

SOME CLASSES OF LOGARITHMICALLY COMPLETELY
MONOTONIC FUNCTIONS RELATED TO
THE GAMMA FUNCTION

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Abstract: In this article, a sufficient condition for some classes of functions involving the gamma function to be logarithmically completely monotonic are established, and then some known results are extended.

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1. Introduction

Recall (see [2, 7, 11, 13, 14, 16, 17]) that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0 \quad (1.1)$$

for all $k \in \mathbb{N}$ on I . If the inequality (1.1) is strict, then f is called strictly logarithmically completely monotonic. The set of the logarithmically complete monotonic functions on I denoted by $\mathcal{L}[I]$.

It is worthwhile to note that there have been a lot of literature on logarithmically completely monotonic functions related to the gamma function, psi function or polygamma function. Although it is not practicable to read

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all of these papers, we still would like to offer some of them, for example [3, 4, 5, 9, 12, 15, 19] and the references therein, to the readers.

It is well known that the classical Euler's gamma function $\Gamma(x)$ is defined for $x > 0$ as

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt. \quad (1.2)$$

The logarithmic derivative of $\Gamma(x)$, denoted by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad (1.3)$$

is called the psi or digamma function, and $\psi^{(n)}(x)$ for $n \in \mathbb{N}$ are known as the polygamma or multigamma functions. These functions play central roles in the theory of special functions and have lots of extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

In [6] and [8], while one studied certain problems of traffic flow, the following double inequality was obtained for $n \in \mathbb{N}$

$$2\Gamma\left(n + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right) \Gamma(n+1) \leq 2^n \Gamma\left(n + \frac{1}{2}\right), \quad (1.4)$$

which can be rearranged for $n > 1$ as

$$1 \leq \left[\frac{\Gamma(1/2)\Gamma(n+1)}{2\Gamma(n+1/2)} \right]^{1/(n-1)} \leq 2. \quad (1.5)$$

In [20], by using the following double inequality due to J. Wendel in [21]:

$$\left(\frac{x}{x+a}\right)^{1-a} \leq \frac{\Gamma(x+a)}{x^a \Gamma(x)} \leq 1 \quad (1.6)$$

for $0 < a < 1$ and $x > 0$, inequality (1.4) was extended and refined as

$$\sqrt{x} \leq \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leq \sqrt{x + \frac{1}{2}} \quad (1.7)$$

for $x > 0$.

The left hand side inequality in (1.6) reminds us to introduce

$$h_a(x) = \frac{(x+a)^{1-a} \Gamma(x+a)}{x \Gamma(x)} \quad (1.8)$$

for $x > 0$ and $a > 0$. F. Qi et al [18] have discussed its logarithmically complete monotonicity.

In order to obtain a refined upper bound in (1.6), F. Qi et al [18] have

studied the logarithmically complete monotonicity of the function

$$f_a(x) = \frac{\Gamma(x+a)}{x^a \Gamma(x)} \quad (1.9)$$

the middle term in (1.6), for $x \in (0, \infty)$ and $a \in (0, \infty)$.

For the given number $b \geq 0$ and $a, r \in \mathbb{R}$ with $r \neq 0$, define

$$h_{a,r}(x) = \left[\frac{(x+a)^{1-a} \Gamma(x+a)}{x \Gamma(x)} \right]^r \quad (1.10)$$

and

$$f_{a,b,r}(x) = \left[\frac{\Gamma(x+a)}{x^a \Gamma(x)} \left(1 + \frac{b}{x}\right)^{x+b} \right]^r \quad (1.11)$$

for $x \in (0, \infty)$.

In [18], the following conclusions were established:

- (1) $h_{a,1}(x) \in \mathcal{L}[(0, \infty)]$ if $0 < a < 1$.
- (2) $[h_{a,1}(x)]^{-1} \in \mathcal{L}[(0, \infty)]$ if $a > 1$.
- (3) $f_{a,0,1}(x) \in \mathcal{L}[(0, \infty)]$ and $\lim_{x \rightarrow 0^+} f_{a,0,1}(x) = \infty$ if $a > 1$.
- (4) $[f_{a,0,1}(x)]^{-1} \in \mathcal{L}[(0, \infty)]$ and $\lim_{x \rightarrow 0^+} f_{a,0,1}(x) = 0$ if $0 < a < 1$.
- (5) $\lim_{x \rightarrow \infty} f_{a,0,1}(x) = 1$ if any $a \in (0, \infty)$.

In this article, a sufficient condition such that $h_{a,r}(x)$ and $f_{a,b,r}(x)$ are strictly logarithmically complete monotonic functions on $(0, \infty)$ will be established, which extend some known results mentioned above.

2. Main Results

Theorem 2.1. *If $0 < a < 1$ and $r > 0$ then the function $h_{a,r}(x)$ defined by (1.10) is strictly logarithmically complete monotonic function on $(0, \infty)$. Moreover,*

$$\lim_{x \rightarrow 0^+} h_{a,r}(x) = \left[\frac{\Gamma(a+1)}{a^a} \right]^r \quad \text{and} \quad \lim_{x \rightarrow \infty} h_{a,r}(x) = 1,$$

for any $a > 0$.

Proof. It is clear that

$$\ln h_{a,r}(x) = r [(1-a) \ln(x+a) + \ln \Gamma(x+a) - \ln \Gamma(x+1)] \quad (2.1)$$

and

$$[\ln h_{a,r}(x)]' = \frac{1-a}{x+a} + \psi(x+a) - \psi(x+1). \quad (2.2)$$

The following formulas are known (see [10], p. 884): For $n \in \mathbb{N}$ and $x \in (0, \infty)$,

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} e^{-xt} dt \tag{2.3}$$

and

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt. \tag{2.4}$$

Hence, for $n \in \mathbb{N}$, easy computation yield

$$\begin{aligned} [\ln h_{a,r}(x)]^{(n)} &= r \left[\frac{(-1)^{n-1}(n-1)!(1-a)}{(x+a)^n} + \psi^{(n-1)}(x+a) - \psi^{(n-1)}(x+1) \right] \\ &= (-1)^n r \left[\frac{(a-1)(n-1)!}{(x+a)^n} + (-1)^n \psi^{(n-1)}(x+a) - (-1)^n \psi^{(n-1)}(x+1) \right] \\ &\quad - (-1)^{2n} \int_0^\infty \frac{t^{n-1}}{1 - e^{-t}} e^{-(x+1)t} dt \\ &= (-1)^n r \left[\int_0^\infty \frac{e^{-(x+a)t} t^{n-1}}{1 - e^{-t}} \left\{ (a-1)(1 - e^{-t}) + 1 - e^{-(a-1)t} \right\} dt \right] \\ &\quad \triangleq (-1)^n \int_0^\infty \frac{e^{-(x+a)t} t^{n-1}}{1 - e^{-t}} g(t) dt, \tag{2.5} \end{aligned}$$

where

$$g(t) := r \left[(a-1)(1 - e^{-t}) + 1 - e^{-(a-1)t} \right].$$

Direct computations show that

$$g'(t) = r \left[(a-1)e^{-t}(1 - e^{at}) \right].$$

It is easy to see that the function $g(t)$ is strictly increasing in $(0, \infty)$ and $\lim_{t \rightarrow 0^+} g(t) = 0$. Consequently, if $0 < a < 1$ and $r > 0$ then $g(t)$ is a positive in $(0, \infty)$. In view of (2.5), we have

$$(-1)^n [\ln h_{a,r}(x)]^{(n)} > 0$$

for $x \in (0, \infty)$.

Moreover, using the differences equation $\Gamma(x+1) = x\Gamma(x)$ and taking limit directly gives

$$\lim_{x \rightarrow 0^+} h_{a,r}(x) = \lim_{x \rightarrow 0^+} \left[\frac{(x+a)^{1-a} \Gamma(x+a)}{\Gamma(x+1)} \right]^r = \left[\frac{\Gamma(a+1)}{a^a} \right]^r.$$

It is well known that, as $x \rightarrow \infty$, the following asymptotic formula holds:

$$x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x}\right) \quad (\text{see [1], p. 257}), \tag{2.6}$$

where a and b are two constant. Using the asymptotic expansion (2.6) yields

$$\begin{aligned} \lim_{x \rightarrow \infty} h_{a,r}(x) &= \lim_{x \rightarrow \infty} \left[\frac{(x+a)^{1-a} \Gamma(x+a)}{\Gamma(x+1)} \right]^r \\ &= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{a}{x}\right)^{-a} \left(1 + \frac{a(a+1)}{2x} + O\left(\frac{1}{x}\right)\right) \right]^r \\ &= 1 \end{aligned}$$

for $a > 0$. This completes the proof. □

Corollary 2.1. *If $a > 1$ and $r < 0$ then $h_{a,r}(x)$ is strictly logarithmically completely monotonic functions on $(0, \infty)$.*

Theorem 2.2. *If $a > 1, r > 0$ and $b \in [0, \infty)$ then the functions $f_{a,b,r}(x)$ defined by (1.11) is strictly logarithmically complete monotonic function on $(0, \infty)$. Moreover,*

$$\lim_{x \rightarrow 0^+} f_{a,b,r}(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} f_{a,b,r}(x) = e^{br}.$$

Proof. Suppose that $a > 1, r > 0$ and $b \in (0, \infty)$. It is clear that

$$\ln f_{a,b,r}(x) = r \left[\ln \Gamma(x+a) - \ln \Gamma(x) - a \ln x + (x+b) \ln \left(1 + \frac{b}{x}\right) \right] \tag{2.7}$$

and

$$[\ln f_{a,b,r}(x)]' = r \left[\psi(x+a) - \psi(x) - \frac{a}{x} + \ln(x+b) - \ln x - \frac{b}{x} \right]. \tag{2.8}$$

Using (2.3) and (2.4), we have

$$\begin{aligned} (-1)^n [\ln f_{a,b,r}(x)]^{(n)} &= (-1)^n r \left[\psi^{(n-1)}(x+a) - \psi^{(n-1)}(x) + \frac{(-1)^n (n-1)! a}{x^n} \right. \\ &\quad \left. + \frac{(-1)^{n-2} (n-2)!}{(x+b)^{n-1}} - \frac{(-1)^{n-2} (n-2)!}{x^{n-1}} - \frac{(-1)^{n-1} b (n-1)!}{x^n} \right] \\ &= r \left[\int_0^\infty \frac{e^{-(x+a)t} t^{n-1}}{1-e^{-t}} dt - \int_0^\infty \frac{e^{-xt} t^{n-1}}{1-e^{-t}} dt + \int_0^\infty a e^{-xt} t^{n-1} dt \right. \\ &\quad \left. + \int_0^\infty t^{n-2} e^{-(x+b)t} dt - \int_0^\infty t^{n-2} e^{-xt} dt + b \int_0^\infty t^{n-1} e^{-xt} dt \right] \\ &\triangleq \int_0^\infty \frac{t^{n-1}}{1-e^{-t}} s(t) e^{-xt} dt + \int_0^\infty t^{n-2} e^{-(x+b)t} g(t) dt, \tag{2.9} \end{aligned}$$

where

$$s(t) := r [e^{-at} - 1 + a(1 - e^{-t})]$$

and

$$g(t) := r \left[bte^{bt} - e^{bt} + 1 \right].$$

It is clear that $s'(t) = ra [1 - e^{(1-a)t}] e^{-t}$ and $g'(t) = rb^2te^{bt}$. Consequently, if $a > 1, r > 0$ and $b > 0$ then $s'(t)$ and $g'(t)$ are positive on $(0, \infty)$, i.e., $s(t)$ and $g(t)$ are strictly increasing in $(0, \infty)$. It is easy to see that $\lim_{t \rightarrow 0^+} s(t) = 0$ and $\lim_{t \rightarrow 0^+} g(t) = 0$, then we have $s(t) > 0$ and $g(t) > 0$ for $t \in (0, \infty)$.

In view of (2.9), we have

$$(-1)^n [\ln f_{a,b,r}(x)]^{(n)} > 0$$

for $x \in (0, \infty)$.

In the case $b = 0$, we consider

$$f_{a,0,r}(x) = \left[\frac{\Gamma(x+a)}{x^a \Gamma(x)} \right]^r. \quad (2.10)$$

By the same above process, we can show that

$$(-1)^n [\ln f_{a,0,r}(x)]^{(n)} > 0$$

for $x \in (0, \infty)$.

Moreover, from

$$f_{a,b,r}(x) = \left[\frac{\Gamma(x+a)}{x^a \Gamma(x)} \left(1 + \frac{b}{x} \right)^{x+b} \right]^r,$$

it follows that

$$\lim_{x \rightarrow 0^+} f_{a,b,r}(x) = \begin{cases} 0, & 0 < a < 1; \\ \infty, & a > 1. \end{cases}$$

And applying (2.6) reveals

$$\frac{\Gamma(x+a)}{x^a \Gamma(x+b)} = 1 + \frac{a(a-1)}{2x} + O\left(\frac{1}{x}\right),$$

we obtain $f_{a,b,r}(x) \rightarrow e^{br}$ as $x \rightarrow \infty$. This completes the proof. \square

Corollary 2.2. *If $0 < a < 1, r < 0$ and $b \in [0, \infty)$ then $f_{a,b,r}(x)$ is strictly logarithmically completely monotonic function on $(0, \infty)$.*

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