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ON PRESERVERS OF A GENERALIZED MATRIX FUNCTION

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Abstract: The general form of an operator preserving the generalized matrix function is described in the paper. The result can be considered as a generalization of the results done for preservers of determinant and permanent of a matrix without the initial presumption of their linearity.

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1. Introduction

Denote by $\mathbb{N}$ the set of natural numbers and by $M_n(\mathcal{F})$ the set of n-squared matrices over a field $\mathcal{F}$, where $n \in \mathbb{N}$. If we consider a subgroup H of the symmetric group S_n of all permutations on an n-element set then by the *generalized matrix function* we mean a mapping $d_\chi : M_n(\mathcal{F}) \to \mathcal{F}$ defined by the prescription

$$d_\chi(A) = \sum_{\sigma \in H} \chi(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where χ is a nontrivial homomorphism from H into $\mathcal{F}$ (so-called *character* of

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degree 1 on H).

This concept studied in several papers (see e.g. [1, 5]) implies some important concepts from the linear algebra field. First, we can recall the concept of the *immanant* of a matrix $A \in M_n(\mathcal{F})$ – see [4], where $H = S_n$ and χ is induced by a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of the integer n, so

$$\operatorname{imm}(A)_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Next the concept of the determinant of a matrix $A \in M_n(\mathcal{F})$, where $H = S_n$ and χ is the alternating character sgn, so

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Finally, recall the concept of the permanent of a matrix $A \in M_n(\mathcal{F})$. Here $H = S_n$ again and $\chi \equiv 1$, so

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Our goal is to study and describe operators which preserve the generalized matrix function $s : M_n(\mathcal{F}) \to \mathcal{F}$ of the form

$$s(A) = \sum_{\sigma \in S_n} \chi(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma(i)}, \tag{1}$$

where $\chi(\sigma) \neq 0$ for every permutation $\sigma \in S_n$.

Definition 1.1. Given a matrix $A \in M_n(\mathcal{F})$, $A = [a_{i,j}]$. We say that an operator $F : M_n(\mathcal{F}) \longrightarrow M_n(\mathcal{F})$ defined by

$$F(A) := [f_{i,j}(a_{i,j})], \quad \text{where} \quad f_{i,j} : \mathcal{F} \to \mathcal{F}, \quad i, j = 1, 2, \ldots, n, \tag{2}$$

preserves the generalized matrix function (1) if and only if the equality

$$s(F(X)) = s(X)$$

holds for every matrix $X \in M_n(\mathcal{F})$.

We can see that we assume no linearity for the operator F and recall that the problem of describing such operators which preserve determinant or permanent of a matrix was already solved by the first author – see the papers [2, 3].

2. Results

First, we show that the operator of the form (2) preserving the generalized matrix function s defined by (2) cannot be nonlinear, in fact, we can even show a little bit more.

Lemma 2.1. *Let $F : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ be an operator of the form (2) preserving the generalized matrix function s defined by (1). Then we can find nonzero elements $d_{i,j} \in \mathcal{F}$ such that for every indices $i, j = 1, 2, \ldots, n$ and each element $x \in \mathcal{F}$ we have*

$$f_{i,j}(x) = d_{i,j} \cdot x \tag{3}$$

and moreover for every permutation $\sigma \in S_n$

$$\prod_{i=1}^{n} d_{i,\sigma(i)} = 1. \tag{4}$$

Proof. Assume that F of the form (2) preserves the generalized matrix function s for any $n \in \mathbb{N}$. For a chosen pair $(k, l) \in \{1, 2, \ldots, n\}^2$ consider a function $f_{k,l} : \mathcal{F} \to \mathcal{F}$. Further, we choose a permutation $\varphi = (\varphi(1), \varphi(2), \ldots, \varphi(n)) \in S_n$ and construct a generalized permutation matrix $B_1 = [b_{i,j}] \in M_n(\mathcal{F})$, where for an element $x \in \mathcal{F}$

$$b_{i,j} = \begin{cases} x, & \text{if} \quad (i,j) = (k,l), \\ 1, & \text{if} \quad j = \varphi(i),\ i \neq k, \\ 0, & \text{in the other cases.} \end{cases}$$

Then we have

$$s(B_1) = \chi(\varphi) \cdot x$$

and also

$$s(F(B_1)) = f_{k,l}(x) \cdot \sum_{\substack{\sigma \in S_n, \\ \sigma(k) = l}} \chi(\sigma) \cdot \prod_{\substack{i=1, \\ i \neq k}}^{n} f_{i,\sigma(i)}(\overline{x}) + \sum_{\substack{\sigma \in S_n, \\ \sigma(k) \neq l}} \chi(\sigma) \cdot \prod_{i=1}^{n} f_{i,\sigma(i)}(\overline{x}),$$

where $\overline{x}$ is either equal 0 or 1. Therefore we can write

$$s(F(B_1)) = a_{k,l} \cdot f_{k,l}(x) + c_{k,l},$$

where

$$a_{k,l} = \sum_{\substack{\sigma \in S_n, \\ \sigma(k) = l}} \chi(\sigma) \cdot \prod_{\substack{i=1, \\ i \neq k}}^{n} f_{i,\sigma(i)}(\overline{x})$$

and

$$c_{k,l} = \sum_{\substack{\sigma \in S_n, \\ \sigma(k) \neq l}} \chi(\sigma) \cdot \prod_{i=1}^{n} f_{i,\sigma(i)}(\overline{x}).$$

Since F is the preserver of s, we obtain from the foregoing

$$a_{k,l} \cdot f_{k,l}(x) + c_{k,l} = \chi(\varphi) \cdot x, \tag{5}$$

from what choosing $x = 0$ consequently

$$a_{k,l} \cdot f_{k,l}(0) + c_{k,l} = 0. \tag{6}$$

If we subtract (6) from (5), we get

$$a_{k,l} \cdot (f_{k,l}(x) - f_{k,l}(0)) = \chi(\varphi) \cdot x.$$

Thanks to $\chi(\varphi) \neq 0$ we necessarily have $a_{k,l} \neq 0$ from that and then we can compute from (5)

$$f_{k,l}(x) = \chi(\varphi) \cdot a_{k,l}^{-1} \cdot x + a_{k,l}^{-1} \cdot (-c_{k,l})$$

or shortly

$$f_{k,l}(x) = d_{k,l} \cdot x + r_{k,l},$$

where

$$d_{k,l} = \chi(\varphi) \cdot a_{k,l}^{-1} \tag{7}$$

and

$$r_{k,l} = a_{k,l}^{-1} \cdot (-c_{k,l}). \tag{8}$$

Since the indices k, l were chosen arbitrarily, it is obvious that the operator $F = [f_{i,j}]$ must be linear, i.e.

$$F(X) = [d_{i,j} \cdot x_{i,j} + r_{i,j}] \tag{9}$$

for every matrix $X = [x_{i,j}] \in M_n(\mathcal{F})$ and each $i, j = 1, 2, \ldots, n$, where the constants $d_{i,j}$, resp. $r_{i,j}$ are determined by (7), resp. (8).

In general, for each matrix $X \in M_n(\mathcal{F})$ we have by (9)

$$\begin{aligned} s(X) &= \sum_{\sigma \in S_n} \chi(\sigma) \cdot \prod_{i=1}^{n} x_{i,\sigma(i)} \\ &= \sum_{\sigma \in S_n} \chi(\sigma) \cdot \prod_{i=1}^{n} (d_{i,\sigma(i)} \cdot x_{i,\sigma(i)} + r_{i,\sigma(i)}) = s(F(X)). \end{aligned}$$

In fact, we compare two polynomials with n^2 variables. On the left hand side, the term $\prod_{i=1}^{n} x_{i,\sigma(i)}$ has the coefficient $\chi(\sigma)$ and on right side, the same term is with the coefficient $\chi(\sigma) \cdot \prod_{i=1}^{n} d_{i,\sigma(i)}$, which means that for every permutation $\sigma \in S_n$ the equality $\prod_{i=1}^{n} d_{i,\sigma(i)} = 1$ holds.

Moreover, on the right side we can find the term $\prod_{i=1,i\neq j}^{n} x_{i,\sigma(i)}$ with the coefficient $r_{j,\sigma(j)} \cdot \prod_{i=1,i\neq j}^{n} d_{i,\sigma(i)}$ for each $j = 1, 2, \ldots, n$. The same term is not on the left hand side, which implies $r_{j,\sigma(j)} = 0$ for each $j = 1, 2, \ldots, n$ and an arbitrary permutation $\sigma \in S_n$. Therefore we can reduce the form of the operator F which preserves the generalized matrix function s onto

$$F(X) = [f_{i,j}(x_{i,j})] = [d_{i,j} \cdot x],$$

where the coefficients $d_{i,j}$ determined by (7) fulfil the identity (4). □

Remark 2.2. a) Due to Lemma 2.1 we can directly see that for the operator $F : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ preserving our generalized matrix function (1) we have

$$f_{i,j}(x) = 0 \quad \text{if and only if} \quad x = 0.$$

b) Note that the operator F from Lemma 2.1 is of the form

$$F(X) = D \bullet X,$$

where $X = [x_{i,j}]$ is an arbitrary matrix from $M_n(\mathcal{F})$, elements of the $D = [d_{i,j}] \in M_n(\mathcal{F})$ are described by (7) and $D \bullet X$ means the Hadamard (or Schur) product of matrices D and X, i.e. the matrix $[d_{i,j} \cdot x_{i,j}] \in M_n(\mathcal{F})$.

We go on with a result telling us a little bit more about the coefficients $d_{i,j}$ from Lemma 2.1.

Lemma 2.3. *Let $F : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ preserve the generalized matrix function s defined by (1). Then for the matrix $D = [d_{i,j}] \in M_n(\mathcal{F})$, where $d_{i,j}$ are determined by (7), we have* $\operatorname{rank}(D) = 1$.

Proof. Since $d_{i,j}$ are nonzero elements of $\mathcal{F}$, for $n = 1$ the result follows from Remark 2.2a). Now, we choose indices p, q, r, s such that $1 \leq p < q \leq n$ and $1 \leq r < s \leq n$ and we consider the matrix

$$B_2 = \begin{pmatrix} d_{p,r} & d_{p,s} \\ d_{q,r} & d_{q,s} \end{pmatrix}.$$

Further, consider permutations $\theta, \omega \in S_n$ such that

$$\theta = (\theta(1), \theta(2), \ldots, \theta(p-1), r, \theta(p+1), \ldots, \theta(q-1), s, \theta(q+1), \ldots, \theta(n)),$$
$$\omega = (\theta(1), \theta(2), \ldots, \theta(p-1), s, \theta(p+1), \ldots, \theta(q-1), r, \theta(q+1), \ldots, \theta(n)).$$

Then by Lemma 2.1

$$\prod_{i=1}^{n} d_{i,\theta(i)} = 1 = \prod_{i=1}^{n} d_{i,\omega(i)},$$

which implies

$$(d_{p,r} \cdot d_{q,s} - d_{p,s} \cdot d_{q,r}) \prod_{\substack{i=1, \\ p \neq i \neq q}}^{n} d_{i,\theta(i)} = 0$$

from what finally $\det(B_2) = 0$, so $\operatorname{rank}(D) = 1$. □

Lemma 2.4. *A matrix $H \in M_n(\mathcal{F})$ with non-zero entries $h_{i,j}$ has* rank *equal to 1 if and only if there exist non-zero elements $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ in the field $\mathcal{F}$ such that for all entries $h_{i,j}$ of the matrix H the equalities*

$$h_{i,j} = u_i \cdot v_j \tag{10}$$

hold.

Proof. See Kalinowski [3]. □

Theorem 2.5. *For an arbitrary $n \in \mathbb{N}$ let $F : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ be an operator of the form (2). Then F preserves the generalized matrix function (1) if and only if there are constants $u_i \neq 0$ and $v_j \neq 0$ for all $i, j = 1, 2, \ldots, n$ in the field $\mathcal{F}$ such that*

$$f_{i,j}(x) = u_i \cdot v_j \cdot x \tag{11}$$

for each $x \in \mathcal{F}$ and all indices $i, j = 1, 2, \ldots, n$ and moreover

$$\prod_{i=1}^{n} u_i \cdot v_{\sigma(i)} = 1 \tag{12}$$

for every permutation $\sigma \in S_n$.

Proof. If an operator $F : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ preserves the generalized matrix function then by Lemma 2.1 it must be of the form (3). If we apply Lemma 2.4 on the matrix D from Lemma 2.1, we get (11) using the result of Lemma 2.3 and also (12) by using (4).

Conversely, consider that an operator $F : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ of the form (1) fulfils (11) and (12). Then for each $X = [x_{i,j}] \in M_n(\mathcal{F})$ we have

$$\begin{aligned} s(F(X)) &= \sum_{\sigma \in S_n} \chi(\sigma) \cdot \prod_{i=1}^{n} u_i \cdot v_{i,\sigma(i)} \cdot x_{i,\sigma(i)} \\ &= \sum_{\sigma \in S_n} \chi(\sigma) \cdot \prod_{i=1}^{n} u_i \cdot v_{i,\sigma(i)} \cdot \prod_{j=1}^{n} x_{j,\sigma(j)} \\ &= \sum_{\sigma \in S_n} \chi(\sigma) \cdot 1 \cdot \prod_{i=1}^{n} x_{i,\sigma(i)} = s(X) \ , \end{aligned}$$

or equivalently, the operator F preserves the generalized matrix function s determined by (1). $\square$

As the direct consequence of the foregoing result we get the following:

Corollary 2.6. *For an arbitrary* $n \in \mathbb{N}$ *let* $F : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ *be an operator of the form (2). Then* F *preserves the generalized matrix function (1) if and only if there are nonsingular diagonal matrices* $M, N \in M_n(\mathcal{F})$, *where* $M = \mathrm{diag}(u_1, u_2, \ldots, u_n)$ *and* $N = \mathrm{diag}(v_1, v_2, \ldots, v_n)$, *such that for an arbitrary matrix* $X \in M_n(\mathcal{F})$

$$F(X) = M \cdot X \cdot N$$

and moreover

$$\det(M) \cdot \det(N) = 1.$$

References

[1] W.S. Cheung, M.A. Dufner, C.K. Li, Multiplicative preservers and induced operators, *Linear Algebra Appl.*, **401** (2005), 307-323.

[2] J. Kalinowski, On operators preserving the permanents, *Intern. Journal of Pure and Applied Mathematics*, **38**, No. 1 (2007), 5-11.

[3] J. Kalinowski, On preservers of determinant over a field, *Demonstr. Mathematica*, **41**, No. 3 (2008), 519-524.

[4] D.E. Littlewood, A.R. Richardson, Group characters and algebras, *Philos. Transactions of the Royal Society*, Ser. A, **233** (1934), 99-124.

[5] M. Marcus, H. Minc, Generalized matrix functions, *Transactions of the American Mathematical Society*, **116** (1965), 316-329.