(\max_{s\in[\sigma(t),\tau(t)]} x(s)\right) = b(t),$$

containing a function f of sublinear rate and depending on the maximum of the unknown $x(t)$ and defined on some interval taken before coming the present time t. We call this equation with "maxima". Criteria for existence of nonoscillating solutions are established under requirement that r, r', q, b should be continuous on some sets, and f has sublinear rate. These differential equations could be seen in lots of mathematical models in theoretical physics, optimal control, chemistry, mechanics of materials, biology, ecology, etc.

AMS Subject Classification: 34K15
Key Words: differential equations with "maxima", second order ODEs, sublinear differential equation, nonoscillation, oscillation

Received: May 20, 2010

§Correspondence author

1. Introduction

We consider the second order functional differential equation with "maxima" that has the form

$$\big(r(t)x'(t)\big)' + q(t)f\left(\max_{s\in[\sigma(t),\tau(t)]} x(s)\right) = b(t), \tag{1}$$

where the real scalar functions σ and τ, determining the interval on which $\max_{s\in[\sigma(t),\tau(t)]} x(s)$ is defined are continuous, monotone increasing and satisfy the inequalities $\sigma(t) \le \tau(t) \le t \in \mathbb{R}_+$.

The study of differential equations with "maxima" begins with the works of A. Magomedov [12], [13], where a linear differential equations with "maxima" was considered as a mathematical model in the theory of optimal control. In most cases we use "maxima" in the right-hand side when the control corresponds to the maximal deviation of the regulated quantity that could be for instance temperature, heat, current density, pressure and so on.

The oscillation properties of the solutions of differential equations with "maxima" were studied by Bainov and his associates (see, e.g. [1]-[6]). However, very little is known about the nonoscillation of equations with forcing terms even in the case of ODEs without "maxima". We refer the reader to see Greaf, Spikes (see. e.g. [7]-[9]), and Kusano, Onose (see, e.g. [11]).

In this paper we establish new criteria for existence of nonoscillatory solutions for equation (1). The functions q, b, r, f satisfy some integral conditions that lead to existence of these solutions applied in many areas of physics, optimal control, biology, etc.

2. Preliminary Notes

Recollect some standard sets $\mathbb{R}_+ \equiv [0, +\infty)$, $\mathbb{R}_+^0 \equiv (0, +\infty)$ and $\mathbb{N} = \{1, 2, ...\}$, that will be used further in the paper, and introduce the conditions:

(H1) $r \in C^1(\mathbb{R}_+, \mathbb{R}_+^0)$, $q \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\sup\big\{q(s) : s \ge t,\ t \in \mathbb{R}_+\big\} > 0$.

(H2) $b \in C(\mathbb{R}_+, \mathbb{R})$.

(H3) $f(z)$ is nondecreasing in $z \in \mathbb{R}$, $f \in C(\mathbb{R}, \mathbb{R})$, and $zf(z) > 0$ for $z \ne 0$.

(H4) The functions $\sigma, \tau \in C(\mathbb{R}_+, \mathbb{R})$ are nondecreasing, $\lim_{t\to+\infty} \sigma(t) = +\infty$, and $\sigma(t) \le \tau(t) \le t$ for all $t \in \mathbb{R}_+$.

(H5) $f(z)$ is sublinear, that is $\limsup_{|x|\to+\infty} \frac{f(x)}{x} < +\infty$.

Further we make use of the following definitions.

Definition 1. The solution $x(t)$ of equation (1) is said to be:

(1) *proper* if it is defined in some interval $[T_x, +\infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq T_x$;

(2) *finally positive* (*finally negative*) if there exists $T \geq 0$ such that $x(t) > 0$ ($x(t) < 0$) is defined for $t \geq T$;

(3) *oscillatory* if it is proper and neither finally positive nor finally negative;

(4) *nonoscillatory* if it is either finally positive or finally negative.

3. Main Results

Lemma 1. *Let the conditions (H1)-(H4) be satisfied.*

1. *If*

$$\liminf_{t\to+\infty} \int_T^t [b(s) - kq(s)]\, ds > 0$$

for each finite number $k > 0$, *and* $T \in \mathbb{R}_+$, (2)

then there does not exist oscillatory solution of equation (1) *that is bounded above.*

2. *If*

$$\liminf_{t\to+\infty} \int_T^t [b(s) + kq(s)]\, ds < 0$$

for each finite number $k > 0$, *and* $T \in \mathbb{R}_+$, (3)

then there does not exist oscillatory solution of equation (1) *that is bounded below.*

Proof. We shall prove the statement 1. The proof of statement 2 is analogous.

Let $x_0(t)$ be an oscillatory solution of equation (1) and $x_0(t) \le M$ for some positive number M and for $t \ge T_0 \ge 0$. Let $t_1 \ge T_0$ be such that $T_0 \le \sigma(t) \le \tau(t) \le t$ for all $t \ge t_1$. Then $\max\limits_{s\in[\sigma(t),\tau(t)]} x_0(s) \le M$ for $t \ge t_1$.

There exists $t_2 \ge t_1$ such that $x_0'(t_2) = 0$, becouse $x_0(t)$ is an oscillatory function. Therefore, after integrating (1) in $[t_2, t]$, we get

$$r(t)x_0'(t) = \int\limits_{t_2}^{t} \left[b(s) - q(s) f\left(\max_{u\in[\sigma(s),\tau(s)]} x_0(u) \right) \right] ds \ge \int\limits_{t_2}^{t} [b(s) - q(s)f(M)]ds.$$

Letting $t \to +\infty$ and using (2) we conclude that $r(t)x_0'(t) > 0$ for all large t which is impossible for the oscillatory function $x_0(t)$. □

Remark 1. The inequality (2) (inequality (3)) is satisfied provided that either:

$$(i) \qquad \int\limits_0^{\infty} q(t)dt < +\infty \qquad \text{and} \qquad \int\limits_0^{+\infty} b(t)dt = +\infty \ \ (-\infty),$$

or

$$(ii) \qquad \lim_{t\to+\infty} \frac{b(t)}{q(t)} = +\infty \ \ (-\infty).$$

Lemma 2. *Let conditions (H1)-(H4) hold.*

1. *If*

$$\lim_{t\to+\infty} \int\limits_0^t [b(s) - kq(s)]ds = +\infty, \tag{4}$$

and

$$\limsup_{t\to+\infty} \int\limits_T^t \frac{1}{r(u)} \int\limits_T^u [b(s) - kq(s)]dsdu = +\infty \tag{5}$$

for each $k > 0$ and $T \in \mathbb{R}_+$ then all proper solutions of equation (1) are unbounded above.

2. *If*

$$\lim_{t\to+\infty} \int\limits_0^t [b(s) + kq(s)]ds = -\infty, \tag{6}$$

and

$$\liminf_{t\to+\infty}\int\limits_T^t \frac{1}{r(u)}\int\limits_T^u [b(s)+kq(s)]dsdu = -\infty \tag{7}$$

for each $k>0$ *and* $T\in\mathbb{R}_+$ *then all proper solutions of equation* (1) *are unbounded below.*

Proof. First we shall prove 1. Let $x_0(t)$ be a proper solution of (1) for $t\ge T_0\ge 0$ which is bounded above. Provided that (4), (5) hold, then there exist a positive number M and a positive $t_0\ge T_0$ such that $x_0(t)\le M$, for all $t\ge t_0$. Let $t_1\ge t_0$ be such that $t_0\le\sigma(t)\le\tau(t)\le t$ for all $t\ge t_1$. Then $\max\limits_{s\in[\sigma(t),\tau(t)]} x_0(s)\le M$, and after integrating (1) from t_1 to t we obtain

$$\begin{aligned} r(t)x_0'(t) &= r(t_1)x_0'(t_1)+\int\limits_{t_1}^t\left[b(s)-q(s)f\left(\max_{u\in[\sigma(s),\tau(s)]}x_0(u)\right)\right]ds \\ &\ge r(t_1)x_0'(t_1)+\int\limits_{t_1}^t[b(s)-q(s)f(M)]ds. \end{aligned}\tag{8}$$

Therefore

$$\lim_{t\to+\infty} r(t)x_0'(t)=+\infty. \tag{9}$$

Hence there exists $T\ge t_1$ such that $r(T)x_0'(T)\ge 0$ and proceeding as above we get

$$r(t)x_0'(t)\ge\int\limits_T^t[b(s)-q(s)f(M)]ds,\quad t\ge T.$$

This implies that

$$x_0(t)\ge x_0(T)+\int\limits_T^t\frac{1}{r(u)}\int\limits_T^u[b(s)-q(s)f(M)]dsdu,\quad t\ge T.$$

Then using (5) obtain that

$$\limsup_{t\to+\infty} x_0(t)=+\infty,$$

which is a contradiction.

The proof of statement 2 is analogous. □

Remark 2. Lemma 1 remains valid if conditions (5) and (7) are replaced by the condition

$$\int_0^{+\infty} \frac{dt}{r(t)} = +\infty. \tag{10}$$

Corollary 1. *Let the conditions (H1)-(H4) and* (10) *hold. Then:*

1. *All proper solutions of* (1) *are unbounded above if condition* (4) *holds.*
2. *All proper solutions of equation* (1) *are unbounded below if condition* (6) *holds.*

Proof. 1. Let condition (4) and (10) hold and $x_0(t)$ be a proper solution of equation (1) which is bounded above. Then (9) is true and there exist $t_2 \geq t_1$ and a number $Q > 0$ such that

$$r(t)x_0'(t) \geq Q, \quad t \geq t_2.$$

This implies that

$$x_0(t) \geq x_0(t_2) + \int_{t_2}^{t} \frac{Qds}{r(s)} \to +\infty \text{ as } t \to +\infty,$$

which is a contradiction.

The proof of statement 2 is analogous. □

Theorem 1. *Assume that:*

1. *Conditions (H1)-(H5) hold and*

$$\int_0^{+\infty} \frac{1}{r(u)} \int_u^{+\infty} q(s)dsdu < +\infty. \tag{11}$$

2. *Either* $b(t) \geq 0$ *for* $t \geq T$ *and* (2) *hold, or* $b(t) \leq 0$ *for* $t \geq T$ *and* (3) *hold.*

Then all proper solutions of equation (1) *are nonoscillatory.*

Proof. Consider the case when $b(t) \geq 0$ for all $t \geq T_0$ and condition (2) hold true. Suppose that there exists an oscillatory solution $x_0(t)$, $t \geq T_0$ of equation (1), where T_0 is sufficiently large. Thus from Lemma 1 we conclude that $x_0(t)$ is unbounded above.

Next we select two sequences $\{\sigma_n\}$ and $\{\tau_n\}$ of zeros of $x_0(t)$ such that $t_0 \leq \sigma_n < \tau_n$,

$$\lim_{n\to+\infty} \sigma_n = \lim_{n\to+\infty} \tau_n = +\infty, \quad x_0(t) > 0,\ t \in (\sigma_n, \tau_n),$$

and

$$M_n = \max_{[\sigma_n,\tau_n]} x_0(t) = \max_{[\sigma_1,\tau_n]} x_0(t), \quad n \in \mathbb{N},$$

where the increasing sequence $\{M_n\}$ tends to infinity as $n \to +\infty$. Let $M_n = x_0(t_n)$ be a sequence of maximal values of $x_0(t)$ in (σ_n, τ_n), $n \in \mathbb{N}$. After integrating (1) from $t \in [\sigma_n, \tau_n]$ to t_n $(t \le t_n)$ obtain

$$r(t)x_0'(t) = \int_t^{t_n} \left[q(s) f\left(\max_{u\in[\sigma(s),\tau(s)]} x_0(u) \right) - b(s) \right] ds$$

$$\le \int_t^{t_n} q(s) f\left(\max_{u\in[\sigma(s),\tau(s)]} x_0(u) \right) ds, \quad (12)$$

where n is taken so large that $\tau(s) \ge \sigma(s) \ge \sigma_1$ for $s \ge \sigma_n$.

Dividing (12) by $r(t)$ and integrating from σ_n to t_n we get

$$M_n = x_0(t_n) \le \int_{\sigma_n}^{t_n} \frac{1}{r(u)} \int_u^{t_n} q(s) f\left(\max_{v\in[\sigma(s),\tau(s)]} x_0(v) \right) dsdu.$$

Note that $t_0 \le \sigma_n \le \sigma(s) \le \tau(s) \le s \le t_n$ and $\max_{v\in[\sigma(s),\tau(s)]} x_0(v) \le M_n$. Hence

$$M_n \le f(M_n) \int_{\sigma_n}^{t_n} \frac{1}{r(u)} \int_u^{t_n} q(s) dsdu$$

and keeping in mind (11) we obtain

$$1 \le \frac{f(M_n)}{M_n} \int_{\sigma_n}^{+\infty} \frac{1}{r(u)} \int_u^{+\infty} q(s) dsdu. \quad (13)$$

Since the sequence $\left\{ \frac{f(M_n)}{M_n} \right\}$ is bounded above the right-hand side of (13) tends to zero as $n \to +\infty$ therefore this is a contradiction.

By similar reasoning we reach a contradiction in the case when $b(t) \le 0$, $t \ge T_0$, and (3) hold. □

Remark 3. Condition (11) holds if either

$$\int_0^{+\infty} \frac{dt}{r(t)} = +\infty \quad \text{and} \quad \int_0^{+\infty} \left(\int_0^t \frac{ds}{r(s)} \right) q(t) dt < +\infty,$$

or

$$\int_0^{+\infty} \frac{dt}{r(t)} < +\infty \quad \text{and} \quad \int_0^{+\infty} q(t)dt < +\infty.$$

Remark 4. Conditions (4) and (6) are stronger than conditions (2) and (3), respectively.

Having in mind the above stated Remark 4 we state a theorem that is a corollary of Theorem 1 and Lemma 2.

Theorem 2. *Let conditions (H1)-(H5) and* (11) *be satisfied.*

1. *If* $b(t) \geq 0$ *for* $t \geq T_0$ *and both conditions* (4) *and* (5) *hold then all proper solutions of equation* (1) *are nonoscillatory and unbounded above.*
2. *If* $b(t) \leq 0$ *for* $t \geq T_0$ *and both conditions* (6) *and* (7) *hold then all proper solutions of equation* (1) *are nonoscillatory and unbounded below.*

From Theorem 1 and Corollary 1 it follows:

Corollary 2. *Let conditions (H1)-(H5),* (10) *and* (11) *hold.*

1. *If* $b(t) \geq 0$ *for* $t \geq T_0$ *and condition* (4) *hold then all proper solutions of equation* (1) *are nonoscillatory and unbounded above.*
2. *If* $b(t) \leq 0$ *for* $t \geq T_0$ *and condition* (6) *holds then all proper solutions of equation* (1) *are nonoscillatory and unbounded below.*

Acknowledgments

The present study was supported by the grant No. 10710 (NIS-2010) of the University of Chemical Technology and Metallurgy – Sofia, Bulgaria.

References

[1] V. Angelov, D. Bainov, On the functional differential equations with "maximums", *Appl. Anal.*, **16** (1983), 187-194.

[2] D. Bainov, D. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay, Adam Hilger*, Bristol, Philadelphia and New York, IOP Publishing (1991).

[3] D. Bainov, V. Petrov, V. Proytcheva, Asymptotic behaviour of second order neutral differential equations with "maxima", *Tamkang Journal of Math.*, **25** (1995), 267-275.

[4] D. Bainov, V. Petrov, V. Proytcheva, Oscillatory and asymptotic behaviour of second order neutral differential equations with "maxima", *Dynamic Syst. Appl.*, **4** (1995), 137-146.

[5] D. Bainov, V. Petrov, V. Proytcheva, Oscillation and nonoscillation of first order neutral differential equations with maxima, *SUT Journal of Mathematics*, **31** (1995), 17-28.

[6] D. Bainov, A. Zahariev, Oscillating and asymptotic properties of a class of functional differential equations with maxima, *Czechslovak Math. Journal*, **34** (1984), 247-251.

[7] J.R. Greaf, P.W. Spikes, A nonoscillatory result for second order ordinary differential equations, *Rend.Accad. Sci. Fis. Mat. Napoli*, **41**, No. 4 (1974), 92-101.

[8] J.R. Greaf, P.W. Spikes, Sufficient conditions for nonoscillation of a second order nonlinear differential equation, *Proc. Amer. Math. Soc.*, **50** (1975), 289-292.

[9] J.R. Greaf, P.W. Spikes, Sufficient conditions for the equation to be nonoscillatory, *Funkcial. Ekvac.*, **18** (1975), 35-40.

[10] John W. Hagood, Brian S. Thomson, Recovering a function from a Dini derivative, *The Mathematical Association of America*, **113** (2006), 34-45.

[11] T. Kusano, H. Onose, A nonoscillation theorem for a second order sublinear retarded differential equation, *Bull. Austral. Math. Soc.*, **15** (1976), 401-406.

[12] A. Magomedov, On some questions about differential equations with maxima, *Izv. Akad. Nauk Azerbaidzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk*, **1** (1977), 104-108, In Russian.

[13] A. Magomedov, Theorem of existence and uniqueness of solutions of linear differential equations with maxima, *Izv. Akad. Nauk Azerbaidzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk*, **5** (1979), 116-118, In Russian.