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ON THE THERMISTOR PROBLEM WITH MIXED BOUNDARY CONDITIONS

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Abstract: In this paper, a standard Galerkin finite element approximate procedure combined with a fixed point algorithm is presented for solving the nonlinear coupled thermistor problem with mixed boundary conditions. The existence and uniqueness of weak solutions are established. The convergence of the approximate solution is analyzed and the corresponding error estimate is given.

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1. Introduction

While studying the modeling of an electrochemical engineering problem, we came across a coupled system between temperature and electrical potential. One of the coupling terms arises from the "Joule effect", i.e. the production of heat by electrical current, see for instance [1, 2, 6, 8, 11], and the references therein, we come across the following coupled nonlinear elliptic problem:

$$\begin{cases} (a) & -\nabla\cdot(\mu(\theta)\nabla u) = f & \text{in } \Omega, \\ (b) & -\nabla\cdot(k(\theta)\nabla\theta) = \mu(\theta)|\nabla u|^2 & \text{in } \Omega, \\ (c) & u = 0, \quad \theta = 0 & \text{on } \Gamma_D, \\ (d) & \frac{\partial u}{\partial \mathbf{n}} = 0, \quad \frac{\partial \theta}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_N, \end{cases} \tag{1}$$

where $u : \Omega \to I\!R$ is the potential, $\theta : \Omega \to I\!R$ is the temperature, $\sigma(\theta)$ is the

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temperature dependent electrical conductivity, $k(\theta)$ is the thermal conductivity od the device, and Ω is a bounded open subset of $I\!R^d$, $2 \leq d \leq 3$. $\partial\Omega = \bar{\Gamma}_D \cap \bar{\Gamma}_N$. $\mathbf{n}$ denotes the outward unit normal and $\partial/\partial\mathbf{n} = \mathbf{n} \cdot \nabla$ is the normal derivative on $\partial\Omega$.

This type of problems have received especial attention recently. Mathematical analysis of the problem with $f = 0$ in Dirichlet boundary conditions case can be found in, for example, [3, 4, 5, 10], and the existence, boundedness and uniqueness of this problem with mixed boundary conditions in the nonstationary case are established in [13], whereas numerical analysis in the nonstationary case is studied in [7]. Standard Galerkin and mixed finite element analysis for this problem with $k = 1$ in Dirichlet boundary conditions case are studied in [12, 15], respectively.

The purpose of this paper is to study the numerical approximation for the system (1). By use of the same technique in [12], we establish existence and uniqueness of the weak solution of the mixed boundary value problem (1), where $k(\theta)$ is not a constant. Then we propose a finite element approximate procedure with a fixed point algorithm and analyze the convergence of the approximate solution. Finally, we give the error estimate.

Throughout this paper, for simplification we assume that $\mu(\theta)$ and $k(\theta)$ satisfy the following conditions:

$$\begin{cases} (a) \quad \mu(s) \in C(\mathbb{R}), \;\; \mu_* \leq \mu(s) \leq \mu^*, \;\; \forall s \in \mathbb{R}, \\ (b) \quad \|\mu'(s)\|_{L^\infty} \leq L, \;\; \forall s \in \mathbb{R}, \\ (c) \quad k(s) \in C(\mathbb{R}), \;\; k_* \leq k(s) \leq k^*, \;\; \forall s \in \mathbb{R}, \\ (d) \quad \|k'(s)\|_{L^\infty} \leq L', \;\; \forall s \in \mathbb{R}. \end{cases} \tag{2}$$

The outlook of this paper is as followed. For convenience, we firstly introduce some preliminaries in Section 2. Then we analyze the existence and uniqueness of the weak solution in Section 3. Next, we introduce a fixed point algorithm to solve the problem (1) in Section 4. Finally we propose a finite element procedure with fixed point algorithm and analyze the convergence of the approximate solution.

2. Some Lemmas

Definition 1. We denote by $\mathcal{R}_q$ for $1 < q < \infty$ the class of regular subsets G in $I\!R^d$ for which the Laplacian operator maps $W_0^{1,q}(G)$ onto $W^{-1,q}(G)$.

Remark 1. A bounded C^1 domain, for example, is of class $\mathcal{R}_q$ for every

$q \in (2, \infty)$, see Theorem 4.6 in [14].

Remark 2. The definition of $\mathcal{R}_q$ here is similar to that in [9].

From now on, we assume $G \in R_q$ for some $q > 2$. For $2 \le p \le q$, we define $M_p \ge 1$ by

$$\inf_{v\in W_0^{1,p'}\backslash\{0\}} \sup_{u\in W_0^{1,p}\backslash\{0\}} \frac{|(\nabla u, \nabla v)|}{\|\nabla u\|_{L^p}\,\|\nabla v\|_{L^{p'}}} = \frac{1}{M_p}, \tag{3}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(G)^d$ or the duality between $L^p(G)^d$ and $L^{p'}(G)^d$, p' is the dual number of p.

Remark 3. We can see that (see [9]) $M_2 = 1$ and $M_{p'} = M_p$, which implies that

$$\begin{aligned} &\inf_{v\in W_0^{1,p'}\backslash\{0\}} \sup_{u\in W_0^{1,p}\backslash\{0\}} \frac{|(\nabla u, \nabla v)|}{\|\nabla u\|_{L^p}\,\|\nabla v\|_{L^{p'}}} \\ =\;& \inf_{u\in W_0^{1,p}\backslash\{0\}} \sup_{v\in W_0^{1,p'}\backslash\{0\}} \frac{|(\nabla u, \nabla v)|}{\|\nabla u\|_{L^p}\,\|\nabla v\|_{L^{p'}}} \\ =\;& \frac{1}{M_p}. \end{aligned} \tag{4}$$

Lemma 1. (see [9]) *Let $G \in R_q$ for some $q > 2$. Then $G \in R_p$ for $2 \le p \le q$, and $M_p \le M_q^\lambda$ if $\frac{1}{p} = \frac{1-\lambda}{2} + \frac{\lambda}{q}$.*

As for the more general operator $\mathcal{A} : W_0^{1,2} \longrightarrow W^{-1,2}$, defined by

$$< \mathcal{A}u, v >= (\mu(\theta)\nabla u, \nabla v), \qquad \forall\, v \in W_0^{1,2}, \tag{5}$$

we have

Lemma 2. *If $2 \le p \le q$ and*

$$M_p \frac{\mu^* - \mu_* 1}{\mu^* + \mu_*} < 1, \tag{6}$$

then the operator $\mathcal{A}$ maps $W_0^{1,p}$ onto $W^{-1,p}$. And the following inf-sup *estimate holds:*

$$\begin{aligned} &\inf_{v\in W_0^{1,p'}\backslash\{0\}} \sup_{u\in W_0^{1,p}\backslash\{0\}} \frac{|(\mu(\theta)\nabla u, \nabla v)|}{\|\nabla u\|_{L^p}\,\|\nabla v\|_{L^{p'}}} \\ =\;& \inf_{u\in W_0^{1,p}\backslash\{0\}} \sup_{v\in W_0^{1,p'}\backslash\{0\}} \frac{|(\mu(\theta)\nabla u, \nabla v)|}{\|\nabla u\|_{L^p}\,\|\nabla v\|_{L^{p'}}} \\ \ge\;& \frac{1}{C_p}, \end{aligned} \tag{7}$$

where

$$\frac{1}{C_p} = \frac{\mu^* + \mu_*}{2M_p}\big(1 - M_p\frac{\mu^* - \mu_*}{\mu^* + \mu_*}\big) > 0. \tag{8}$$

Moreover, condition (6) can be satisfied if

$$\frac{1}{p} > \frac{1}{2} - \big(\frac{1}{2} - \frac{1}{q}\big)\frac{\log(\mu^* + \mu_*) - \log(\mu^* - \mu_*)}{\log M_q}. \tag{9}$$

Proof. It is enough to establish the *inf-sup* estimates (7). We can see firstly that

$$\inf_{v\in W_0^{1,p'}\setminus\{0\}} \sup_{u\in W_0^{1,p}\setminus\{0\}} \frac{|(\mu(\theta)\nabla u, \nabla v)|}{\|\nabla u\|_{L^p}\|\nabla v\|_{L^{p'}}} = \inf_{u\in W_0^{1,p}\setminus\{0\}} \sup_{v\in W_0^{1,p'}\setminus\{0\}} \frac{|(\mu(\theta)\nabla u, \nabla v)|}{\|\nabla u\|_{L^p}\|\nabla v\|_{L^{p'}}}.$$

Next, we write

$$(\mu(\theta)\nabla u, \nabla v) = \frac{\mu^* + \mu_*}{2}(\nabla u, \nabla v) + \big(\big(\mu(\theta) - \frac{\mu^* + \mu_*}{2}\big)\nabla u, \nabla v\big).$$

Notice that

$$\big|\big(\big(\mu(\theta) - \frac{\mu^* + \mu_*}{2}\big)\nabla u, \nabla v\big)\big| \leq \frac{\mu^* - \mu_*}{2}\|\nabla u\|_{L^p}\|\nabla v\|_{L^{p'}},$$

we get

$$\begin{aligned}
&\inf_{v\in W_0^{1,p'}\setminus\{0\}} \sup_{u\in W_0^{1,p}\setminus\{0\}} \frac{|(\mu(\theta)\nabla u, \nabla v)|}{\|\nabla u\|_{L^p}\|\nabla v\|_{L^{p'}}}\\
= &\inf_{u\in W_0^{1,p}\setminus\{0\}} \sup_{v\in W_0^{1,p'}\setminus\{0\}} \frac{|(\mu(\theta)\nabla u, \nabla v)|}{\|\nabla u\|_{L^p}\|\nabla v\|_{L^{p'}}}\\
\geq &\frac{\mu^* + \mu_*}{2M_p} - \frac{\mu^* - \mu_*}{2}\\
= &\frac{\mu^* + \mu_*}{2M_p}\big(1 - M_p\frac{\mu^* - \mu_*}{\mu^* + \mu_*}\big),
\end{aligned}$$

if (6) satisfies, which is (7).

Finally, by Lemma 1, (9) implies (6). □

Remark 4. By (9), it is easy to see that $p \to 2$ as $\mu_*/\mu^* \to 0$, and $p = q$ when $\frac{\mu^*+\mu_*}{\mu^*-\mu_*} \geq M_q$.

Similar to Theorem 1 in [9], we now can get the following lemma.

Lemma 3. *Let $G \in R_q$, for any given θ, $u \in W_0^1(G)$ satisfies (1(a)), then there exists a $p \in (2, q]$ satisfying (9) and a constant $C_p > 0$ defined by (8) such* *that* $u \in W_0^{1,p}(G)$ *and the following estimate holds*

$$\|\nabla u\|_{L^p} \le C_p \|f\|_{W^{-1,p}}. \tag{10}$$

Remark 5. If $f \in L^2(\Omega)$, by Lemma 3 and the Sobolev inequality, there is a $p \in (2, q]$ such that

$$\|\nabla u\|_{L^p} \le C\|f\|_{L^2}. \tag{11}$$

3. Variational Formulation

Throughout this work, we assume that $f \in L^2(\Omega)$, and define the spaces: $V = \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0\}$. Then the variational formulation of problem (1) can be defined as:

$$\begin{cases} \text{Find } (u,\theta) \in V \times V \text{ such that} \\ (a) \quad (\mu(\theta)\nabla u, \nabla v) = (f, v), & \forall\, v \in V, \\ (b) \quad (k(\theta)\nabla\theta, \nabla\eta) = (\mu(\theta)|\nabla u|^2, \eta), & \forall\, \eta \in V \cap L^\infty(\Omega), \end{cases} \tag{12}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)^d$ or the duality between $L^p(\Omega)^d$ and $L^{p'}(\Omega)^d$, p' is the dual number of p.

By use of Lemma 3 and Remark 5, for any $\eta \in V$, and p defined as in Lemma 3, and satisfying that

$$p \ge 12/5, \quad \text{if } d = 3, \tag{13}$$

we have

$$|(\mu(\theta)|\nabla u|^2, \eta)| \le \mu^* \|\nabla u\|_{L^p}^2 \|\eta\|_{L^{p/(p-2)}} \le C\|f\|_{L^2}^2 \|\nabla\eta\|_{L^2}. \tag{14}$$

Therefore, by the density of $V \cap L^\infty(\Omega)$ in V, problem (12) can be written equivalently as:

$$\begin{cases} \text{Find } (u,\theta) \in V \times V \text{ such that} \\ (a) \quad (\mu(\theta)\nabla u, \nabla v) = (f, v), & \forall\, v \in V, \\ (b) \quad (k(\theta)\nabla\theta, \nabla\eta) = (\mu(\theta)|\nabla u|^2, \eta), & \forall\, \eta \in V. \end{cases} \tag{15}$$

We are now going to show existence of a solution of problem (15). For any given $\xi \in L^2(\Omega)$, we denote by $u_\xi \in V$ the solution of

$$(\mu(\xi)\nabla u_\xi, \nabla v) = (f, v), \qquad \forall\, v \in V. \tag{16}$$

Next, we define by $\theta_\xi \in V$ the solution of

$$(k(\xi)\nabla\theta_\xi, \nabla\eta) = (\mu(\xi)|\nabla u_\xi|^2, \eta), \qquad \forall\, \eta \in V. \tag{17}$$

By (17), (2) and (14), we have

$$\|\nabla\theta_\xi\|_{L^2} \le C_0 \|f\|_{L^2}^2, \tag{18}$$

where constant $C_0 > 0$ depends only on Ω, k_* and μ^*.

We define the map $T: L^2(\Omega) \to V$ by

$$T(\xi) = \theta_\xi, \qquad \forall\, \xi \in L^2(\Omega). \tag{19}$$

It is obviously compact since V is compactly imbedded in $L^2(\Omega)$. The continuity of T can be obtained similarly to that in [10]. Moreover, let B_R be the bounded closed ball of $L^2(\Omega)$ defined by

$$B_R = \left\{ \eta \in L^2(\Omega) \,\middle|\, \|\eta\|_{L^2} \le C_0 C(\Omega) \|f\|_{L^2}^2 \right\}, \tag{20}$$

where $C(\Omega)$ is the Poincaré-Friedrichs constant such that

$$\|\eta\|_{L^2} \le C(\Omega) \|\nabla \eta\|_{L^2}, \qquad \forall\, \eta \in V.$$

Then, T maps B_R into itself. Hence, the solvability of problem (15) comes from the Schauder Fixed Point Theorem.

Theorem 1. (Existence) *Assume that conditions (2a-d) hold, and p, defined in Lemma 3, then problem (15) is equivalent to problem (12), and has a solution (u, θ). Furthermore, there exists a constant $C > 0$ dependent only on Ω, k_*, μ^*, μ_* and p such that*

$$\|\nabla u\|_{L^p} \le C \|f\|_{L^2} \tag{21}$$

and

$$\|\nabla \theta\|_{L^{\tilde{p}}} \le C \|f\|_{L^2}^2, \tag{22}$$

where

$$\tilde{p} = \begin{cases} dp/(2d - p) \ge 2, & \text{if} \quad p < 2d \\ \text{any number in } (2, \infty), & \text{if} \quad p \ge 2d. \end{cases} \tag{23}$$

Proof. It is only needed to prove (22). In fact, by Sobolev inequality, we have

$$\begin{aligned} \|\nabla \theta\|_{L^{\tilde{p}}} &\le \frac{1}{k_*} \|k(\theta) \nabla \theta\|_{L^{\tilde{p}}} \le \frac{C_1}{k_*} \|\nabla \cdot (k(\theta) \nabla \theta)\|_{L^{p/2}} \\ &= \frac{C_1}{k_*} \|\mu(\theta) |\nabla u|^2\|_{L^{p/2}} \le \frac{C_1 \mu^*}{k_*} \|\nabla u\|_{L^p}^2 \\ &\le C \|f\|_{L^2}^2, \end{aligned}$$

where $C > 0$ is a constant dependent only on Ω, k_*, μ^* and p. ∎

Next, we study the uniqueness of the problem (15).

Assume that the problem (15) has two solutions (u_1, θ_1) and (u_2, θ_2), and let $\tilde{u} = u_1 - u_2$ and $\tilde{\theta} = \theta_1 - \theta_2$. Then, by (15), we have

$$(\mu(\theta_1) \nabla \tilde{u}, \nabla v) = ([\mu(\theta_2) - \mu(\theta_1)] \nabla u_2, \nabla v), \qquad \forall\, v \in V, \tag{24}$$

and

$$(k(\theta_1)\nabla\tilde{\theta},\nabla\eta) = ([k(\theta_2)-k(\theta_1)]\nabla\theta_2,\nabla\eta) + (\mu(\theta_1)|\nabla u_1|^2 - \mu(\theta_2)|\nabla u_2|^2,\eta), \quad \forall\, \eta\in V. \tag{25}$$

By (24), (2), (21) and the Sobolev inequality, for

$$p \geq 3, \qquad \text{if} \quad d=3, \tag{26}$$

we have

$$\begin{aligned}\|\nabla\tilde{u}\|_{L^2} &\leq 1/\mu_*\|[\mu(\theta_2)-\mu(\theta_1)]\nabla u_2\|_{L^2}\\ &\leq L/\mu_*\|\tilde{\theta}\|_{L^{2p/(p-2)}}\|\nabla u_2\|_{L^p} \leq C\|f\|_{L^2}\|\nabla\tilde{\theta}\|_{L^2},\end{aligned} \tag{27}$$

where $C>0$ is a constant dependent on Ω, μ_* and L.

Let $\eta=\tilde{\theta}$ in (25), we have

$$\begin{aligned}(k(\theta_1)\nabla\tilde{\theta},\nabla\tilde{\theta}) &= ([k(\theta_2)-k(\theta_1)]\nabla\theta_2,\nabla\tilde{\theta})\\ &\quad +(\mu(\theta_1)|\nabla u_1|^2-\mu(\theta_2)|\nabla u_2|^2,\tilde{\theta})\\ &= R_1+R_2.\end{aligned}$$

From (23) and (26) we know that

$$\tilde{p}\geq 3, \qquad \text{if} \quad d=3. \tag{28}$$

Using (2), (27), (22) and (28) we have

$$\begin{aligned}R_1 &= ([k(\theta_1)-k(\theta_2)]\nabla\theta_2,\nabla\tilde{\theta})\\ &\leq \|[k(\theta_1)-k(\theta_2)]\nabla\theta_2\|_{L^2}\|\nabla\tilde{\theta}\|_{L^2} \leq L'\|\tilde{\theta}\nabla\theta_2|\|_{L^2}\|\nabla\tilde{\theta}\|_{L^2}\\ &\leq L'\|\nabla\theta_2\|_{L^{\tilde{p}}}\|\tilde{\theta}\|_{L^{2\tilde{p}/(\tilde{p}-2)}}\|\nabla\tilde{\theta}\|_{L^2} \leq C\|f\|^2_{L^2}\|\nabla\tilde{\theta}\|^2_{L^2}.\end{aligned}$$

On the other hand, by use of the Hölder inequality, the Sobolev inequality, (2), (27) and (21), we have

$$\begin{aligned}R_2 &= (\mu(\theta_1)|\nabla u_1|^2-\mu(\theta_2)|\nabla u_2|^2,\tilde{\theta})\\ &\leq \|\mu(\theta_1)|\nabla u_1|^2-\mu(\theta_2)|\nabla u_2|^2\|_{L^{2p/(2+p)}}\|\tilde{\theta}\|_{L^{2p/(p-2)}}\\ &\leq C\Big\{\|\mu(\theta_1)\nabla\tilde{u}\cdot\nabla(u_1+u_2)\|_{L^{2p/(2+p)}}\\ &\qquad +\|[\mu(\theta_1)-\mu(\theta_2)]|\nabla u_2|^2\|_{L^{2p/(2+p)}}\Big\}\|\nabla\tilde{\theta}\|_{L^2}\\ &\leq C\Big\{\mu^*\|\nabla\tilde{u}\|_{L^2}\|\nabla(u_1+u_2)\|_{L^p}\\ &\qquad +L\|\tilde{\theta}\|_{L^{2p/(p-2)}}\|\nabla u_2\|^2_{L^p}\Big\}\|\nabla\tilde{\theta}\|_{L^2}\\ &\leq C\|f\|^2_{L^2}\|\nabla\tilde{\theta}\|^2_{L^2}.\end{aligned}$$

Hence, we obtain

$$\|\nabla\tilde{\theta}\|^2_{L^2} \leq \tilde{C}\|f\|^2_{L^2}\|\nabla\tilde{\theta}\|^2_{L^2}, \tag{29}$$

where $\tilde{C}$ is a constant dependent on Ω, μ^*, μ_*, k^*, k_*, L and L'.

Therefore, if

$$\tilde{C}\|f\|_{L^2}^2 < 1, \tag{30}$$

then it should hold that $\tilde{\theta} = 0$, and which implies that $\tilde{u} = 0$ by (27).

Theorem 2. (Uniqueness) *Assume that the assumptions* (2) *hold. If* p *satisfies* (26)*, and condition* (30) *holds, then the problem* (15) *has a unique solution.*

4. A Fixed Point Algorithm

From numerical point of view, it is interesting to introduce an iterative scheme to solve problem (15). The scheme proposed in this section is based on a fixed point algorithm.

For an arbitrary θ^0, and $n = 1, 2, \cdots$, we can get $\{(u^n, \theta^n)\}$ by:

$$\begin{cases} \text{Find } (u^n, \theta^n) \in V \times V \text{ such that} \\ (a) \quad (\mu(\theta^{n-1})\nabla u^n, \nabla v) = (f, v), & \forall\, v \in V, \\ (b) \quad (k(\theta^{n-1})\nabla \theta^n, \nabla \eta) = (\mu(\theta^{n-1})|\nabla u^n|^2, \eta), & \forall\, \eta \in V. \end{cases} \tag{31}$$

Similarly to Theorem 1, we can prove

Theorem 3. *The solution of* (31) $\{(u^n, \theta^n)\}$ *satisfies that,*

$$\|\nabla u^n\|_{L^p} \le C\|f\|_{L^2}, \qquad \forall\, n \ge 1, \tag{32}$$

$$\|\nabla \theta^n\|_{L^{\tilde{p}}} \le C\|f\|_{L^2}^2, \qquad \forall\, n \ge 1, \tag{33}$$

where p *and* $\tilde{p}$ *are the same as in Theorem 1. Moreover, we have*

$$\|u^n\|_{1+\sigma} + \|\theta^n\|_{1+\sigma} \le K, \quad \forall\, n \ge 1. \tag{34}$$

Theorem 4. *If problem* (15) *has a unique solution* (u, θ)*, and* p *is the same as in Theorem 2, then the sequence* $\{(u^n, \theta^n)\}$ *defined by* (31) *converges in* $V \times V$ *to* (u, θ).

Proof. If the results were not true, then there exist some small constant $\varepsilon_0 > 0$ and an infinite subsequence of $\{(u^n, \theta^n)\}$, denoted by $\{(u^{n_i}, \theta^{n_i})\}$, such that

$$\|\nabla(u - u^{n_i})\|_{L^2} + \|\nabla(\theta - \theta^{n_i})\|_{L^2} \ge \varepsilon_0, \qquad \forall\, i. \tag{35}$$

On the other hand, $\{(u^{n_i}, \theta^{n_i})\} \in W^{1,p}(\Omega) \times W^{1,\tilde{p}}(\Omega)$, with p and $\tilde{p} > 2$. Since the space $W^{1,p}(\Omega) \times W^{1,\tilde{p}}(\Omega)$ is compact in $H^1(\Omega) \times H^1(\Omega)$, and notice that any limit of $\{(u^n, \theta^n)\}$ should satisfy (15) and problem (15) has a unique solution. Then we can get a subsequence of $\{(u^{n_i}, \theta^{n_i})\}$ which converges to

(u, θ) in $V \times V$, which leads a contradiction to (35). Hence, we complete the proof. □

To further study $u - u^n$ and $\theta - \theta^n$, by (15) and (31), by use of the same techniques of (27) and (29), we can deduce that

$$\|\nabla(u - u^n)\|_{L^2} \leq C\|f\|_{L^2}\|\nabla(\theta - \theta^{n-1})\|_{L^2},$$
$$\|\nabla(\theta - \theta^n)\|_{L^2} \leq \tilde{C}\|f\|_{L^2}^2\|\nabla(\theta - \theta^{n-1})\|_{L^2},$$

where $\tilde{C}$ is same as in (29). Thus, we have

Theorem 5. *If condition* (30) *holds, then the fixed point algorithm* (31) *works with the linear convergence rate, and the following estimates hold:*

$$\|\nabla(u - u^n)\|_{L^2} \leq C\|f\|_{L^2}\tilde{M}(f)^{n-1}\|\nabla(\theta - \theta^0)\|_{L^2}, \tag{36}$$

and

$$\|\nabla(\theta - \theta^n)\|_{L^2} \leq \tilde{M}(f)^n\|\nabla(\theta - \theta^0)\|_{L^2}, \tag{37}$$

where $\tilde{M}(f) = \tilde{C}\|f\|_{L^2}^2 < 1$.

5. Finite Element Approximation

For simplicity we assume that Ω is a polygonal (or polyhedral) domain discretized by a quasi uniform mesh of Ne triangles (or tetrahedrons) or convex quadrilaterals (or hexahedrons), with mesh parameter h. Let S_h be the Lagrangian finite element space of $C^0(\Omega)$ piecewise linear polynomials, $V_h = S_h \cap V$.

The Galerkin approximation to problem (31) reads:
Given θ_h^0 as an approximation of θ^0, for $n = 1, 2, \cdots$, $\{(u_h^n, \theta_h^n)\}$ can be calculated by:

$$\begin{cases} \text{Find } (u_h^n, \theta_h^n) \in V_h \times M_h \text{ such that} \\ (a) \quad (\mu(\theta_h^{n-1})\nabla u_h^n, \nabla v_h) = (f, v_h), & \forall\, v_h \in V_h, \\ (b) \quad (k(\theta_h^{n-1})\nabla \theta_h^n, \nabla \eta_h) = (\mu(\theta_h^{n-1})|\nabla u_h^n|^2, \eta_h), & \forall\, \eta_h \in V_h. \end{cases} \tag{38}$$

To analyze problem (38), we introduce the standard elliptic projections $(\tilde{u}_h^n, \tilde{\theta}_h^n) \in V_h \times V_h$ defined by

$$(\nabla(\tilde{u}_h^n - u^n), \nabla v_h) = 0, \qquad \forall\, v_h \in V_h, \tag{39}$$

$$(\nabla(\tilde{\theta}_h^n - \theta^n), \nabla \eta_h) = 0, \qquad \forall\, \eta_h \in V_h. \tag{40}$$

It is well known that

Lemma 4. *There exists a constant* $C > 0$ *independent of* h *and* n *such* *that the following estimates hold:*

$$\|u^n - \tilde{u}_h^n\|_{L^2} + h\|\nabla(u^n - \tilde{u}_h^n)\|_{L^2} \le Ch^{1+\sigma}\|u^n\|_{1+\sigma}, \tag{41}$$

$$\|\theta^n - \tilde{\theta}_h^n\|_{L^2} + h\|\nabla(\theta^n - \tilde{\theta}_h^n)\|_{L^2} \le Ch^{1+\sigma}\|\theta^n\|_{1+\sigma}, \tag{42}$$

where $1/2 < \sigma < 1$.

The following inverse property of finite element space V_h is useful.

Lemma 5. *For any* $\eta_h \in M_h$, *we have*

$$\|\eta_h\|_{L^\infty} \le M(h)\|\nabla \eta_h\|_{L^2}, \tag{43}$$

where

$$M(h) = \begin{cases} M, & \text{if} \quad d = 1, \\ M|\log h|^{1/2}, & \text{if} \quad d = 2, \\ Mh^{-1/2}, & \text{if} \quad d = 3, \end{cases} \tag{44}$$

and M *is a constant independent of* h.

For the errors of $u_h^n - \tilde{u}_h^n$ and $\theta_h^n - \tilde{\theta}_h^n$, we have:

Lemma 6. *There exists a constant* $C > 0$ *dependent only on* Ω, k_*, k^* *and* L *such that*

$$\|\nabla(u_h^n - \tilde{u}_h^n)\|_{L^2} \le C\Big\{\|f\|_{L^2}\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + h^\sigma\|u^n\|_{1+\sigma}\Big\}. \tag{45}$$

Proof. By (38(a)) and (31(a)), we have, $\forall\, v_h \in V_h$

$$\begin{aligned} &(\mu(\theta_h^{n-1})\nabla(u_h^n - \tilde{u}_h^n), \nabla v_h) \\ &= ([\mu(\theta^{n-1}) - \mu(\theta_h^{n-1})]\nabla u^n + \mu(\theta_h^{n-1})(\nabla u^n - \nabla \tilde{u}_h^n), \nabla v_h). \end{aligned}$$

Let $v_h = u_h^n - \tilde{u}_h^n$, we can get

$$\begin{aligned} &\|\nabla(u_h^n - \tilde{u}_h^n)\|_{L^2} \\ &\le 1/\mu_*\Big\{L\|\nabla u^n\|_{L^p}\|\theta^{n-1} - \theta_h^{n-1}\|_{L^{2p/(p-2)}} + \mu^*\|\nabla(u^n - \tilde{u}_h^n)\|_{L^2}\Big\} \\ &\le C\Big\{\|\nabla(u^n - \tilde{u}_h^n)\|_{L^2} + \|f\|_{L^2}\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}\Big\}. \end{aligned}$$

Thus, Lemma 4 leads to (45). □

Lemma 7. *The following estimate*

$$\begin{aligned} &\|\nabla(\theta_h^n - \tilde{\theta}_h^n)\|_{L^2} \\ &\le C\|f\|_{L^2}^2\Big\{1 + M(h)\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}\Big\}\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} \\ &\quad + Ch^\sigma\Big\{[\|f\|_{L^2} + M(h)h^\sigma\|u^n\|_{1+\sigma}]\|u^n\|_{1+\sigma} + \|\theta^n\|_{1+\sigma}\Big\} \end{aligned} \tag{46}$$

holds with $C > 0$ *dependent only on* Ω, μ^*, μ_*, k^*, k_*, L, L', *and* $M(h)$ *defined by* (44).

Proof. By (38(b)), (40) and (31(b)), we have

$$\begin{aligned}(k(\theta_h^{n-1})\nabla(\theta_h^n-\tilde{\theta}_h^n),\nabla(\theta_h^n-\tilde{\theta}_h^n))\\ &=([k(\theta^{n-1})-k(\theta_h^{n-1})]\nabla\theta^n,\nabla(\theta_h^n-\tilde{\theta}_h^n))\\ &\quad-(k(\theta_h^{n-1})\nabla(\tilde{\theta}_h^n-\theta^n),\nabla(\theta_h^n-\tilde{\theta}_h^n))\\ &\quad+(\mu(\theta_h^{n-1})|\nabla u_h^n|^2-\mu(\theta^{n-1})|\nabla u^n|^2,(\theta_h^n-\tilde{\theta}_h^n))\\ &=E_1+E_2+E_3.\end{aligned}\tag{47}$$

Firstly, we estimate the bound of E_1 and E_2,

$$\begin{aligned}|E_1| &= ([k(\theta^{n-1})-k(\theta_h^{n-1})]\nabla\theta^n,\nabla(\theta_h^n-\tilde{\theta}_h^n))\\ &\le \|[k(\theta^{n-1})-k(\theta_h^{n-1})]\nabla\theta^n\|_{L^2}\|\nabla(\theta_h^n-\tilde{\theta}_h^n)\|_{L^2}\\ &\le L'\|(\theta^{n-1}-\theta_h^{n-1})\nabla\theta^n\|_{L^2}\|\nabla(\theta_h^n-\tilde{\theta}_h^n)\|_{L^2}\\ &\le L'\|\theta^{n-1}-\theta_h^{n-1}\|_{L^{\tilde{p}/(\tilde{p}-2)}}\|\nabla\theta^n\|_{L^{\tilde{p}}}\|\nabla(\theta_h^n-\tilde{\theta}_h^n)\|_{L^2}\\ &\le C\|f\|_{L^2}^2\|\nabla(\theta^{n-1}-\theta_h^{n-1})\|_{L^2}\|\nabla(\theta_h^n-\tilde{\theta}_h^n)\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}|E_2| &= (k(\theta_h^{n-1})\nabla(\tilde{\theta}_h^n-\theta^n),\nabla(\theta_h^n-\tilde{\theta}_h^n))\\ &\le k^*\|\nabla(\tilde{\theta}_h^n-\theta^n)\|_{L^2}\|\nabla(\theta_h^n-\tilde{\theta}_h^n)\|_{L^2}\\ &\le Ch^\sigma\|\theta^n\|_{1+\sigma}\|\nabla(\theta_h^n-\tilde{\theta}_h^n)\|_{L^2}.\end{aligned}$$

Then, we estimate the bound of E_3.

$$\begin{aligned}|E_3| &= (\mu(\theta_h^{n-1})|\nabla u_h^n|^2-\mu(\theta^{n-1})|\nabla u^n|^2,\theta_h^n-\tilde{\theta}_h^n)\\ &= ([\mu(\theta_h^{n-1})-\mu(\theta^{n-1})]|\nabla u^n|^2,\theta_h^n-\tilde{\theta}_h^n)\\ &\quad+(\mu(\theta_h^{n-1})(|\nabla u_h^n|^2-|\nabla u^n|^2),\theta_h^n-\tilde{\theta}_h^n)\\ &= R_1+R_2.\end{aligned}\tag{48}$$

By the Hölder inequality and the Sobolev inequality,

$$\begin{aligned}R_1 &\le L\|\theta^{n-1}-\theta_h^{n-1}\|_{L^{2p/(p-2)}}\|\nabla u^n\|_{L^p}^2\|\theta_h^n-\tilde{\theta}_h^n\|_{L^{2p/(p-2)}}\\ &\le C\|f\|_{L^2}^2\|\nabla(\theta^{n-1}-\theta_h^{n-1})\|_{L^2}\|\nabla(\theta_h^n-\tilde{\theta}_h^n)\|_{L^2}.\end{aligned}\tag{49}$$

Since $a^2-b^2=(a-b)^2+2b(a-b)$, then R_2 can be split into:

$$\begin{aligned}R_2 &= (\mu(\theta_h^{n-1})|\nabla(u_h^n-u^n)|^2,\theta_h^n-\tilde{\theta}_h^n)\\ &\quad+2(\mu(\theta_h^{n-1})\nabla u^n\cdot\nabla(u_h^n-u^n),\theta_h^n-\tilde{\theta}_h^n)\\ &= R_{21}+R_{22}.\end{aligned}\tag{50}$$

By the Hölder inequality and the inverse inequality (43),

$$\begin{aligned}R_{21} &\le \mu^*\|\nabla(u^n-u_h^n)\|_{L^2}^2\|\theta_h^n-\tilde{\theta}_h^n\|_{L^\infty}\\ &\le \mu^*M(h)\|\nabla(u^n-u_h^n)\|_{L^2}^2\|\nabla(\theta_h^n-\tilde{\theta}_h^n)\|_{L^2}\ .\end{aligned}\tag{51}$$

To estimate R_{22}, we have

$$\begin{aligned} R_{22} &\le 2\mu^*\|\nabla u^n\|_{L^p}\|\nabla(u_h^n - u^n)\|_{L^2}\|\theta_h^n - \tilde\theta_h^n\|_{L^{2p/(p-2)}} \\ &\le C\|f\|_{L^2}\|\nabla(u^n - u_h^n)\|_{L^2}\|\nabla(\theta_h^n - \tilde\theta_h^n)\|_{L^2}. \end{aligned} \tag{52}$$

Combine (48)–(52), and notice that

$$\begin{aligned} \|\nabla(u^n - u_h^n)\|_{L^2} &\le \|\nabla(u^n - \tilde u_h^n)\|_{L^2} + \|\nabla(u_h^n - \tilde u_h^n)\|_{L^2} \\ &\le C\Big\{\|f\|_{L^2}\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + h^\sigma\|u^n\|_{1+\sigma}\Big\}, \end{aligned} \tag{53}$$

we obtain (46). □

Let us now make an inductive hypothesis: for sufficiently small h,

$$M(h)\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} < 1, \quad \forall\, n \ge 1. \tag{54}$$

In fact, when $n = 1$, we can choose θ_h^0 as the standard elliptic projection of θ^0, thus, $M(h)\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} \le Ch^{\sigma-1/2}\|\theta^0\|_{H^{1+\sigma}} < 1$ for sufficiently small h. If (54) holds for $n-1$. Then, we have

$$\begin{aligned} &\|\nabla(\theta^n - \theta_h^n)\|_{L^2} \\ \le\ &\|\nabla(\theta^n - \tilde\theta_h^n)\|_{L^2} + \|\nabla(\theta_h^n - \tilde\theta_h^n)\|_{L^2} \\ \le\ &\hat C\|f\|_{L^2}^2\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} \\ &+Ch^\sigma\Big\{[\|f\|_{L^2} + M(h)h^\sigma\|u^n\|_{1+\sigma}]\|u^n\|_{1+\sigma} + \|\theta^n\|_{1+\sigma}\Big\} \end{aligned} \tag{55}$$

where $\hat C$ is a constant dependent only on Ω, μ^*, μ_*, k^*, k_*, L L'. If

$$\hat C\|f\|_{L^2}^2 = \hat M(f) < 1, \tag{56}$$

notice that (34), for sufficiently small h such that $h^{\sigma-1/2}\max_n\|u^n\|_{1+\sigma} \le C\|f\|_{L^2}$. We can obtain

$$\begin{aligned} &\|\nabla(\theta^n - \theta_h^n)\|_{L^2} \\ \le\ &\hat M(f)^n\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} \\ &+Ch^\sigma/(1-\hat M(f))\Big\{\|f\|_{L^2}\max_n\|u^n\|_{1+\sigma} + \max_n\|\theta^n\|_{1+\sigma}\Big\} \\ \le\ &\hat M(f)^n h^\sigma\|\theta^0\|_{1+\sigma} + CKh^\sigma/(1-\hat M(f)). \end{aligned} \tag{57}$$

Thus, when we choose h sufficiently small, the inductive hypothesis (54) holds. Meanwhile, we have

Theorem 6. *Let (u^n, θ^n) and (u_h^n, θ_h^n) be the solutions of problems* (31) *and* (38) *respectively, then, the following error estimates hold:*

$$\left\{\begin{aligned} &\text{(a)}\quad \|\nabla(u^n - u_h^n)\|_{L^2} \le C\|f\|_{L^2}h^\sigma\Big\{\hat M(f)^{n-1}\|\theta^0\|_{1+\sigma} + 1\Big\}, \\ &\text{(b)}\quad \|\nabla(\theta^n - \theta_h^n)\|_{L^2} \le \hat M(f)^n h^\sigma\|\theta^0\|_{1+\sigma} + CKh^\sigma/(1-\hat M(f)), \end{aligned}\right. \tag{58}$$

where C is a constant independent of n, h and f, and $\hat M(f) < 1$ is defined by (56).

Now, let $C^* = \max\{\tilde{C}, \hat{C}\}$ where $\tilde{C}$ and $\hat{C}$ are defined by (29) and (55) respectively. Thus,

$$C^*\|f\|_{L^2}^2 = M^*(f) < 1 \tag{59}$$

implies (30) and (56). Hence, we get the main result

Theorem 7. *Under the assumptions (2) , if condition (59) holds. Then, problem* (15) *has a unique solution* (u,θ), *the finite element solution sequence* $\{(u_h^n,\theta_h^n)\}$ *of* (38) *converges to* (u,θ) *and the following estimates hold,*

$$\begin{cases} (a) \quad \|\nabla(u-u_h^n)\|_{L^2} \\ \qquad \le C\|f\|_{L^2}\Big\{(M^*(f))^{n-1}[\|\nabla(\theta-\theta^0)\|_{L^2} + h^\sigma\|\theta^0\|_{1+\sigma}]h^\sigma\Big\}, \\ (b) \quad \|\nabla(\theta-\theta_h^n)\|_{L^2} \\ \qquad \le (M^*(f))^n\Big\{\|\nabla(\theta-\theta^0)\|_{L^2} + h^\sigma\|\theta^0\|_{1+\sigma}\Big\} \\ \qquad\quad + CKh^\sigma/(1-M^*(f)), \end{cases} \tag{60}$$

where C *is a constant independent of* n, h *and* f, *and* $M^*(f) < 1$ *is defined by* (59).

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