International Journal of Pure and Applied Mathematics
Volume 63 No. 4 2010, 501-505

THE LARGEST CONJUGACY SIZES AND SOLVABILITY OF FINITE GROUPS

Liguo He[1 §], Zhe Zhang[2], Hong Li[3]
[1,2,3]Department of Mathematics
Shenyang University of Technology
Shenyang, 110870, P.R. CHINA
[1]e-mail: cowleyhe@yahoo.com.cn

Abstract: In this note, it is proved that if the largest conjugacy class size of the finite group G is less than or equal to 19, then G is solvable. We observe that the largest conjugacy class size of the alternating group A_5 of degree 5 is 20.

AMS Subject Classification: 20C15
Key Words: solvable group, conjugacy class size

1. Introduction

It is well-known that the conjuacy class sizes of finite group have important influence on the structure of the group. For examples, N. Itô proved in [8] that if the set of class sizes of G is $\{1, m\}$, then G is nilpotent, $m = p^a$ for some prime p and $G = P \times A$ where $A \leq Z(G)$ and P is a Sylow p-subgroup of G; he still shows in [9] that G is solvable when the set of class sizes of G is $\{1, m, n\}$. A.R. Camina proves in [5] that G is nilpotent if the set of class sizes is $\{1, p^a, q^b, p^a q^b\}$ with distinct primes p, q. A. Beltran and M.J. Felip show in [2, 1] that if the set of class sizes is $\{1, m, n, mn\}$ with $(m, n) = 1$, then G is solvable and furthermore is nilpotent. Indeed they also study in [3] the group theory properties in the situation where the set is $\{1, m, n, mk\}$ with $k|n$, $(m, n) = 1, m > 1$, and $n > 1$.

In this note, we investigate how the largest conjugacy class size influences group structure. In fact, we prove the following result.

Received: June 20, 2010

§Correspondence author

Theorem. *Let G be a finite group with the largest conjugacy class size less than or equal to 19. Then G is a solvable group.*

Observe that the alternating group A_5 has the largest class size 20, the upper bound 19 is best possible in order to guarantee the group's solvability. We also study the connections between the largest conjugacy class size and nilpotenty and supersolvability, respectively.

All groups considered here are of finite order. For $x \in G$, x^G denotes a conjugacy class of x in G and $|x^G|$ the size of x^G. We shall freely use the following basic properties of class size, for $N \trianglelefteq G, x \in N$ and $y \in G$, $|x^N|$ divides $|x^G|$ and $|(yN)^{G/N}|$ divides $|y^G|$.

2. Proofs of Main Results

We list some useful results in order to prove the main conclusions. The next result is from [6].

Lemma 2.1. *Let G be a finite group. Then*

1. *If none of conjugacy class sizes of G can be divided by 4, then G is solvable.*
2. *If all prime divisors of conjugacy class sizes of G are square-free, then G is supersolvable.*

Lemma 2.2. *The conjugacy class sizes of a non-Abelian simple group are not of prime power except the identity element 1.*

Proof. See Theorem 3.9 of [7]. □

Lemma 2.3. *Let G be a finite non-Abelian simple group. If the prime power p^k divides $|G|$, then p^k divides some conjugacy class size of G.*

Proof. See the theorem in [4]. □

Now we proceed to prove the main results.

Theorem 2.4. *Let G be a non-Abelian finite group with the largest conjugacy class size less than or equal to 19. Then G is not a simple group.*

Proof. Assume that this is false and G is a counterexample. If each class size is divisible by 4, then Lemma 2.1 shows that G is solvable and thus Abelian, a contradiction.

Now we assume $x_0 \in G$ such that 4 divides $|x_0^G|$. If $|x_0^G| \neq 12$, then under our hypothesis, $|x_0^G| = 4, 8$, or 16. Lemma 2.2 implies that G is not simple, a contradiciton.

Suppose that $|x_0^G| = 12$. If none conjugacy size of G equals 15, then Lemma 2.2 implies that the conjugacy class sizes of noncentral elements are $\{6, 10, 12, 14, 18\}$, which are all even number. The conjugacy class equation shows that the center of G is not trivial, this is a contradiction since G is a non-Abelian simple group. Note that G is of even order.

Now we are reduced to the case where the non-Abelian simple G must have class sizes $\{1, 12, 15\}$, can have class sizes $\{6, 10, 14, 18\}$. By using Lemma 2.3, we may get that $|G|_2 = 2^2, |G|_3 = 3$ or 3^2, $|G|_5 = 5$, $|G|_7 = 7$. Here $|G|$ denotes the order of G, and $|G|_p$, the p-part of $|G|$, i.e., the biggest p-power divisor of $|G|$. By routine caculation, it follows that $|G| = 60, 180, 420$ or 1260. It is known that the groups of orders 180 or 420 are not simple; and the groups of order 60 are isomorphic to the alternating group A_5, whose largest class size is 20. Thus the only possibilty is the case when $|G| = 1260$.

We shall apply the classification theorem of finite simple groups to G. Since $|A_6| = 360 < 1260 < |A_7| = 2520$, it follows that G is not isomorphic to any alternating group. Since $1260 (= 2^2 \times 3^2 \times 5 \times 7)$ does not contain cubic divisors, we get that G is not isomorphic to a simple group of Lie type unless $A_n(q)$ and ${}^2B_2(q)$. Recall that $|A_n(q)| = q^{n(n+1)/2}\Pi_{i=1}^{n}(q^{i+1} - 1)$. When $n = 1$, $|A_1(9)| = 720 < 1260 < 1320 = |A_1(11)|$, which shows that G is not isomorphic to $A_1(q)$. If $n \geq 2$, then $A_n(q)$ possess cubic divisors, thus G is not isomorphic to $A_n(q)$. Recall that $|{}^2B_2(q)| = q^2(q^2+1)(q-1)$ and $q = 2^{2n+1}$. If $n = 0$, then ${}^2B_2(2)$ is solvable; otherwise G has cubic divisors. Thus G is not isomorphic to ${}^2B_2(q)$. By checking the order expressions of sporadict simple groups, it follows that the sporadic simple group of minimal order is M_{11} whose order is 7920 (> 1260), hence G is not isomorphic to a sporadict simple groups. The whole proof is complete. □

Corollary 2.5. *Let G be a finite group with the largest conjugacy class size less than or equal to 19. Then G is solvable.*

Proof. If G is Abelian, G is obvious solvable. Otherwise, the above theorem shows that G is not simple, then there exist normal subgroup N of G. Observe that N and G/N satisfy the condition of the theorem, the inductive hypothesis is applied to concluding that N and G/N are solvable, thus G is solvable, as desired. □

Remark 2.6. Since the largest class size of the alternating group A_5 is 20, the upper bound 19 in the above theorem is sharp.

It is an elementary fact that if all class sizes of G are 1, then G is Abelian. Further we prove the following result.

Theorem 2.7. *If all conjugacy class sizes of G are less than or equal to 2, then G is nilpotent.*

Proof. If all class sizes are 1, then G is Abelian, as desired. Otherwise, there exist $x \in G$ suc that $|x^G| = 2$. Note that the centralizer $C_G(x)$ (of x in G) is the normal maximal subgroup. By the inductive argument, $C_G(x)$ is nilpotent. Let $x_1, x_2, \cdots, x_n$ be all noncentral elements of G, then $G = C_G(x_1)C_G(x_2)\cdots C_G(x_n)$. Since $n \geq 2$, it follows that G is nilpotent. The proof is finished. □

Remark 2.8. Since the largest class size of the symmetric group S_3 is 3, we get that the upper bound 2 in the above theorem is sharp. Note that S_3 is not nilpotent.

Theorem 2.9. *If all conjugacy class sizes of G are no more than 3, then G is supersolvable.*

Proof. Because the divisors of the class sizes of G are square-free, it follows via Lemma 1 that G is supersolvable. □

Remark 2.10. Considering the largest class size of the nonsupersolvable group S_4 is 4, the upper bound 3 in the above theorem is sharp. Observe that the supersolvable group $\langle(12)(34), (1345)\rangle$ possesses conjugacy classes of size 5.

Acknowledgments

This research is supported by the Natural Science Foundation of Liaoning Education Department (Grant No.2008516).

References

[1] A. Beltran, M.J. Felipe, Variations on a theorem by Alan Camina on conjugacy class sizes, *J. Algebra*, **296** (2006), 253-266; **320** (2008), 4317-4319.

[2] A. Beltran, M.J. Felipe, Some conjugacy length conditions implying solvablity of finite groups, *J. Group Theory*, **9** (2006), 787-797.

[3] A. Beltran, M.J. Felipe, Structures of finite groups under some arithmetical conditions on class sizes, *J. Algebra*, **319** (2008), 897-910.

[4] Jianxing Bi, An arithmetical property of the conjugacy classes of the finite simple groups, *J. Liaoning University*, Natural Science Edition, **35**, No. 1 (2008), 5-6.

[5] A.R. Camina, Arithmetical conditions on the conjugacy class numbers of a finite group, *J. London Math. Soc.*, **2**, No. 5 (1972), 127-132.

[6] D. Chillag, M. Herzog, On the length of conjugacy classes of finite groups, *J. Algebra*, **131** (1990), 110-125.

[7] I.M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York (1976).

[8] N. Itô, On the finite groups with given conjugate types I, *Nagoyama Math.*, **6** (1953), 17-28.

[9] N. Itô, On the finite groups with given conjugate types II, *Nagoyama Math.*, **7** (1970), 231-251.