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ORTHOGONAL STABILITY OF 2 DIMENSIONAL MIXED TYPE ADDITIVE AND QUARTIC FUNCTIONAL EQUATION

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Abstract: In this paper, the authors investigate the orthogonal stability of 2 dimensional mixed type additive and quartic functional equation of the form

$$7[f(2x+y)+f(2x-y)] = 28[f(x+y)+f(x-y)]-3[f(2y)-2f(y)]+14[f(2x)-4f(x)],$$

with $x \perp y$, where $\perp$ is orthogonality in the sense of Ratz.

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1. Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [31] in 1940, concerning the stability of group homomorphisms.

Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$

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satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ *for all* $x, y \in G_1$*, then there exists a homomorphism* $H : G_1 \to G_2$ *with* $d(h(x), H(x)) < \epsilon$ *for all* $x \in G_1$*?*

The case of approximately additive functions was solved by D.H. Hyers [13] under the assumption that G_1 and G_2 are Banach spaces. In 1951 and in 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by T. Aoki [4] and Th.M. Rassias [22]. The Hyers-Ulam-Aoki-Rassias stability originates from this historical backgrounds, see [1, 5, 6, 10, 23, 28]. In 1982, J.M. Rassias [20, 21] provided a generalizations of the Hyers stability theorem which allows the Cauchy difference to be bounded. The stability phenomenon that was proved by J.M. Rassias is called the Ulam-Gavruta-Rassias stability by [26]. Very recently J.M. Rassias [27] introduced a new concept on stability called J.M. Rassias mixed type product-sum of powers of norms stability.

Now, we introduce some basic concepts of orthogonality and orthogonality normed spaces.

Definition 1.1. (see [12]) A vector space X is called an *orthogonality vector space* if there is a relation $x \perp y$ on X such that:

(i) $x \perp 0$, $0 \perp x$ for all $x \in X$;

(ii) if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(iii) $x \perp y$, $ax \perp by$ for all $a, b \in \mathbb{R}$;

(iv) if P is a two-dimensional subspace of X, then:

(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;

(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.

Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector $x,\ y$ define $x \perp y$ iff $x,\ y$ are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x,\ y \in X$. The pair $(x, \perp)$ is called an *orthogonality space.* It becomes *orthogonality normed space* when the orthogonality space is equipped with a norm.

The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), \quad x \perp y, \tag{1.1}$$

in which $\perp$ is an abstract orthogonally was first investigated by S. Gudder and D. Strawther [12]. R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1.1) in [11].

Definition 1.2. Let X be an orthogonality space and Y be a real Banach space. A mapping $f : X \to Y$ is called *orthogonally quadratic* if it satisfies the so called orthogonally Euler-Lagrange quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.2}$$

for all $x, y \in X$ with $x \perp y$. The orthogonality Hilbert space for the orthogonally quadratic functional equation (1.2) was first investigated by F. Vajzovic [32].

Several other functional equations and its stability in orthogonality spaces was discussed in [7, 16, 17, 19, 26, 27]. Very recently, M. Eshaghi Gordji [8], obtained the general solution and the generalized Hyers-Ulam-Rassias stability of a functional equation deriving from quartic and additive functions of the form

$$\begin{aligned} &f(2x+y)+f(2x-y)\\ &\quad = 4[f(x+y)+f(x-y)]-\frac{3}{7}[f(2y)-2f(y)]+2f(2x)-8f(x). \end{aligned} \tag{1.3}$$

In this paper, the authors discussed the orthogonal stability of 2 dimensional mixed type additive and quartic functional equation of the form

$$\begin{aligned} &7[f(2x+y)+f(2x-y)]\\ &\quad = 28[f(x+y)+f(x-y)]-3[f(2y)-2f(y)]+14[f(2x)-4f(x)] \end{aligned} \tag{1.4}$$

with $x \perp y$ and investigates its Hyers-Ulam-Aoki-Rassias stability of (1.4), where $\perp$ is orthogonality in the sense of Ratz. Note that the function $f(x) = ax + bx^4$ is the solution of the functional equation (1.4).

Definition 1.3. A mapping $f : A \to B$ is called orthogonal additive and quartic respectively, if it satisfies the mixed type functional equation (1.4) for all $x, y \in A$, with $x \perp y$ where A be an orthogonality space and B be a real Banach space.

Through out this paper, let $(A, \perp)$ denote an orthogonality normed space with norm $\|\cdot\|_A$ and $(B, \|\cdot\|_B)$ is a Banach space. We define

$$\begin{aligned} D\ f\ (x,y) &= 7[f(2x+y)+f(2x-y)]-28[f(x+y)+f(x-y)]\\ &\quad +3[f(2y)-2f(y)]-14[f(2x)-4f(x)] \end{aligned}$$

for all $x, y \in A$, with $x \perp y$.

2. Hyers-Ulam-Aoki-Rassias Stability of (1.4)

In this section, we present the Hyers-Ulam-Aoki-Rassias stability of the orthogonal functional equation (1.4).

Theorem 2.1. *Let α and s $(s < 1)$ be nonnegative real numbers. Let $f_a : A \to B$ be an odd mapping satisfying*

$$\|D\ f_a\,(x, y)\|_B \leq \alpha\,\{\|x\|_A^s + \|y\|_A^s\} \tag{2.1}$$

for all $x, y \in A$, with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : A \to B$ such that

$$\|f_a(y) - L(y)\|_B \leq \frac{\alpha}{3(2-2^s)}\,\|y\|_A^s \tag{2.2}$$

for all $y \in A$. The function $L(y)$ is defined by

$$L(y) = \lim_{n\to\infty} \frac{f_a(2^n y)}{2^n} \tag{2.3}$$

for all $y \in A$.

Proof. Replacing (x, y) by $(0, 0)$ in (2.1), we get $f_a(0) = 0$. Setting (x, y) by $(0, y)$ in (2.1), we obtain

$$\|3f_a(2y) - 6f_a(y)\|_B \leq \alpha\,\|y\|_A^s \tag{2.4}$$

for all $y \in A$. Since $y \perp 0$, we have

$$\left\|\frac{f_a(2y)}{2} - f_a(y)\right\|_B \leq \frac{\alpha}{6}\,\|y\|_A^s \tag{2.5}$$

for all $y \in A$. Now replacing y by $2y$ and dividing by 2 in (2.5) and summing resulting inequality with (2.5), we arrive to

$$\left\|\frac{f_a(2^2 y)}{2^2} - f_a(y)\right\|_B \leq \frac{\alpha}{6}\left\{1 + \frac{2^s}{2}\right\}\|y\|_A^s \tag{2.6}$$

for all $y \in A$. In general, using induction on a positive integer n, we obtain that

$$\begin{aligned}\left\|\frac{f_a(2^n y)}{2^n} - f_a(y)\right\|_B &\leq \frac{\alpha}{6}\sum_{k=0}^{n-1}\frac{2^{s\,k}}{2^k}\,\|y\|_A^s \\ &\leq \frac{\alpha}{6}\sum_{k=0}^{\infty}\frac{2^{s\,k}}{2^k}\,\|y\|_A^s\end{aligned} \tag{2.7}$$

for all $y \in A$. In order to prove the convergence of the sequence $\{f_a(2^n y)/2^n\}$, replace y by $2^m y$ and divide by 2^m in (2.7), for any $n, m > 0$, we obtain

$$\begin{aligned}\left\|\frac{f_a\,(2^n 2^m y)}{2^{(n+m)}} - \frac{f_a(2^m y)}{2^m}\right\|_B &= \frac{1}{2^m}\left\|\frac{f_a\,(2^n 2^m y)}{2^n} - f_a\,(2^m y)\right\|_B \\ &\leq \frac{1}{2^m}\frac{\alpha}{6}\ \sum_{k=0}^{n-1}\frac{2^{s\,k}}{2^k}\,\|2^m y\|_A^s\end{aligned}$$

$$\leq \frac{\alpha}{6} \sum_{k=0}^{\infty} \frac{1}{2^{(1-s)(k+m)}} \|y\|_A^s . \tag{2.8}$$

As $s < 1$, the right hand side of (2.8) tends to 0 as $m \to \infty$ for all $y \in A$. Thus $\{f_a(2^n y)/2^n\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $L : A \to B$ such that

$$L(y) = \lim_{n\to\infty} \frac{f_a(2^n y)}{2^n}, \quad \forall y \in A.$$

Letting $n \to \infty$ in (2.7), we arrive the formula (2.2) for all $y \in A$. To prove L satisfies (1.4), replace (x, y) by $(2^n x, 2^n y)$ in (2.1) and divide by 2^n, it follows that

$$\begin{aligned} \frac{1}{2^n}\Big\| 7[f_a(2^n(2x + y)) + f_a(2^n(2x - y))] - 28[f_a(2^n(x + y)) + f_a(2^n(x - y))] \\ + 3[f_a(2^n 2y) - 2f_a(2^n y)] - 14[f_a(2^n\ 2x) - 4f_a(2^n x)]\Big\|_B \\ \leq \frac{\alpha}{2^n} \{\|2^n x\|_A^s + \|2^n y\|_A^s\} . \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned} 7[L(2x + y) + L(2x - y)] = 28[L(x + y) + L(x - y)] - 3[L(2y) - 2L(y)] \\ + 14[L(2x) - 4L(x)] \end{aligned}$$

for all $x, y \in A$ with $x \perp y$. Therefore $L : A \to B$ is an orthogonally additive mapping which satisfies (1.4). To prove the uniqueness of L, let L' be another orthogonally additive mapping satisfying (1.4) and the inequality (2.2). Then

$$\begin{aligned} \left\|A(y) - A'(y)\right\|_B &= \frac{1}{2^n} \left\|A(2^n y) - A'(2^n y)\right\|_B \\ &\leq \frac{1}{2^n} \left(\|L(2^n y) - f_a(2^n y)\|_B + \left\|f_a(2^n y) - L'(2^n y)\right\|_B\right) \\ &\leq \frac{2\ \alpha}{3[2 - 2^s]} \frac{1}{2^{n(1-s)}} \|y\|_A^s \\ &\to\ 0 \ \text{ as } n \to \infty \end{aligned}$$

for all $y \in A$. Therefore L is unique. This completes the proof of the theorem. □

Theorem 2.2. *Let β and s ($s < 4$) be nonnegative real numbers. Let $f_q : A \to B$ be an even mapping satisfying*

$$\|D\ f_q(x, y)\|_B \leq \beta \{\|x\|_A^s + \|y\|_A^s\} \tag{2.9}$$

for all $x, y \in A$, with $x \perp y$. Then there exists a unique orthogonally quartic *mapping $M : A \to B$ such that*

$$\|f_q(y) - M(y)\|_B \le \frac{\beta}{16 - 2^s} \|y\|_A^s \tag{2.10}$$

for all $y \in A$. The function $M(y)$ is defined by

$$M(y) = \lim_{n\to\infty} \frac{f_q(2^n y)}{16^n} \tag{2.11}$$

for all $y \in A$.

Proof. Replacing (x, y) by $(0, 0)$ in (2.9), we get $f_q(0) = 0$. Setting (x, y) by $(0, y)$ in (2.9), we obtain

$$\|3f_q(2y) - 48f_q(y)\|_B \le \beta \|y\|_A^s \tag{2.12}$$

for all $y \in A$. Since $y \perp 0$, we have

$$\left\| \frac{f_q(2y)}{16} - f_q(y) \right\|_B \le \frac{\beta}{48} \|y\|_A^s \tag{2.13}$$

for all $y \in A$. Now replacing y by $2y$ and dividing by 16 in (2.13) and summing resulting inequality with (2.13), we arrive to

$$\left\| \frac{f_q(2^2 y)}{16^2} - f_q(y) \right\|_B \le \frac{\beta}{48} \left\{ 1 + \frac{2^s}{16} \right\} \|y\|_A^s \tag{2.14}$$

for all $y \in A$. In general, using induction on a positive integer n, we obtain that

$$\begin{aligned} \left\| \frac{f_q(2^n y)}{16^n} - f_q(y) \right\|_B &\le \frac{\beta}{48} \sum_{k=0}^{n-1} \frac{2^{s\,k}}{16^k} \|y\|_A^s \\ &\le \frac{\beta}{48} \sum_{k=0}^{\infty} \frac{2^{s\,k}}{16^k} \|y\|_A^s \end{aligned} \tag{2.15}$$

for all $y \in A$. In order to prove the convergence of the sequence $\{f_q(2^n y)/16^n\}$, replace y by $2^m y$ and divide by 16^m in (2.15), for any $n, m > 0$. We obtain

$$\begin{aligned} \left\| \frac{f_q(2^n 2^m y)}{16^{(n+m)}} - \frac{f_q(2^m y)}{16^m} \right\|_B &= \frac{1}{16^m} \left\| \frac{f_q(2^n 2^m y)}{16^n} - f_q(2^m y) \right\|_B \\ &\le \frac{1}{16^m} \frac{\beta}{48} \sum_{k=0}^{n-1} \frac{2^{s\,k}}{16^k} \|2^m y\|_A^s \\ &\le \frac{\beta}{48} \sum_{k=0}^{\infty} \frac{1}{2^{(4-s)(k+m)}} \|y\|_A^s . \end{aligned} \tag{2.16}$$

As $s < 4$, the right hand side of (2.16) tends to 0 as $m \to \infty$ for all $y \in A$. Thus $\{f_q(2^n y)/16^n\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $M : A \to B$ such that

$$M(y) = \lim_{n\to\infty} \frac{f_q(2^n y)}{16^n}, \quad \forall y \in A.$$

Letting $n \to \infty$ in (2.15), we arrive to the formula (2.10) for all $y \in A$. To prove M satisfies (1.4) and it is unique the proof is similar to that of Theorem 2.1. □

Now we are ready to prove our main theorem.

Theorem 2.3. *Let θ and s $(s < 1)$ be nonnegative real numbers. Let $f : A \to B$ be a mapping satisfying*

$$\|D\ f\,(x, y)\|_B \le \theta\,\{\|x\|_A^s + \|y\|_A^s\} \tag{2.17}$$

for all $x, y \in A$, with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : A \to B$ and a unique orthogonally quartic mapping $M : A \to B$ such that

$$\|f(y) - L(y) - M(y)\|_B \le \left[\frac{\theta}{3(2-2^s)} + \frac{\theta}{16-2^s}\right] \|y\|_A^s \tag{2.18}$$

for all $y \in A$. The functions $L(y)$ and $M(y)$ are defined in (2.3) and (2.11) respectively for all $y \in A$.

Proof. Let $f_e(y) = \frac{f_q(y)+f_q(-y)}{2}$ for all $y \in A$, then $f_e(0) = 0$. Hence

$$\begin{aligned}\|Df_e(x, y)\| &\le \frac{\theta}{2}\,\{(||x||_A^s + ||y||_A^s) + (|| - x||_A^s + || - y||_A^s)\}\\ &\le \theta(||x||_A^s + ||y||_A^s). \end{aligned}\tag{2.19}$$

By Theorem 2.2, we have

$$\|f_e(y) - M(y)\|_B \le \frac{\theta}{16-2^s}\,\|y\|_A^s \tag{2.20}$$

for all $y \in A$. Also, let $f_o(y) = \dfrac{f_a(y) - f_a(-y)}{2}$ for all $y \in A$, then $f_o(0) = 0$. Hence

$$\begin{aligned}\|Df_o(x, y)\| &\le \frac{\theta}{2}\,\{(||x||_A^s + ||y||_A^s) + (|| - x||_A^s + || - y||_A^s)\}\\ &\le \theta(||x||_A^s + ||y||_A^s). \end{aligned}\tag{2.21}$$

By Theorem 2.1, we have

$$\|f_o(y) - L(y)\|_B \le \frac{\theta}{3(2-2^s)}\,\|y\|_A^s \tag{2.22}$$

for all $y \in A$. Define

$$f(y) = f_e(y) + f_o(y) \tag{2.23}$$

for all $y \in A$. From (2.20), (2.22) and (2.23), we arrive to

$$\begin{aligned}\|f(y) - L(y) - M(y)\|_B &= \|f_e(y) + f_o(y) - L(y) - M(y)\|_B \\ &\leq \|f_e(y) - M(y)\|_B + \|f_o(y) - L(y)\|_B \\ &\leq \left[\frac{\theta}{3(2-2^s)} + \frac{\theta}{16-2^s}\right] \|y\|_A^s\end{aligned}$$

for all $y \in A$. Hence the theorem is proved. □

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