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MULTIPOINT BOUNDARY VALUE PROBLEMS FOR DISCRETE NONLINEAR SYSTEMS AT RESONANCE

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Abstract: In this paper we provide sufficient conditions for the existence of solutions to multipoint boundary value problems for discrete nonlinear systems at resonance. We gain insight into the relationship between the nonlinearity and the solution space of the associated linear problem with the Lyapunov-Schmidt technique. Using degree theory we obtain results for systems, thereby extending previous work that used techniques applicable only to scalar equations.

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1. Introduction and Statement of Results

In this paper, we provide sufficient conditions for the existence of solutions to the discrete, nonlinear, multipoint boundary value problem

$$x(t+1) = A(t)x(t) + f(t, x(t)), \quad t \in \{0, \dots, N-1\}, \tag{1}$$

subject to

$$B_0\, x(0) + B_1\, x(1) + \cdots + B_N\, x(N) = 0. \tag{2}$$

Here $x\colon \{0, \dots, N\} \to \mathbf{R}^n$, $B_k \in \mathbf{R}^{n\times n}$, and $f\colon \{0, \dots, N-1\} \times \mathbf{R}^n \to \mathbf{R}^n$.

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Multipoint boundary value problems can arise in a number of settings. In [17], Miura uses a three point boundary value problem to compute stable solitary wave solutions of the FitzHugh-Nagumo equations. The FitzHugh-Nagumo equations are a reduction of the Hodgkin-Huxley model which was originally used to model the propogation of action potentials along the giant axon of a squid and has since been used to study various excitable cells including neurons and heart tissue [14]. The deflection of an equally loaded, three layered "sandwich beam" is modeled by a third order differential equation coupled with a three point boundary condition in [1]. Although the examples mentioned here are taken from continuous problems, the discrete analogs of these problems will naturally contain multipoint boundary conditions. Our boundary condition, (2), allows for n general, non-separated, linear conditons on x. Of course the special case of separated boundary conditions can be addressed by choosing all but n of the matrices B_k to be 0, and then making each of the remaining n matrices, B_{k_i}, 0 in every entry outside of row i.

The linear problems associated with (1) play a critical role in our study of the nonlinear boundary value problem (1)-(2). The homogeneous linear problem associated with (1) is defined by

$$x(t+1) = A(t)x(t) \qquad t \in \{0, \ldots, N-1\}, \tag{3}$$

and the nonhomogeneous linear problem associated with (1) is defined by

$$x(t+1) = A(t)x(t) + y(t) \qquad t \in \{0, \ldots, N-1\},\ y \in \mathbf{R}^n. \tag{4}$$

We study (1)-(2) under conditions of resonance; specifically we allow for the existence of a one-parameter family of solutions of (3) subject to (2). The case in which 0 is the only solution of the problem (3)-(2) (i.e. the linear part of the problem is invertible) was treated in [23], where it is shown that if the nonlinearity satisfies a "sublinear" growth condition, then the boundary value problem (1)-(2) will have at least one solution. (More precisely, the relevant result, Proposition 3.2 as stated in [23] applies to scalar-like equations only but the proof given applies just as well to true systems.)

Our main result overcomes the resonance by restricting the behavior of the nonlinear term $f(t, x)$ for large x. The proof makes use of the Lyapunov-Schmidt technique, see [5, 6, 7, 11, 18, 19], and degree theory, see [26].

Our methods and results extend [9], which treated discrete nonlinear scalar two-point boundary value problems and [23], which appeared later and allowed multipoint boundary conditions but continued to require that x and f be (essentially) scalar. In the resonance case, this (essentially) scalar hypothesis is required by the proofs given in [9] and [23]. The techniques used in [23] were applied to ordinary differential equations subject to multipoint boundary conditions in [25]. Again, in the resonance case, the proofs given in [25] depend on the scalar hypothesis. Recently, [21] extended [25] to encompass more general nonlinearities, however the underlying differential equation remained scalar. Our results constitute an extension of [9] and [23] because they treat true systems. Related two-point boundary value problems on time scales were treated in [12, 13]; [13] dealt with a scalar dynamic equation and [12] studied systems with a weak nonlinearity. The reader interested in other work in the area of discrete boundary value problems may consult [2, 3, 4, 8, 10, 15, 16, 19, 20, 22, 24, 26, 27].

We will impose the following hypotheses on the system (1)-(2).

H1.1. For all $t \in \{0, \dots, N-1\}$, $A(t)$ is invertible.

Let $\widetilde{X} = \{x\colon \{0, \dots, N\} \to \mathbf{R}^n\}$ and define $\beta\colon \widetilde{X} \to \mathbf{R}^n$ by the left hand side of (2). In most cases, the multipoint boundary conditions will not involve x at every time step t, i.e. many of the matrices B_t will be zero. We assume that the boundary conditions impose n linearly independent conditions on x.

H1.2. The rank of the linear map β is n.

Hypothesis 1.2 is equivalent to the condition $\bigcap_{i=0}^{N} \ker\left(B_i^T\right) = \{0\}$.

Next we impose bounds on the growth of f and on variations in f for $|x|$ large. Specifically, we assume that:

H1.3. $f(t, \cdot)$ is continuous on $\mathbf{R}^n$ for each $t \in \{0, \dots, N-1\}$ and there exists $b > 0$ such that $|f(t, x)| \le b$ for all $t \in \{0, \dots, N-1\}$ and $x \in \mathbf{R}^n$.

H1.4. There is a real number $r_0 \ge 0$ and a decreasing function $\delta\colon [r_0, \infty) \to [0, \infty)$ satisfying:

(i) $\lim_{r \to \infty} \delta(r) = 0$

(ii) If $t \in \{0, \dots, N-1\}$, $|x_0| \ge r_0 + s$, and $|x_1| \le s$, then $|f(t, x_0) - f(t, x_0 + x_1)| \le \delta(|x_0| - s)s$.

H1.5. The linear multipoint boundary value problem (3)-(2) has a one dimensional solution space.

The crucial hypothesis needed to guarantee existence of solutions to (1)-(2) relates the large-x behavior of the nonlinear term f to certain quantities associated with the linear problem (3)-(2). This requires several definitions.

Let Φ be the fundamental matrix solution associated with the linear difference equation. That is, for $0 \le l \le t \le N$ define $\Phi(t,l) \in \mathbf{R}^{n\times n}$ by

$$\Phi(t,l) = \begin{cases} A(t-1)\cdot \ldots \cdot A(l) & \text{if } t \ge l+1, \\ I & \text{if } t = l. \end{cases} \tag{5}$$

Let $D \in \mathbf{R}^{n\times n}$ be defined by

$$D = \sum_{t=0}^{N} B_t \Phi(t,0). \tag{6}$$

Hypothesis H1.5 guarantees that D has a 1 dimensional kernel. Utilizing this we choose nonzero vectors $c, d \in \mathbf{R}^n$ satisfying:

$$Dd = 0 \tag{7}$$

and

$$c^T D = 0. \tag{8}$$

Define $\phi\colon \{0,\ldots,N\} \to \mathbf{R}^n$ by

$$\phi(t) = \Phi(t,0)d, \tag{9}$$

then ϕ spans the solution space of (3)-(2), which is proven in [23]. As in [23], define $\psi:\ \{0,\ldots,N-1\} \to \mathbf{R}^n$ by

$$\psi(t) = \sum_{k=t+1}^{N} \left[B_k \Phi(k,t+1)\right]^T c. \tag{10}$$

In [23] it is shown that for a given $y\colon \{0,\ldots,N-1\} \to \mathbf{R}^n$, the nonhomogeneous linear boundary value problem (4)-(2) has a solution if and only if $\sum_{t=0}^{N-1} \psi(t)^T y(t) = 0$.

Before stating our final hypothesis, we need one additional definition. Define $J\colon \mathbf{R} \to \mathbf{R}$ by:

$$J(r) = \sum_{t=0}^{N-1} \psi(t)^T f(t, r\phi(t)). \tag{11}$$

H1.6. $J_1 = \lim_{r\to-\infty} J(r)$ and $J_2 = \lim_{r\to\infty} J(r)$ exist and are of opposite sign.

Our main result can now be stated.

Theorem 1. *If $A(t)$, B_t, and f are as above and satisfy hypotheses H1.1–H1.6, then the problem* (1)-(2) *has a solution.*

Theorem 1 is an easy consequence of Theorem 2, which is formulated without any reference to difference equations and may be of independent interest. We state and prove Theorem 2 and use it to prove Theorem 1 in Section 2. We provide an example in Section 3.

Before proceeding it is interesting to observe that the roles of the functions ϕ and ψ appearing in H1.6 are analogous to the roles of the vectors d and c. Let $X = \ker(\beta), Y = \{y : \{0, \ldots, N-1\} \to \mathbf{R}^n\}$, and define $L : X \to Y$ by

$$(Lx)(t) = x(t+1) - A(t)x(t). \tag{12}$$

ϕ and d span the kernels of L and the $n \times n$ matrix D respectively while ψ and c define linear functionals that annihilate the images of L and D. Furthermore, ϕ and ψ are determined by problems, dual in some sense, involving d and c. Indeed, it follows from (5), (9) and (10) that ϕ satisfies the initial value problem

$$\phi(0) = d, \tag{13}$$

$$\phi(t+1) = A(t)\phi(t) \quad t \in \{0, \ldots, N-1\}, \tag{14}$$

and ψ satisfies the "terminal value" problem

$$\psi(N-1)^T = c^T B_N, \tag{15}$$

$$\psi(t)^T = \psi(t+1)^T A(t+1) + c^T B_{t+1} \quad t \in \{0, \ldots, N-2\}. \tag{16}$$

2. A General Setting

Our results about discrete boundary value problems follow from the next theorem. The crucial hypotheses concern the behavior of F on and near the kernel of L. We require that $F(x)$ move from one side of the image(L) to the other as x moves along $\ker(L)$ and that variations in F with respect to bounded perturbations in x approach 0 for $x \in \ker(L)$ and $|x| \to \infty$.

Theorem 2. *Let $(X, |\ |)$ and $(Y, |\ |)$ be finite dimensional normed linear spaces with the same dimension; $L\colon X \to Y$ be a linear map with $\ker(L) = \operatorname{span}\{k\}$ for some vector k with $|k| = 1$; and $\psi^*\colon Y \to \mathbf{R}$ be a linear functional with the property that $y \in \operatorname{image}(L)$ if and only if $\psi^* y = 0$. Let $F\colon X \to Y$ be continuous on X and bounded by b. Then the equation*

$$Lx = F(x) \tag{17}$$

has a solution in X if the following two hypotheses are satisfied.

H2.1 *For every $s > 0$, there is real number $R_s \geq 0$ and a map $\mu_s\colon [R_s, \infty) \to [0, \infty)$ with the following properties:*

(i) $\lim_{r\to\infty} \mu_s(r) = 0$

(ii) *If $|r| \geq R_s$ and $|x_1| \leq s$, then $|F(rk) - F(rk + x_1)| \leq \mu_s(|r|)s$.*

H2.2 *The limits*

$$\alpha_1 = \lim_{r\to-\infty} \psi^* F(rk), \text{ and} \tag{18a}$$

$$\alpha_2 = \lim_{r\to\infty} \psi^* F(rk) \tag{18b}$$

exist and are of opposite sign.

Proof. Let $X_0 = \ker(L) = \operatorname{span}\{k\}$, $|k| = 1$ and let X_1 be any complement of X_0. Let P_0 and P_1 be the projections associated with the decomposition $X = X_0 \oplus X_1$. Let $Y_1 = \operatorname{image}(L)$, a subspace of codimension 1 in Y and let Y_0 be any complement of Y_1. Let Q_0 and Q_1 be the projections associated with the decomposition $Y = Y_0 \oplus Y_1$. These complements exist and the projections are bounded since X and Y are finite dimensional. We will denote the norms of these projections and of any linear operator by $\|\ \|$. Let $\overline{L}$ be the restriction of L to the subspace X_1. Note that $\overline{L}\colon X_1 \to Y_1$ is bijective and thus has a bounded inverse $\overline{M}\colon Y_1 \to X_1$.

For any $x \in X$, $P_1 x \in X_1$ so $\overline{M}LP_1 x = P_1 x$. Therefore $\overline{M}L x = \overline{M}L(P_0 x + P_1 x) = \overline{M}LP_1 x = P_1 x$. That is,

$$\overline{M}L = P_1. \tag{19}$$

Similarly, for all $y \in Y$, $Q_1 y \in Y_1$ so $L\overline{M}Q_1 y = Q_1 y$. Thus

$$L\overline{M}Q_1 = Q_1. \tag{20}$$

We claim that with $x_0 \in X_0$, $x_1 \in X_1$, and $x = x_0 + x_1$, equation (17) is equivalent to the system of equations

$$0 = \psi^* F(x_0 + x_1) \quad \text{and} \tag{21a}$$

$$x_1 = \overline{M}Q_1 F(x_0 + x_1). \tag{21b}$$

The necessity (which we do not need in the sequel but which motivates the proof) follows from first applying ψ^* and then applying $\overline{M}Q_1$ to both sides of (17). Conversely, suppose that (x_0, x_1) solves the system (21). Then, from (21a) $F(x_0 + x_1) \in Y_1$, so $Q_1 F(x_0 + x_1) = F(x_0 + x_1)$. Furthermore,

$$\begin{aligned} L(x_0 + x_1) &= L x_1 \\ &= L\overline{M}Q_1 F(x_0 + x_1) \qquad \text{by (21b)} \end{aligned}$$

$$= Q_1 F(x_0 + x_1) \qquad \text{by (20)}$$
$$= F(x_0 + x_1).$$

Define $H = (H_0, H_1) : \mathbf{R} \times X_1 \to \mathbf{R} \times X_1$ by

$$H_0(r, x_1) = \psi^* F(rk + x_1), \tag{22a}$$
$$H_1(r, x_1) = x_1 - \overline{M} Q_1 F(rk + x_1). \tag{22b}$$

Letting $x_0 = rk$ in (21) we see that $x = rk + x_1$ satisfies $L\,x = F(x)$ if $H(r, x_1) = (0, 0)$.

We will show that H has a root in $\mathbf{R} \times X_1$ by constructing an open set $B \subset \mathbf{R} \times X_1$, where the topological degree of H with respect to B and 0, denoted by $deg[H, B, 0]$, is equal to 1. Background in degree theory can be found in [26].

By H2.2, α_1 and α_2 are of opposite sign. Without loss of generality (replace k with $-k$ if necessary) we may assume that $\alpha_1 < 0$ and $\alpha_2 > 0$.

To construct the set B we first pick a real number s such that

(C1) $s > \|\overline{M} Q_1\| b$.

Making use of hypotheses H2.1 and H2.2, choose a real number $r_1 \geq R_s$ that satisfies the following conditions:

(C2) $\frac{\alpha_2}{2} < \psi^* F(r_1 k) < \alpha_2$

(C3) $\alpha_1 < \psi^* F(-r_1 k) < \frac{\alpha_1}{2}$

(C4) $r_1 > \max\{-\alpha_1, \alpha_2\}$

(C5) $\mu_s(r_1) < \dfrac{\min\{-\alpha_1, \alpha_2\}}{2\|\psi^*\| s}$.

Let $B \subset \mathbf{R} \times X_1$ be defined by $B = B_0 \times B_1$ where

$$B_0 = (-r_1, r_1) \text{ and} \tag{23a}$$
$$B_1 = \{x_1 \in X_1 : |x_1| < s\}. \tag{23b}$$

Denote the closure of B by $\overline{B}$. We now construct a homotopy G between H and the identity on $\overline{B}$.

Define $G = (G_{\mathbf{R}}, G_{X_1}) \colon \overline{B} \times [0, 1] \to \mathbf{R} \times X_1$ by

$$G_{\mathbf{R}}(r, x_1, \lambda) = (1 - \lambda) r + \lambda \psi^* F(rk) - \lambda \psi^* [F(rk) - F(rk + x_1)], \tag{24a}$$
$$G_{X_1}(r, x_1, \lambda) = x_1 - \lambda \overline{M} Q_1 F(rk + x_1). \tag{24b}$$

For each $\lambda \in [0,1]$, let $G_\lambda = G(\cdot,\cdot,\lambda)$. Clearly G is continuous on $\overline{B} \times [0,1]$, and G_0 is the identity map on $\overline{B}$. Since B contains $(0,0) \in \mathbf{R} \times X_1$, $deg[G_0, B, 0] = 1$.

We will show that for all (r, x_1) in the boundary of B, and all $\lambda \in [0,1]$, $G(r, x_1, \lambda) \neq (0,0)$.

Suppose $|x_1| = s,\ r \in [-r_1, r_1]$, and $\lambda \in [0,1]$. Then

$$|G_{X_1}(r, x_1, \lambda)| = |x_1 - \lambda \overline{M} Q_1 F(rk + x_1)| \geq s - \|\overline{M} Q_1\| b > 0 \quad \text{by } \textbf{(C1)}. \tag{25}$$

Suppose $r = r_1, x_1 \in \overline{B_1}$, and $\lambda \in [0,1]$. Then

$$\begin{aligned} |G_R(r, x_1, \lambda)| &= |(1-\lambda) r_1 + \lambda \psi^* F(r_1 k) - \lambda \psi^* [F(r_1 k) - F(r_1 k + x_1)]| \\ &\geq \min\{r_1, \psi^* F(r_1 k)\} - \|\psi^*\| \mu_s(r_1) s \quad \text{by } \textbf{H2.1} \\ &> \frac{\alpha_2}{2} - \|\psi^*\| \mu_s(r_1) s \quad \text{by } \textbf{(C2)} \text{ and } \textbf{(C4)} \\ &> 0 \quad \text{by } \textbf{(C5)}. \end{aligned}$$

A similar argument shows that if $r = -r_1,\ x_1 \in \overline{B}$, and $\lambda \in [0,1]$, then $|G_R(r, x_1, \lambda)| > 0$.

Since G is never 0 on the boundary of B, we have $1 = deg[G_0, B, 0] = deg[G_1, B, 0] = deg[H, B, 0]$, and thus there exists $(\hat{r}, \hat{x_1}) \in B$ such that $H(\hat{r}, \hat{x_1}) = (0,0)$. $Lx = F(x)$ is then solved by $x = \hat{r}k + \hat{x_1}$. □

Theorem 1 about discrete multipoint boundary value problems is a direct consequence of Theorem 2. Its proof follows.

Proof. Let $\widetilde{X}$ and β be as in the statement of Theorem 1. Let $X = \ker(\beta) \subseteq \widetilde{X}$ and $Y = \{y : \{0, \ldots, N-1\} \to \mathbf{R}^n\}$. Suppose that hypotheses H1.1-H1.6 are met. H1.2 implies that $\dim(X) = (N+1)n - n = Nn = \dim(Y)$. $L\colon\ X \to Y$ is defined by

$$(Lx)(t) = x(t+1) - A(t)x(t), \qquad t \in \{0, \ldots, N-1\}. \tag{26}$$

Then x satisfies the linear problem (3)-(2) if and only if $x \in \ker(L)$, and thus by H1.5 the kernel of L is one dimensional. Choose any norm $|\ |$ on $\mathbf{R}^n$ and then define norms on X and Y by taking the maximum over $t \in \{0, \ldots, N\}$ for $x \in X$ and over $t \in \{0, \ldots, N-1\}$ for $y \in Y$. We denote these maximum norms on X and Y by $|\ |_\infty$. Define $F\colon X \to Y$ by

$$F(x)(t) = f(t, x(t)) \qquad t \in \{0, \ldots, N-1\}. \tag{27}$$

Then x solves (1)-(2) if and only if $Lx = F(x)$.

We now proceed to verify the remaining hypotheses of Theorem 2. Since we are using the maximum norm on Y, H1.3 guarantees that F is continuous and bounded.

Let $k, |k|_\infty = 1$, be the non-zero vector in $\ker(L)$ guaranteed by H1.5. It follows from H1.1 that $t \in \{0, \dots, N\}$ implies that $k(t) \neq 0$. Let

$$m = \min_{t \in \{0,\dots,N\}} |k(t)|. \tag{28}$$

Then $0 < m \leq 1$. Let r_0 be the real number guaranteed by H1.4, and suppose $s > 0$. Choose $R_s \geq \frac{r_0+s}{m}$ and define $\mu_s\colon [R_s, \infty) \to [0, \infty)$ by:

$$\mu_s(r) = \delta(mr - s).$$

Then μ_s is well defined and approaches 0 as $r \to \infty$.

Now suppose $|r| \geq R_s$ and $|x_1| \leq s$.

For any $t \in \{0, \dots, N-1\}$, $|rk(t)| \geq |r|m \geq R_s m \geq r_0 + s$; and thus by H1.4 and the fact that δ is decreasing we have that:

$$\begin{aligned} |F(rk) - F(rk + x_1)|_\infty &= \max_{t \in \{0,\dots,N-1\}} |f(t, rk(t)) - f(t, rk(t) + x_1(t))| \\ &\leq \max_{t \in \{0,\dots,N-1\}} \delta(|rk(t)| - s)s \\ &\leq \delta(|r|m - s)s \\ &= \mu_s(|r|)s. \end{aligned}$$

Therefore, H2.1 is valid with this choice of R_s and μ_s.

In the discussion preceding H1.6, we stated (using Proposition 4.2 from [23]) that (4)-(2) has a solution for a given y, if and only if $\sum_{t=0}^{N-1} \psi(t)^T y(t) = 0$, where ψ is defined in equation (10). Therefore, if we define the linear functional $\psi^*\colon Y \to \mathbf{R}$, by $\psi^* y = \sum_{t=0}^{N-1} \psi(t)^T y(t)$, then $y \in \text{image}(L)$ if and only $\psi^* y = 0$. H2.2 then follows immediately from H1.6 with $\phi(t) = k(t)$.

Since all the hypotheses of Theorem 2 are met, there exists $x \in X$ satisfying (17). This x solves the problem (1)-(2). □

3. Example

The example presented in this section illustrates one way in which some of the hypotheses of Theorem 1 can be met. The nonlinear term $f(x)$ has "radial limits," i.e. $\lim_{r\to\infty} f(rx)$ exists for all $x \in \mathbf{R}^n$, and thus these limits certainly exist for $x \in \ker(L)$. Also, f is differentiable and $\|Df(x)\| \to 0$ as $|x| \to \infty$, yielding the estimate needed for H1.4.

We examine the solvability of the autonomous nonlinear three point boundary value problem:

$$x(t+1) = Ax(t) + f(x(t)) \quad t \in \{0, 1, \ldots 9\} \tag{29}$$

subject to

$$B_0 x(0) + B_5 x(5) + B_{10} x(10) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{30}$$

Here the constant matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $f : \mathbf{R}^2 \to \mathbf{R}^2$ is defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{x_1 - x_2}{1+\sqrt{x_1^2+x_2^2}} + v_1 \\ \frac{x_1+x_2}{1+\sqrt{x_1^2+x_2^2}} + v_2 \end{bmatrix}$, where v_1 and v_2 are constants. The matrices that define the boundary conditions are given by:

$$B_0 = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}, \; B_5 = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } B_{10} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

Note that $x = 0$ obviously solves (29)-(30) if $v_1 = v_2 = 0$. We will show that the problem can be solved for all $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in a certain unbounded set in $\mathbf{R}^2$.

Since A is constant, $\Phi(t, s) = A^{t-s}$ and the matrix D is given by

$$D = B_0 + B_5 A^5 + B_{10} A^{10} = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}.$$

For $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^2$, we choose $|x| = \max\{|x_1|, |x_2|\}$. Therefore $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a unit vector that spans the kernel of D, and thus the function $\phi : \{0, 1, \ldots 10\} \to \mathbf{R}^2$ defined by $\phi(t) = A^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ spans the solution space of the associated linear problem. Therefore H1.5 is valid.

A is invertible so it remains to verify hypotheses H1.2, H1.3, H1.4, and H1.6. $\ker(B_0^T) \bigcap \ker(B_5^T) \bigcap \ker(B_{10}^T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and thus the linear map β has full rank, validating H1.2.

Let $\hat{f}(x) = \hat{f}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{x_1 - x_2}{1+\sqrt{x_1^2+x_2^2}} \\ \\ \frac{x_1+x_2}{1+\sqrt{x_1^2+x_2^2}} \end{bmatrix}$. Then $f(x) = \hat{f}(x) + v$. Since f is continuous, $|\hat{f}(x)| < \sqrt{2}$ for all x and v is constant, H1.3 holds.

For all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $Df\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ is given by the matrix

$$\sigma\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\begin{bmatrix} \sqrt{x_1^2+x_2^2}+x_2^2+x_2x_1 & -(\sqrt{x_1^2+x_2^2}+x_1^2+x_2x_1) \\ \\ \sqrt{x_1^2+x_2^2}+x_2^2-x_2x_1 & \sqrt{x_1^2+x_2^2}+x_1^2-x_2x_1 \end{bmatrix},$$

where

$$\sigma\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{\sqrt{x_1^2+x_2^2}(1+\sqrt{x_1^2+x_2^2})^2}.$$

The matrix norm compatible with the max norm on $\mathbf{R}^2$ is the maximum of the sum of the absolute values of the entries of each row, and thus $|x| > 0$ implies that

$$\begin{aligned} \left\|Df\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\right\| &\leq \frac{\left(\left|\sqrt{x_1^2+x_2^2}+x_2^2+x_2x_1\right| + \left|\sqrt{x_1^2+x_2^2}+x_1^2+x_2x_1\right|\right)}{\sqrt{x_1^2+x_2^2}(1+\sqrt{x_1^2+x_2^2})^2} \\ &\leq \frac{2}{(1+\sqrt{x_1^2+x_2^2})^2} + \frac{x_1^2+x_2^2+2|x_1x_2|}{\sqrt{x_1^2+x_2^2}(1+\sqrt{x_1^2+x_2^2})^2} \\ &= \frac{2}{(1+\sqrt{x_1^2+x_2^2})^2} + \frac{(|x_1|+|x_2|)^2}{\sqrt{x_1^2+x_2^2}(1+\sqrt{x_1^2+x_2^2})^2} \\ &\leq \frac{2}{(1+\sqrt{x_1^2+x_2^2})^2} + \frac{2(x_1^2+x_2^2)}{\sqrt{x_1^2+x_2^2}(1+\sqrt{x_1^2+x_2^2})^2} \\ &= \frac{2}{(1+\sqrt{x_1^2+x_2^2})}. \end{aligned}$$

Let $\delta : (0,\infty) \to (0,\infty)$ be defined by $\delta(r) = \frac{2}{1+r}$. Then $|x| \geq r > 0$ implies $\left\|Df\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\right\| \leq \delta(r)$. Since $\lim_{r\to\infty} \delta(r) = 0$, then H1.4 follows by the (vector) mean value theorem.

Next note that

$$\lim_{r\to\infty} \hat{f}(rx) = \lim_{r\to\infty} \begin{pmatrix} \frac{r(x_1-x_2)}{1+r\sqrt{x_1^2+x_2^2}} \\ \\ \frac{r(x_1+x_2)}{1+r\sqrt{x_1^2+x_2^2}} \end{pmatrix} = \frac{1}{\sqrt{x_1^2+x_2^2}} \begin{pmatrix} x_1 - x_2 \\ \\ x_1 + x_2 \end{pmatrix}.$$

Define $\gamma(x) = \lim_{r\to\infty} \hat{f}(rx)$. Observe that γ is an odd function of x and that $\lim_{r\to\infty} f(rx) = \gamma(x) + v$.

The last item that needs to be verified is that the limits J_1 and J_2 exist and satisfy $J_1 J_2 < 0$. Since

$$\begin{aligned} J_1 &= \lim_{r\to-\infty} \sum_{t=0}^{N-1} \psi(t)^T f(r\phi(t)) \\ &= \sum_{t=0}^{N-1} \psi(t)^T \lim_{r\to-\infty} f(r\phi(t)) \\ &= \sum_{t=0}^{N-1} \psi(t)^T (-\gamma(\phi(t)) + v) \quad \text{(since } \gamma \text{ is odd)} \end{aligned}$$

and

$$\begin{aligned} J_2 &= \lim_{r\to\infty} \sum_{t=0}^{N-1} \psi(t)^T f(r\phi(t)) \\ &= \sum_{t=0}^{N-1} \psi(t)^T \lim_{r\to\infty} f(r\phi(t)) \\ &= \sum_{t=0}^{N-1} \psi(t)^T (\gamma(\phi(t)) + v), \end{aligned}$$

then $J_1 J_2 = -\left(\sum_{t=0}^{N-1} \psi(t)^T \gamma(\phi(t))\right)^2 + \left(\sum_{t=0}^{N-1} \psi(t)^T v\right)^2$, and thus we need

$$\left|\sum_{t=0}^{N-1} \psi(t)^T v\right| < \left|\sum_{t=0}^{N-1} \psi(t)^T \gamma(\phi(t))\right|. \tag{31}$$

To help interpret the inequality (31), let

$$\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}, \gamma(x) = \begin{bmatrix} \gamma_1(x) \\ \gamma_2(x) \end{bmatrix}, \mathbf{\Psi} = \begin{bmatrix} \psi_1(0) \\ \psi_2(0) \\ \psi_1(1) \\ \psi_2(1) \\ \vdots \\ \psi_1(9) \\ \psi_2(9) \end{bmatrix}, \ \mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_1 \\ v_2 \\ \vdots \\ v_1 \\ v_2 \end{bmatrix},$$

and $\mathbf{\Gamma} = \begin{bmatrix} \gamma_1(\phi(0)) \\ \gamma_2(\phi(0)) \\ \gamma_1(\phi(1)) \\ \gamma_2(\phi(1)) \\ \vdots \\ \gamma_1(\phi(9)) \\ \gamma_2(\phi(9)) \end{bmatrix}$.

Then (31) is equivalent to

$$|\mathbf{\Psi} \cdot \mathbf{V}| < |\mathbf{\Psi} \cdot \mathbf{\Gamma}|. \tag{32}$$

Since $\mathbf{\Psi}$ and $\mathbf{\Gamma}$ are fixed and independent of $\mathbf{V}$, we can choose v_1 and v_2, so that $|\mathbf{V}|_2 < \frac{|\mathbf{\Psi}\cdot\mathbf{\Gamma}|}{|\mathbf{\Psi}|_2}$ and (32) will be satisfied. Specifically, since $|\mathbf{V}|_2 = \sqrt{10}|v|_2$, then choosing v_1 and v_2 so that $|v|_2 < \frac{|\mathbf{\Psi}\cdot\mathbf{\Gamma}|}{\sqrt{10}|\mathbf{\Psi}|_2}$ will guarantee that (32) holds and thus H1.6a is satisfied.

Although sufficient, it is not necessary that $|v|_2 < \frac{|\mathbf{\Psi}\cdot\mathbf{\Gamma}|}{\sqrt{10}|\mathbf{\Psi}|_2}$. If we let $u_1 = \mathbf{\Psi} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$ and $u_2 = \mathbf{\Psi} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$, then $\mathbf{\Psi} \cdot \mathbf{V} = u_1 v_1 + u_2 v_2$ and the inequality (32) is equivalent to

$$|u_1 v_1 + u_2 v_2| < |\mathbf{\Psi} \cdot \mathbf{\Gamma}| \tag{33}$$

which describes a tube in $\mathbf{R}^2$. Therefore the inequality (31) can also hold for $|v|$ large, as long as v lies inside the tube given by (33).

Since hypotheses H1.1-H1.6 hold, the boundary value problem (29)-(30) has a solution.

4. Conclusion and Future Directions

A variety of different assumptions and techniques have been used to treat discrete boundary value problems. One segment of the literature has studied nonlinear problems at resonance in the context of scalar equations. In this paper we show one way to extend these results to systems. Our method requires the nonlinearity, $f(t, x)$, to be bounded and for $|f(t, x_0) - f(t, x_0 + x_1)| \to 0$ for

$|x_1|$ bounded and $|x_0| \to \infty$. Natural extensions of our work might include loosening these restrictions on f and generalizing the type of boundary conditions considered.

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