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MODELING THE LAW OF THE ITERATED LOGARITHM IN OPEN QUEUEING NETWORKS

Saulius Minkevičius
Institute of Mathematics and Informatics
4, Akademijos, Vilnius, 08663, LITHUANIA

Abstract: An open queueing network model in heavy traffic has been developed. The law of the iterated logarithm for the total queue length of customers in an open queueing network have been presented.

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1. Introduction

One can apply the law of the iterated logarithm to the waiting time of a customer, virtual waiting time of a customer, and the queue length of customers to get more important probabilistic characteristics of the queueing theory in heavy traffic (see, for example, [2], [3], [4]).

A single phase case, where intervals of time between the arrival of customers to queue are independent identically distributed random variables and there is a single device, working independently of the output in heavy traffic, has been completely investigated. But there are only several papers on the theory of open queueing networks in heavy traffic and no proof of the theorems on laws of the iterated logarithm for the main probabilistic characteristics of an open queueing network in heavy traffic (for example, sojourn time of a customer and a virtual waiting time of a customer).

So in this paper, we prove theorem on the law of the iterated logarithm for the total queue length of customers in the queueing network.

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First we consider open queueing networks with the “first come, first served” service discipline at each station and general distributions of interarrival and service time. The basic components of the queueing network are arrival processes, service processes, and routing processes.

Particularly, there are k nodes with the node j having a single server and a waiting room of unlimited capacity. The external input stream to the node j is a renewal process, the interarrival time of this process with the mean $\lambda_j = \left(M\left[z_n^{(j)}\right]\right)^{-1} > 0$, and finite variance $a_j = D\left(z_n^{(j)}\right) > 0$, $j = 1, 2, ..., k$. These external input streams at the various nodes are assumed to be independent. The service times at the node j are independent and have a common distribution with the mean $\mu_j = \left(M\left[S_n^{(j)}\right]\right)^{-1} > 0$ and finite variance $\sigma_j = D\left(S_n^{(j)}\right) > 0$, $j = 1, 2, ..., k$. The service times at the node j are also independent of all customer arrivals at the node j. A customer leaving the node j is immediately and independently routed to the node i with probability p_{ji}; and the customer departs the system from the node j with probability $p_j = 1 - \sum_{i=1}^{N} p_{ji}$.

The matrix $P^* \equiv [p_{ij}]$ is called a switching matrix. The $k \times k$ matrix $P = (p_{ij})$ is assumed to have a spectral radius strictly smaller than a unit. Observe that this system is quite general, encompassing the tandem system, acyclic networks of $GI/G/1$ queues, and networks of $GI/G/1$ queues with feedback.

In the context of the queueing network, the random variables $z_n^{(j)}$ function as interarrival times (from outside the network) at the station j, while $S_n^{(j)}$ is the n-th service time at the station j, and $\Phi_n^{(j)}$ is a routing indicator for the n-th customer served at the station j. If $\Phi_n^{(i)} = j$ (which occurs with probability p_{ij}), then the n-th customer served at the station i is routed to the station j. When $\Phi_n^{(i)} = 0$, the associated customer leaves the network. The matrix P is called a routing matrix.

To construct renewal processes generated by the interarrival and service times, we assume

$$z_j(0) = 0, \quad z_j(l) = \sum_{m=1}^{l} z_m^{(j)}, \quad S_j(0) = 0,$$

$$S_j(l) = \sum_{m=1}^{l} S_m^{(j)}, \qquad l \geq 1, \quad j = 1, 2, \ldots, k.$$

We now define $A_j(t) = \max\left(l \geq 0 : z_j(l) \leq t\right)$ and $X_j(t) = \max(l \geq 0 : S_j(l) \leq t)$, and denote $\tau_j(t)$ as the total number of customer service departure from the j-th station of the network until time t, $\tilde{\tau}_j(t)$ as the total number of customer arrival at the j-th station of the network until time t, $\hat{\tau}_j(t)$ as the total number of customers after service at the j-th station of departure network until time t, $\tau_{ij}(t)$ as the total number of customers after service that depart from the i-th station of the network and arrive at the j-th station of the network until time t, $p_{ij}^t = \frac{\tau_{ij}(t)}{\tau_i(t)}$ as a part of the total number of customers which, after service at the i-th station of the network, arrive at the j-th station; $i, j = 1, 2, \ldots, k$ and $t > 0$.

At first let us define $Q_j(t)$ as the queue length of customers at the j-th station of the queueing network in time t, and $v(t)$ as the total queue length of customers in the open queueing network in time t; $\hat{\beta}_j = \lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} - \mu_j > 0$, $\beta = \sum_{i=1}^{k} \hat{\beta}_i = \sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} \mu_i \cdot p_i > 0$, $\hat{\sigma}_j^2 = (\lambda_j)^3 \cdot Dz_n^{(j)} + \sum_{i=1}^{k} (\mu_i)^3 \cdot DS_n^{(i)} \cdot (p_{ij})^2 + (\mu_j)^3 \cdot DS_n^{(j)} > 0$, $\hat{\sigma}^2 = \sum_{i=1}^{k} (\lambda_i)^3 \cdot Dz_n^{(i)} + \sum_{i=1}^{k} (\mu_i)^3 \cdot DS_n^{(i)} \cdot (p_i)^2 > 0$, $j = 1, 2, \ldots, k$.

We assume that the following condition is fulfilled:

$$\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} > \mu_j, \quad j = 1, 2, \ldots, k. \tag{1}$$

Note that this condition quarantees that, with probability one there exists a queue lentgh of customers and this queue length of customers is constantly growing. In addition, we assume throughout that

$$\max_{1 \le j \le k} \sup_{n \ge 1} M \left\{ \left(z_n^{(j)} \right)^{2+\varepsilon} \right\} < \infty \text{ for some } \varepsilon > 0, \tag{2}$$

$$\max_{1 \le j \le k} \sup_{n \ge 1} M \left\{ \left(S_n^{(j)} \right)^{2+\varepsilon} \right\} < \infty \text{ for some } \varepsilon > 0. \tag{3}$$

Conditions (2) and (3) imply the Lindeberg condition for the respective sequences, and are easier to verify in practice (usually $\varepsilon = 1$ works).

To prove the main theorem, we apply the following lemma:

Lemma. *If conditions (1)-(3) hold, then for* $\varepsilon > 0$

$$P\left(\overline{\lim_{t \to \infty}} \frac{\sup\limits_{0 \le s \le t} (X_j(s) - \tau_j(s))}{a(t)} > \varepsilon \right) = 0,$$

$j = 1, 2, \ldots, k$ and $a(t) = \sqrt{2t \ln \ln t}$.

Proof. The proof of the lemma is similar to that of the lemma in [4]. □

2. Main Result

One of the result of the paper is the following theorem on the law of the iterated logarithm for the total queue length of customers in an open queueing network.

Theorem. *If conditions (1)-(3) are fulfilled, then*

$$P\left(\overline{\lim_{t\to\infty}}\frac{v(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)}=1\right)=P\left(\underline{\lim_{t\to\infty}}\frac{v(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)}=-1\right)=1.$$

Proof. First denote

$$w_j(t)=X_j(t)\cdot\left\{\sum_{i=1}^{k}\left|p_{ji}^{t}-p_{ji}\right|\right\},\quad \gamma(t)=\sum_{i=1}^{k}\{A_i(t)-X_i(t)\cdot p_i\},$$

$j=1,2,\ldots,$ k and $t>0$.

Note that

$$\hat{\tau}_j(t)=\tau_j(t)-\sum_{i=1}^{k}\tau_{ji}(t)=\tau_j(t)\cdot\left(1-\sum_{i=1}^{k}p_{ji}^{t}\right),\ j=1,2,\ldots,k \text{ and } t>0. \quad (4)$$

Therefore, we have that for $t>0$

$$\begin{aligned}v(t)=\sum_{i=1}^{k}Q_i(t)=\sum_{i=1}^{k}A_i(t)-\sum_{i=1}^{k}\hat{\tau}_i(t)=\sum_{i=1}^{k}\{A_i(t)-X_i(t)\cdot p_i\}\\ +\sum_{i=1}^{k}\{X_i(t)\cdot p_i-\hat{\tau}_i(t)\}.\end{aligned} \quad (5)$$

According to (4), we get

$$\begin{aligned}X_j(t)\cdot p_j-\hat{\tau}_j(t)=X_j(t)\cdot p_j-X_j(t)\cdot\left(1-\sum_{i=1}^{k}p_{ji}^{t}\right)+(X_j(t)-\tau_j(t))\\ \left(1-\sum_{i=1}^{k}p_{ji}^{t}\right)\le w_j(t)+\sup_{0\le s\le t}\left(X_j(s)-\tau_j(s)\right),\\ j=1,2,\ldots,k \text{ and } t>0.\end{aligned} \quad (6)$$

From this and (5) it follows that for $t>0$

$$v(t)\le\gamma(t)+\sum_{i=1}^{k}\left\{w_i(t)+\sup_{0\le s\le t}\left(X_i(s)-\tau_i(s)\right)\right\}. \quad (7)$$

Fix $\varepsilon > 0$ as a small constant. So (7) implies that for $\varepsilon > 0$

$$P\left(\frac{v(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)} > 1+2\varepsilon\right) \le P\left(\frac{\gamma(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)} > 1+\varepsilon\right)$$

$$+P\left(\frac{\sum_{i=1}^{k}\left\{w_i(t)+\sup_{0\le s\le t}(X_i(s)-\tau_i(s))\right\}}{\hat{\sigma}\cdot a(t)} > \varepsilon\right) \le P\left(\frac{\gamma(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)} > 1+\varepsilon\right)$$

$$+\sum_{i=1}^{k}\left\{P\left(\frac{w_i(t)}{\hat{\sigma}\cdot a(t)} > \frac{\varepsilon}{2k}\right)+P\left(\frac{\sup_{0\le s\le t}(X_i(s)-\tau_i(s))}{\hat{\sigma}\cdot a(t)} > \frac{\varepsilon}{2k}\right)\right\}. \quad (8)$$

Besides, we have for $\varepsilon > 0$ that (see (8))

$$P\left(\overline{\lim_{t\to\infty}}\frac{v(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)} > 1+2\varepsilon\right) \le P\left(\overline{\lim_{t\to\infty}}\frac{\gamma(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)} > 1+\varepsilon\right)+$$

$$\sum_{i=1}^{k}\left\{P\left(\overline{\lim_{t\to\infty}}\frac{w_i(t)}{\hat{\sigma}\cdot a(t)} > \frac{\varepsilon}{2k}\right)+P\left(\overline{\lim_{t\to\infty}}\frac{\sup_{0\le s\le t}(X_i(s)-\tau_i(s))}{\hat{\sigma}\cdot a(t)} > \frac{\varepsilon}{2k}\right)\right\}. \quad (9)$$

Using the law of the iterated logarithm in renewal processes (see [1]), we get for $\varepsilon > 0$

$$P\left(\overline{\lim_{t\to\infty}}\frac{\gamma(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)} > 1+\varepsilon\right)=0. \quad (10)$$

Now we prove for $\varepsilon > 0$ that

$$P\left(\overline{\lim_{t\to\infty}}\frac{w_j(t)}{\hat{\sigma}\cdot a(t)} > 1+\varepsilon\right)=0, \quad j=1,2,\ldots,k. \quad (11)$$

Note that for $\varepsilon > 0$

$$P\left(\frac{w_j(t)}{a(t)} > \varepsilon\right)$$

$$\le \sum_{i=1}^{k}P\left(\frac{X_i(t)\cdot|p_{ji}^t-p_{ji}|}{a(t)} > \varepsilon\right) \le \sum_{i=1}^{k}P\left(\frac{X_i(t)-\mu_i\cdot t}{a(t)} > 1+\varepsilon\right)$$

$$+\sum_{i=1}^{k}P\left(\left\{\frac{X_i(t)\cdot|p_{ji}^t-p_{ji}|}{a(t)} > \varepsilon\right\}\cap\{X_i(t)\le(1+\varepsilon)\cdot a(t)+\mu_i\cdot t\}\right) \le$$

$$\sum_{i=1}^{k}\left\{P\left(\frac{\{(1+\varepsilon)\cdot a(t)+\mu_i\cdot t\}\cdot|p_{ji}^t-p_{ji}|}{a(t)} > \varepsilon\right)+P\left(\frac{X_i(t)-\mu_i\cdot t}{a(t)} > 1+\varepsilon\right)\right\}$$

$$\leq \sum_{i=1}^{k}\left\{P\left(\frac{|p_{ji}^t - p_{ji}|}{a(t)} > \frac{\varepsilon}{(1+\varepsilon)\cdot a(t) + \mu_i \cdot t}\right) + P\left(\frac{X_i(t) - \mu_i \cdot t}{a(t)} > 1+\varepsilon\right)\right\}$$

$$\leq \sum_{i=1}^{k}\left\{P\left(|p_{ji}^t - p_{ji}| > 0\right) + P\left(\frac{X_i(t) - \mu_i \cdot t}{a(t)} > 1+\varepsilon\right)\right\},$$

$$j = 1, 2, \ldots, k \text{ and } t > 0. \quad (12)$$

Again, applying the law of the iterated logarithm in renewal processes, we get that for $\varepsilon > 0$

$$P\left(\varlimsup_{t\to\infty} \frac{X_j(t) - \mu_j \cdot t}{a(t)} > 1+\varepsilon\right) = 0, \quad j = 1, 2, \ldots, k. \quad (13)$$

Using the lemma from [4], we obtain that

$$P\left(\varlimsup_{t\to\infty} |p_{ji}^t - p_{ji}| > 0\right) = \lim_{\delta\to 0} P\left(\varlimsup_{t\to\infty} |p_{ji}^t - p_{ji}| > \delta\right) = 0,$$

$$j = 1, 2, \ldots, k. \quad (14)$$

Applying (12)-(14), we get (11). Using this, (9)-(10), and the lemma, we achieve that for $\varepsilon > 0$

$$P\left(\varlimsup_{t\to\infty} \frac{v(t) - \beta \cdot t}{\hat{\sigma} \cdot a(t)} > 1+\varepsilon\right) = 0. \quad (15)$$

Now we prove that for $0 < \varepsilon < 1$

$$P\left(\varlimsup_{t\to\infty} \frac{v(t) - \beta \cdot t}{\hat{\sigma} \cdot a(t)} < 1-\varepsilon\right) = 0. \quad (16)$$

Note that (see (3))

$$\hat{\tau}_j(t) \leq X_j(t) \cdot \left(1 - \sum_{i=1}^{k} p_{ji}^t\right) \leq X_j(t) \cdot \left(1 - \sum_{i=1}^{k} p_{ji}\right)$$

$$+ X_j(t) \cdot \left(\sum_{i=1}^{k} |p_{ji}^t - p_{ji}|\right) = X_j(t) \cdot p_j + w_j(t),$$

$$j = 1, 2, \ldots, k \text{ and } t > 0. \quad (17)$$

This and (6) imply that for $t > 0$

$$v(t) = \sum_{i=1}^{k} \{A_i(t) - \hat{\tau}_i(t)\} \geq \sum_{i=1}^{k} \{A_i(t) - X_i(t) \cdot p_i\} - \sum_{i=1}^{k} w_i(t). \quad (18)$$

Thus, for $0 < \varepsilon < 1/2$ (see (18))

$$P\left(\frac{v(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)}<1-2\varepsilon\right)\le P\left(\frac{\sum_{i=1}^{k}\{A_i(t)-X_i(t)\cdot p_i\}-\beta\cdot t}{\hat{\sigma}\cdot a(t)}<1-\varepsilon\right)$$

$$+P\left(\frac{\sum_{i=1}^{k}w_i(t)}{\hat{\sigma}\cdot a(t)}>\varepsilon\right)\le\sum_{i=1}^{k}P\left(\frac{A_i(t)-X_i(t)\cdot p_i-\beta_i\cdot t}{\hat{\sigma}\cdot a(t)}<1-\varepsilon\right)$$

$$+\sum_{i=1}^{k}P\left(\frac{w_i(t)}{\hat{\sigma}\cdot a(t)}>\frac{\varepsilon}{k}\right). \quad (19)$$

Again, using the law of the iterated logarithm in renewal processes (see, for example, [1]), we have for $0<\varepsilon<1$ that

$$P\left(\overline{\lim_{t\to\infty}}\frac{A_j(t)-X_j(t)\cdot p_j-\beta_j\cdot t}{\hat{\sigma}_j\cdot a(t)}<1-\varepsilon\right)=0,\quad j=1,2,\ldots,k. \quad (20)$$

By this and (11) we obtain that for $0<\varepsilon<1$

$$P\left(\overline{\lim_{t\to\infty}}\frac{v(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)}<1-\varepsilon\right)=0. \quad (21)$$

So, since $\varepsilon>0$ is free, we obtain from (15) and (21) that

$$P\left(\overline{\lim_{t\to\infty}}\frac{v(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)}=1\right)=1. \quad (22)$$

The proof that

$$P\left(\underline{\lim_{t\to\infty}}\frac{v(t)-\beta\cdot t}{\hat{\sigma}\cdot a(t)}=-1\right)=1$$

is similar to the proof of (22).

Therefore, the proof of the theorem is complete. □

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