International Journal of Pure and Applied Mathematics
Volume 63 No. 4 2010, 449-460

A NEW KIND OF HARDY-HILBERT TYPE INTEGRAL INEQUALITY

Ayşe Gülsüm Bayram[1 §], Hüseyin Yildirim[2]
[1,2]Department of Mathematics
Faculty of Science and Arts
Afyon Kocatepe University
Afyonkarahisar, 03200, TURKEY
[1]e-mail: aysegbayram@yahoo.com
[2]e-mail: hyildir@aku.edu.tr

Abstract: We establish a new extension of Hardy-Hilbert type integral inequality by using the technique of analysis and the constant factor which is related to the Riemann Zeta function.

AMS Subject Classification: 26D15, 49C99
Key Words: Hardy-Hilbert-type integral inequality, Hölder's inequality, Riemann Zeta function

1. Introduction

Hilbert's inequality has a great interest in analysis and its applications (see [8], [9]). The original Hilbert's inequality can be stated as follows:

If $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(x)\,dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)\,dx \int_0^\infty g^2(x)\,dx \right\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible (see [3]). The inequality (1.1) has been extended by Hardy as (see [2]):

Received: May 6, 2010
§Correspondence author

If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)$,$g(x)\geq 0$, such that $0<\int\limits_0^\infty f^p(x)\,dx<\infty$ and $0<\int\limits_0^\infty g^q(x)\,dx<\infty$, then we have the following Hardy-Hilbert's integral inequality:

$$\int\limits_0^\infty\int\limits_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\int\limits_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int\limits_0^\infty g^q(x)\,dx\right\}^{\frac{1}{q}}, \quad (1.2)$$

where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible (see [3]).

In the papers [3] and [7], the following inequality of the form

$$\int\limits_0^\infty\int\limits_0^\infty \frac{(\ln x-\ln y)\,f(x)\,g(y)}{x-y}dxdy$$

$$<\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2\left\{\int\limits_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int\limits_0^\infty g^q(x)\,dx\right\}^{\frac{1}{q}} \quad (1.3)$$

was established, and the coefficient $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2$ is also the best possible.

$$\int\limits_0^\infty\int\limits_0^\infty \frac{f(x)\,g(y)}{\max\{x,y\}}dxdy\leq pq\left\{\int\limits_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int\limits_0^\infty g^q(x)\,dx\right\}^{\frac{1}{q}}, \quad (1.4)$$

where the constant factor pq is the best possible (see [3]).

Theorem 1. *If $\lambda>0$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f,g\geq 0$ such that $0<\int\limits_0^\infty t^{p-1-\lambda}f^p(t)\,dt<\infty$, $0<\int\limits_0^\infty t^{q-1-\lambda}g^q(t)\,dt<\infty$, then*

$$\int\limits_0^\infty\int\limits_0^\infty \frac{f(x)\,g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy$$

$$\leq\frac{pq}{\lambda}\left\{\int\limits_0^\infty t^{p-1-\lambda}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int\limits_0^\infty t^{q-1-\lambda}g^q(x)\,dx\right\}^{\frac{1}{q}}, \quad (1.5)$$

where the constant factor $\frac{pq}{\lambda}$ is the best possible (see [12]).

Theorem 2. *Let $f(x),g(x)\geq 0$, $0<\int\limits_0^\infty f^2(x)\,dx<\infty$ and $0<$*

$\int\limits_0^\infty g^2(x)\,dx < \infty$. *Then*

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)\,g(y)}{x+y+\max\{x,y\}}dxdy < c\left(\int\limits_0^\infty f^2(x)\,dx \int\limits_0^\infty g^2(x)\,dx\right)^{\frac{1}{2}}, \tag{1.6}$$

where $c = \sqrt{2}\left(\pi - 2\arctan\sqrt{2}\right) \approx 1.7408$ *(see [16]).*

Owing to the importance of the Hardy-Hilbert's integral inequality and the Hardy-Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) appear in a great deal of papers (see [8], [10], [12], [16], [7], [1], [5], [4], [14], [15], [6], [11]). The aim of this paper is to built a new Hardy-Hilbert type integral inequality by introducing a proper integral kernel function and by using the technique of analysis and to discuss the constant factor which is related to the Riemann Zeta function.

Let $0 < \alpha < 1$ and n is a positive integer. Define a function ζ^* by

$$\zeta^*(n,\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha+k)^n}. \tag{1.7}$$

And further define the function ζ_p by

$$\zeta_p = n!\left(\zeta^*\left(n+1,\frac{1}{p}\right) + \zeta^*\left(n+1,1-\frac{1}{p}\right)\right) \qquad (n \in N_0),$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. It is obvious that $\zeta_p = \zeta_q$.

Lemma 1. *Let* $0 < \alpha < 1$ *and* n *is a nonnegative integer. Then*

$$\int\limits_0^1 t^{\alpha-1}\left(\ln\frac{1}{t}\right)^n \frac{1}{1+t}dt = n!\zeta^*(n+1,\alpha),$$

where ζ^* *is defined by* (1.7) *(see [6]).*

Lemma 2.

$$\int\limits_0^\infty u^{\alpha-1}\left|\ln\frac{1}{u}\right|^n \frac{1}{1+u}du = n!\left\{\zeta^*(n+1,\alpha) + \zeta^*(n+1,1-\alpha)\right\},$$

where ζ^* *is defined by* (1.7).

2. Main Results

In this section, we will prove our assertions by using the above theorems and lemmas.

Theorem 3. *Let $f, g \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, n is a nonnegative integer. If $0 < \int_0^\infty f^p(x)\,dx < \infty$ and $0 < \int_0^\infty g^q(y)\,dy < \infty$ and $\lambda > 0$, then*

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^n f(x)\, g(y)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dxdy \leq n!\left\{\zeta^*\left(n+1, \frac{1}{p}+\lambda\right) + \zeta^*\left(n+1, \frac{1}{q}+\lambda\right)\right\} \times \left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(y)\,dy\right\}^{\frac{1}{q}}, \quad (2.1)$$

the coefficient $n!\left\{\zeta^*\left(n+1, \frac{1}{p}+\lambda\right) + \zeta^*\left(n+1, \frac{1}{q}+\lambda\right)\right\}$ is the best possible.

Proof. First of all we can write

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^n f(x)\, g(y)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dxdy = \int_0^\infty \int_0^\infty \left(\frac{|\ln x - \ln y|^n}{x+y}\right)^{\frac{1}{p}} \times \left(\frac{x}{y}\right)^{\frac{1}{pq}} \frac{f(x)}{\left(\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}\right)^{\frac{1}{p}}} \left(\frac{|\ln x - \ln y|^n}{x+y}\right)^{\frac{1}{q}} \left(\frac{y}{x}\right)^{\frac{1}{pq}} \frac{g(y)}{\left(\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}\right)^{\frac{1}{q}}} dxdy.$$

Then, we can use Hölder inequality

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^n f(x)\, g(y)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dxdy$$

$$\leq \left(\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{q}} \frac{f^p(x)}{\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dxdy \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} \left(\frac{y}{x}\right)^{\frac{1}{p}} \frac{g^q(y)}{\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dxdy \right)^{\frac{1}{q}}.$$

$$= \left(\int_0^\infty f^p(x) \left(\int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{q}} \frac{1}{\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dy \right) dx \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty g^q(y) \left(\int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} \left(\frac{y}{x}\right)^{\frac{1}{p}} \frac{1}{\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dx \right) dy \right)^{\frac{1}{q}}$$

$$= I_1 \times I_2. \quad (2.2)$$

Here we consider following equality

$$M = \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{q}} \frac{1}{\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dy.$$

In this case we have

$$M = \int_0^x \frac{|\ln x - \ln y|^n}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{q}} \frac{1}{\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dy$$

$$+ \int_x^\infty \frac{|\ln x - \ln y|^n}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{q}} \frac{1}{\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} dy = M_1 + M_2. \quad (2.3)$$

Since in the case of M_1, $x \geq y$, implies $\frac{x}{y} \geq \frac{y}{x}$, and hence

$$\max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\} = \left(\frac{x}{y}\right)^\lambda,$$

then we have

$$M_1 = \int_0^x \frac{|\ln \frac{x}{y}|^n}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{q}} \frac{1}{\left(\frac{x}{y}\right)^{\lambda}} dy = \int_0^x \frac{|\ln \frac{x}{y}|^n}{x\left(1+\frac{y}{x}\right)} \left(\frac{x}{y}\right)^{\frac{1}{q}-\lambda} dy.$$

Setting $\frac{y}{x} = u$, we have

$$M_1 = \int_0^1 u^{\lambda-\frac{1}{q}} |\ln \frac{1}{u}|^n \frac{1}{1+u} du = n!\zeta^* \left(n+1, \lambda + \frac{1}{p}\right). \tag{2.4}$$

Following the same way,

$$M_2 = \int_x^{\infty} \frac{|\ln \frac{x}{y}|^n}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{q}} \frac{1}{\max\left\{\left(\frac{x}{y}\right)^{\lambda}, \left(\frac{y}{x}\right)^{\lambda}\right\}} dy = n!\zeta^* \left(n+1, \lambda + \frac{1}{q}\right). \tag{2.5}$$

Combining (2.3), (2.4) and (2.5) we have the following inequality

$$I_1 \leq \left(M \int_0^{\infty} f^p(x)\, dx\right)^{\frac{1}{p}}, \tag{2.6}$$

where

$$M = n!\left(\zeta^* \left(n+1, \lambda + \frac{1}{p}\right) + \zeta^* \left(n+1, \lambda + \frac{1}{q}\right)\right). \tag{2.7}$$

Following the same steps for the second integral on the right hand side of (2.2) we have,

$$I_2 \leq \left(M \int_0^{\infty} g^q(y)\, dy\right)^{\frac{1}{q}}. \tag{2.8}$$

Combining (2.2), (2.6), (2.7) and (2.8) we can write,

$$\int_0^{\infty}\int_0^{\infty} \frac{|\ln x - \ln y|^n f(x)\, g(y)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^{\lambda}, \left(\frac{y}{x}\right)^{\lambda}\right\}} dxdy \leq$$

$$n!\left\{\zeta^* \left(n+1, \frac{1}{p} + \lambda\right) + \zeta^* \left(n+1, \frac{1}{q} + \lambda\right)\right\} \left\{\int_0^{\infty} f^p(x)\, dx\right\}^{\frac{1}{p}} \left\{\int_0^{\infty} g^q(y)\, dy\right\}^{\frac{1}{q}}.$$

Now, we have to show that $n!\left\{\zeta^* \left(n+1, \frac{1}{p} + \lambda\right) + \zeta^* \left(n+1, \frac{1}{q} + \lambda\right)\right\}$ in (2.1) is the best possible. $\forall \varepsilon > 0$, define two functions by $f_{\varepsilon}(x) = x^{-\frac{1+\varepsilon}{p}}$ and

$g_\varepsilon(y) = y^{-\frac{1+\varepsilon}{q}}$. We can easily see that,

$$\int_\varepsilon^\infty f_\varepsilon^p(x)\,dx = \frac{1}{\varepsilon^{1+\varepsilon}} \text{ and } \int_\varepsilon^\infty g_\varepsilon^q(y)\,dy = \frac{1}{\varepsilon^{1+\varepsilon}}.$$

If $n!\left\{\zeta^*\left(n+1,\frac{1}{p}+\lambda\right)+\zeta^*\left(n+1,\frac{1}{q}+\lambda\right)\right\}$ is not the best possible, then there exists $C>0$, such that

$$\int_\varepsilon^\infty\int_\varepsilon^\infty \frac{|\ln x-\ln y|^n f_\varepsilon(x)\,g_\varepsilon(y)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dxdy \le C\left(\int_\varepsilon^\infty f_\varepsilon^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_\varepsilon^\infty g_\varepsilon^q(y)\,dy\right)^{\frac{1}{q}}$$

$$= \frac{C}{\varepsilon^{1+\varepsilon}} < \frac{n!\left\{\zeta^*\left(n+1,\frac{1}{p}+\lambda\right)+\zeta^*\left(n+1,\frac{1}{q}+\lambda\right)\right\}}{\varepsilon^{1+\varepsilon}}. \quad (2.9)$$

On the other hand, we have

$$\int_\varepsilon^\infty\int_\varepsilon^\infty \frac{|\ln x-\ln y|^n f_\varepsilon(x)\,g_\varepsilon(y)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dxdy$$

$$= \int_\varepsilon^\infty\int_\varepsilon^\infty \frac{x^{-\frac{1+\varepsilon}{p}}\left(|\ln x-\ln y|^n y^{-\frac{1+\varepsilon}{q}}\right)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dxdy$$

$$= \int_\varepsilon^\infty\left\{\int_\varepsilon^\infty \frac{\left(|\ln x-\ln y|^n y^{-\frac{1+\varepsilon}{q}}\right)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dy\right\}\left\{x^{-\frac{1+\varepsilon}{p}}\right\}dx$$

$$= \int_\varepsilon^\infty\left\{\int_\varepsilon^x \frac{\left(|\ln x-\ln y|^n y^{-\frac{1+\varepsilon}{q}}\right)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dy + \int_x^\infty \frac{\left(|\ln x-\ln y|^n y^{-\frac{1+\varepsilon}{q}}\right)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dy\right\}$$

$$\times\left\{x^{-\frac{1+\varepsilon}{p}}\right\}dx = \int_\varepsilon^\infty\left\{\int_\varepsilon^x \frac{\left(|\ln \frac{x}{y}|^n y^{-\frac{1+\varepsilon}{q}}\right)}{(x+y)\left(\frac{x}{y}\right)^\lambda}dy + \int_x^\infty \frac{\left(|\ln \frac{x}{y}|^n y^{-\frac{1+\varepsilon}{q}}\right)}{(x+y)\left(\frac{y}{x}\right)^\lambda}dy\right\}\left\{x^{-\frac{1+\varepsilon}{p}}\right\}dx$$

setting $\frac{y}{x}=t$, we have

$$=\int\limits_{\varepsilon}^{\infty}\left\{\int\limits_{\frac{\varepsilon}{x}}^{1}\frac{\left(|\ln\frac{1}{t}|^{n}(xt)^{-\frac{1+\varepsilon}{q}}\right)}{x(1+t)\left(\frac{1}{t}\right)^{\lambda}}xdt+\int\limits_{1}^{\infty}\frac{|\ln\frac{1}{t}|^{n}(xt)^{-\frac{1+\varepsilon}{q}}}{x(1+t)t^{\lambda}}xdt\right\}\left\{x^{-\frac{1+\varepsilon}{p}}\right\}dx$$

$$=\int\limits_{\varepsilon}^{\infty}\left\{\int\limits_{\frac{\varepsilon}{x}}^{1}|\ln\frac{1}{t}|^{n}t^{\lambda-\frac{1+\varepsilon}{q}}\frac{1}{1+t}dt+\int\limits_{1}^{\infty}|\ln\frac{1}{t}|^{n}t^{-\lambda-\frac{1+\varepsilon}{q}}\frac{1}{1+t}dt\right\}\left\{x^{-\frac{1+\varepsilon}{p}-\frac{1+\varepsilon}{q}}\right\}dx$$

$$=\frac{1}{\varepsilon^{1+\varepsilon}}\left\{\int\limits_{\frac{\varepsilon}{x}}^{1}|\ln\frac{1}{t}|^{n}t^{\lambda-\frac{1+\varepsilon}{q}}\frac{1}{1+t}dt+\int\limits_{1}^{\infty}|\ln\frac{1}{t}|^{n}t^{-\lambda-\frac{1+\varepsilon}{q}}\frac{1}{1+t}dt\right\}.$$

When ε is small enough, we can obtain

$$\int\limits_{\varepsilon}^{\infty}\int\limits_{\varepsilon}^{\infty}\frac{|\ln x-\ln y|^{n}f_{\varepsilon}(x)\,g_{\varepsilon}(y)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^{\lambda},\left(\frac{y}{x}\right)^{\lambda}\right\}}dxdy$$
$$>\frac{1}{\varepsilon^{1+\varepsilon}}\left\{n!\left\{\zeta^{*}\left(n+1,\frac{1}{p}+\lambda\right)+\zeta^{*}\left(n+1,\frac{1}{q}+\lambda\right)\right\}+o(1)\right\},\quad (2.10)$$

as $(\varepsilon\to 0)$.

It is obvious that, when ε is small enough, the inequality (2.9) is in contradiction with the inequality (2.10). Therefore $n!\left\{\zeta^{*}\left(n+1,\frac{1}{p}+\lambda\right)+\zeta^{*}(n+1,\frac{1}{q}+\lambda)\right\}$ in (2.1) is the best possible. This gives the proof of the theorem. □

Theorem 4. *If* $0<\int\limits_{0}^{\infty}f^{2}(x)\,dx<\infty$ *and* $0<\int\limits_{0}^{\infty}g^{2}(y)\,dy<\infty$ *and* $\lambda>0$ *then*

$$\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\frac{|\ln x-\ln y|^{n}f(x)\,g(y)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^{\lambda},\left(\frac{y}{x}\right)^{\lambda}\right\}}dxdy$$
$$\leq 2n!\zeta^{*}\left(n+1,\frac{1}{2}+\lambda\right)\left\{\int\limits_{0}^{\infty}f^{2}(x)\,dx\right\}^{\frac{1}{2}}\left\{\int\limits_{0}^{\infty}g^{2}(y)\,dy\right\}^{\frac{1}{2}}.\quad (2.11)$$

$2n!\zeta^{*}\left(n+1,\frac{1}{2}+\lambda\right)$ *is the best possible.*

3. Some Equivalent Forms

Theorem 5. *Let n be a nonnegetive integer, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f is a nonnegative real function such that $0 < \int\limits_0^\infty f^p(x)\,dx < \infty$, then*

$$\int\limits_0^\infty \left(\int\limits_0^\infty \frac{|\ln x - \ln y|^n f(x)}{(x+y)\max\left\{ \left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda \right\}} dx \right)^p dy$$
$$\leq \left[n! \left\{ \zeta^* \left(n+1, \frac{1}{p} + \lambda \right) + \zeta^* \left(n+1, \frac{1}{q} + \lambda \right) \right\} \right]^p \int\limits_0^\infty f^p(x)\,dx, \quad (3.1)$$

where the constant factor $\left[n! \left\{ \zeta^* \left(n+1, \frac{1}{p} + \lambda \right) + \zeta^* \left(n+1, \frac{1}{q} + \lambda \right) \right\} \right]^p$ is the best possible.

Proof. First we show that the inequality (3.1) is equivalent to (2.1). Setting a real function $g(y)$ as

$$g(y) = \left(\int\limits_0^\infty \frac{|\ln x - \ln y|^n f(x)}{(x+y)\max\left\{ \left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda \right\}} dx \right)^{p-1} \qquad y \in (0, \infty).$$

By using (2.1), we have

$$\int\limits_0^\infty \left(\int\limits_0^\infty \frac{|\ln x - \ln y|^n f(x)}{(x+y)\max\left\{ \left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda \right\}} dx \right)^p dy$$
$$= \int\limits_0^\infty \int\limits_0^\infty \frac{|\ln x - \ln y|^n f(x)\, g(y)}{(x+y)\max\left\{ \left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda \right\}} dx dy \leq$$
$$n! \left\{ \zeta^* \left(n+1, \frac{1}{p} + \lambda \right) + \zeta^* \left(n+1, \frac{1}{q} + \lambda \right) \right\} \left\{ \int\limits_0^\infty f^p(x)\,dx \right\}^{\frac{1}{p}} \left\{ \int\limits_0^\infty g^q(y)\,dy \right\}^{\frac{1}{q}}$$
$$= n! \left\{ \zeta^* \left(n+1, \frac{1}{p} + \lambda \right) + \zeta^* \left(n+1, \frac{1}{q} + \lambda \right) \right\} \left\{ \int\limits_0^\infty f^p(x)\,dx \right\}^{\frac{1}{p}}$$

$$\times\left[\int_0^\infty\left(\int_0^\infty \frac{|\ln x-\ln y|^n f(x)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dx\right)^{q(p-1)}dy\right]^{\frac{1}{q}}$$

$$=n!\left\{\zeta^*\left(n+1,\frac{1}{p}+\lambda\right)+\zeta^*\left(n+1,\frac{1}{q}+\lambda\right)\right\}\left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left[\int_0^\infty\left(\int_0^\infty \frac{|\ln x-\ln y|^n f(x)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dx\right)^{p}dy\right]^{\frac{1}{q}}. \quad (3.2)$$

The inequality (3.1) is valid after some simplifications on (3.2). On the other hand, assume that the inequality (3.1) keeps valid, then applying in turn Hölder's inequality and (3.1), we have

$$\int_0^\infty\int_0^\infty \frac{|\ln x-\ln y|^n f(x)\,g(y)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dxdy$$

$$=\int_0^\infty\left\{\int_0^\infty \frac{|\ln x-\ln y|^n f(x)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dx\right\}g(y)\,dy$$

$$\leq\left\{\int_0^\infty\left\{\int_0^\infty \frac{|\ln x-\ln y|^n f(x)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dx\right\}^p dy\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)\,dy\right\}^{\frac{1}{q}} \quad (3.3)$$

$$\leq\left[(n!\left\{\zeta^*\left(n+1,\frac{1}{p}+\lambda\right)+\zeta^*\left(n+1,\frac{1}{q}+\lambda\right)\right\})^p\right]^{\frac{1}{p}}$$

$$\times\left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)\,dy\right\}^{\frac{1}{q}}$$

$$=n!\left\{\zeta^*\left(n+1,\frac{1}{p}+\lambda\right)+\zeta^*\left(n+1,\frac{1}{q}+\lambda\right)\right\}$$

$$\times \left\{ \int_0^\infty f^p(x)\,dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y)\,dy \right\}^{\frac{1}{q}}. \quad (3.4)$$

If the constant factor $\left[n!\left\{\zeta^*\left(n+1,\frac{1}{p}+\lambda\right)+\zeta^*\left(n+1,\frac{1}{q}+\lambda\right)\right\}\right]^p$ in (3.1) is not the best possible, then it is known from (3.4) that the constant factor

$$n!\left\{\zeta^*\left(n+1,\frac{1}{p}+\lambda\right)+\zeta^*\left(n+1,\frac{1}{q}+\lambda\right)\right\}$$

in (2.1) is also not the best possible. This is a contradiction. This gives the proof of the theorem. □

Theorem 6. *If* $0<\int_0^\infty f^2(x)\,dx<\infty$ *and* $\lambda>0$, *then*

$$\int_0^\infty \left(\int_0^\infty \frac{|\ln x-\ln y|^n f(x)}{(x+y)\max\left\{\left(\frac{x}{y}\right)^\lambda,\left(\frac{y}{x}\right)^\lambda\right\}}dx \right)^2 dy$$

$$\leq \left[2n!\zeta^*\left(n+1,\frac{1}{2}+\lambda\right)\right]^2 \int_0^\infty f^2(x)\,dx. \quad (3.5)$$

The constant factor $\left[2n!\zeta^*\left(n+1,\frac{1}{2}+\lambda\right)\right]^2$ *is the best possible. Inequality* (3.5) *is equivalent to* (2.11).

References

[1] M.Z. Gao, L. Hsu, A survey of various refinements and generalizations of Hilbert's inequalities, *J. Math. Res and Exp.*, **25**, No. 2 (2005), 227-243

[2] G.H. Hardy, Note on a theorem of Hilbert concerning series positive terms, *Proc. London Math. Soc.*, Records of Proc. XLV-XLIV, **23**, No. 2 (1925).

[3] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge (1952).

[4] L. He, M.Z. Gao, Y. Zhou, On new extensions of Hilbert's integral inequality, *Int. J. Math. Math. Sci.* (2008), Art. ID 297508, 1-8.

[5] K. Hu, On Hilbert's inequality, *Chin. Ann. Math. Ser. B*, **13**, No. 1 (1992), 35-39.

[6] Y. Jin, *Applied Integral Tables*, Hefei: Chinese Science and Technology University Press (2006).

[7] J. Kuang, *Applied Inequalities*, 3-rd Edition, Shandong Science and Technology Press (2004), 534-535.

[8] D.S. Mitrinović, J.E. Pecarić, A.M. Fink, *Inequalities Involving Functions and Their Integral and Derivatives*, Boston, Kluwer Academic Publishers (1991).

[9] D.S. Mitrinović, J.E. Pecarić, A.M. Fink, *Classical and New Inequalities in Analysis*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecth (1993).

[10] L. Peng, M.Z. Gao, A Hardy-Hilbert type integral inequality, *Int. Journal of Math. Analysis*, **33**, No. 3 (2009), 1607-1616.

[11] W.T. Sulaiman, New kinds of Hardy Hilbert's integral inequalities, *Appl. Math. Lett.*, **23** (2010), 361-365.

[12] B. Sun, Best generalization of Hilbert type inequality, *J. Inequal. Pure Appl. Math.*, **7**, No. 1 (2006), Article ID 113.

[13] L.X. Wang, D. Fang, *Mathematical Handbook*, People's Education Press (1979).

[14] B.C. Yang, On a new inequality similar to Hardy-Hilbert's inequality, *Math. Inequal. Appl.*, **6**, No. 1 (2003), 37-44.

[15] B.C. Yang, A new Hilbert-type integral inequality and its generalization, *J. Jilin Univ. Sci.*, **43**, No. 5 (2005), 580-584.

[16] L. Yongjin, W. Jing, H. Bing, A new Hilbert-type integral inequality and the equivalent form, *Int. J. Math. Sci.* (2006), Article ID 45378, 1-6.