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HEAVY TRAFFIC ANALYSIS FOR THE SOJOURN TIME PROCESS IN MULTIPHASE QUEUEING SYSTEMS

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Abstract: The object of this research in the queueing theory is functional limit theorems under the various conditions of heavy traffic in multiphase queueing systems (MQS). In this paper, limit theorems are proved for values of important probabilistic characteristics of the queueing system investigated as well as sojourn time of a customer. Also, we prove a lemma that sojourn time of a customer can be approximated by some novel recurrent functional.

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1. Introduction

In this paper, limit theorems are considered by investigating the sojourn time of a customer in MQS. The MQS is a queueing system when a customer does not visit same queueing node twice (see, for example, [9]). Therefore, such a system is a special case of the open Jackson network.

The works on sojourn time for the MQS and open Jackson networks in heavy traffic are sparse. Papers [2], [3], [8] and [9] investigated the distribution of sojourn times in MQS with identical service times. [4] and [13] presented the proof of an expression for the stationary distribution of the diffusion approximation for sojourn times in open Jackson networks. Papers [14] and [15] investigated the sojourn time distribution in networks of priority queues. In [5] and [7], applying the method of strong approximations, sojourn time processes

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in open and multiclass feedforward queueing networks are investigated. Papers [14] and [6] deal with the simulation of sojourn time distribution in open queueing networks.

Let the sojourn time of a customer in the phases of a queueing system be unrestricted, the principle of service being "first come, first served". All the random variables studied are defined on one basic probability space $(\Omega, \mathfrak{F}, \mathbf{P})$.

We present some definitions in the theory of weak convergence of probability measures (see, for example, [1]).

Let $\mathbf{S}$ be a metric space. Consider probability measures defined on a class Φ of Borel sets of space $\mathbf{S}$. If the probability measures $\mathbf{P}_n$ and $\mathbf{P}$ satisfy the relation $\int_{\mathbf{S}} f d\mathbf{P}_n \Rightarrow \int_{\mathbf{S}} f d\mathbf{P}$ for each bounded continuous function f on $\mathbf{S}$, we say that $\mathbf{P}_n$ weakly converges to $\mathbf{P}$ and we write $\mathbf{P}_n \Rightarrow \mathbf{P}$. Let $\mathbf{X}$ map the probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ into a metric space $\mathbf{S}$. If $\mathbf{X}$ is measurable (in the sense that $\mathbf{X}^{-1}\Phi \subset \mathfrak{F}$), then we call $\mathbf{X}$ a random element. The distribution of random element $\mathbf{X}$ is a probability measure $\mathbf{P_X} = P \circ \mathbf{X}^{-1}$ on $(\mathbf{S}, \Phi)$,

$$\mathbf{P_X}(A) = P(\omega; \mathbf{X}(\omega) \in A).$$

We say that a sequence $\mathbf{X}_n$ of random elements converges in distribution to the random element $\mathbf{X}$ and write $\mathbf{X}_n \Rightarrow \mathbf{X}$ if the distribution $\mathbf{P}_n$ of elements $\mathbf{X}_n$ weakly converges to the distribution $\mathbf{P}$ of the element $\mathbf{X}$. Also, let $\mathbf{C}$ be a metric space consisting of real continuous functions in [0, 1] with a uniform metric

$$\rho(x, y) = \sup_{0 \le t \le 1} |x(t) - y(t)|, \quad x, y \in \mathbf{C}.$$

Let $\mathbf{D}$ be a space of all real-valued right-continuous functions in [0,1] having left limits and endowed with the Skorokhod topology induced by the metric d (under which $\mathbf{D}$ is complete and separable).

In this paper we will constantly use Theorem 4.1 from the book [1].

Theorem 1.1. *If* $\mathbf{X}_n \Rightarrow \mathbf{X}$ *and* $\rho(\mathbf{X}_n, \mathbf{Y}_n) \Rightarrow 0$, *then* $\mathbf{Y}_n \Rightarrow \mathbf{X}$.

In this paper, the limit theorem for the sojourn time of MQS in conditions of heavy traffic is proved. The main tool for the analysis of MQS in heavy traffic is a functional limit theorem for partial sums of independent identically distributed random variables (the proof can be found in [1]).

2. Statement of the Problem

We investigate here a k-phase queue (i.e., after a customer has been served in the j-th phase of the queue, he goes to the $(j+1)$-st phase of the queue, and, after the customer has been served in the k-th phase of the queue, he leaves the queue). Let us denote by t_n the time of arrival of the n-th customer; by $S_n^{(j)}$– the service time of the n-th customer in the j-th phase; $z_n = t_{n+1} - t_n$; by $\tau_{j,n+j}$ - departure of the n-th customer from the j-th phase of the queue, $j = 1, 2, \cdots, k$. Let interarrival times (z_n) to the MQS and service times $(S_n^{(j)})$ in each phase of the queue for $j = 1, 2, \cdots, k$ be mutually independent identically distributed random variables.

Next, denote by $W_n^{(j)}$ the waiting time of the n-th customer in the j-th phase of the queue; $T_{j,n} = \sum_{i=1}^{j}(W_n^{(i)} + S_n^{(i)})$ stands for the sojourn time of the n-th customer (time, which the n-th customer spent in the queueing system until the j-th phase), $j = 1, 2, \ldots, k$.

Suppose that the sojourn time of a customer until each phase of the MQS is unlimited, the service principle of customers is "first come, first served".

Let us define $\delta_{j,n+1} = S_{n-(j-1)}^{(j)} - z_n$, $S_{j,n} = \sum_{i=1}^{n-1} \delta_{j,i}$, $S_{0,n} \equiv 0$, $\hat{S}_{j,n} = S_{j-1,n} - S_{j,n}$, $x_{j,n} = \tau_{j,n} - t_n$, $x_{0,n} \equiv 0$, $\hat{x}_{j,n+1} = x_{j,n} - \delta_{j,n+1}$, $\hat{x}_{0,n} \equiv 0$, $y_{j,n} = \hat{x}_{j,n+(j-2)} - \hat{x}_{j-1,n+(j-2)}$, $\hat{y}_{j,n} = \hat{x}_{j,n} - \hat{x}_{j-1,n}$, $\hat{\delta}_{j,n} = \delta_{j,n+(j-2)} - \delta_{j-1,n+(j-2)}$, $\alpha_j = M\delta_{j,1}$, $\alpha_0 \equiv 0$, $S_n^{(0)} = z_n$, $j = 1, 2, \ldots, k$, $\hat{\delta}_n = \max\limits_{1\le j\le k}\max\limits_{0\le l\le 2n} |\delta_{j,l}|$, $[x]$ as the integer part of number x.

As in [10] and [11] we consider a sequence of MQS: $\delta_{j,m}^n$ are identically distributed in the n-th system, $j = 1, 2, \ldots, k$, $m \ge 1$, $n \ge 1$.

We denote $G_{j,n}(x) = P(S_{n,m}^{(j)} \le x)$, $j = 0, 1, 2, \ldots, k$. Let

$$DS_{n,1}^{(j)} \to B_j^2 > 0 \tag{1}$$

and $S_{n,m}^{(j)}$ satisfy Lindeberg conditions:

for every $\varepsilon > 0$

$$\lim_{n\to\infty} \int\limits_{|x|>\varepsilon\cdot\sqrt{n}} x^2 dG_{j,n}(x) = 0, \quad j = 1, 2, \ldots, k. \tag{2}$$

Let $\beta_j^2 = B_0^2 + \beta_j^2 > 0$, $j = 1, 2, \ldots, k$. For simplicity, in what follows we shall omit the index n.

It was proved in [10] that

$$\hat{x}_{j,n} = \max_{0\le l\le n}(\hat{x}_{j-1,l} - S_{j,l}) + S_{j,n}, \quad \hat{x}_{0,n} \equiv 0,$$

$$x_{j,n} = \max(x_{j-1,n-1} + \delta_{j,n};\ x_{j,n-1} + \delta_{j,n}), \quad x_{0,n} \equiv 0,$$

$$x_{j,n+1} = \max_{0\le l_1 < l_2 < \ldots < l_j \le n}\left(\sum_{m=l_1}^{l_2-1}\delta_{1,m} + \sum_{m=l_2}^{l_3-1}\delta_{2,m} + \cdots + \sum_{m=l_j}^{n}\delta_{j,m}\right), \tag{3}$$

$$x_{j,n} = \max_{0\le l\le n-1}(x_{j-1,l} - S_{j,l}) + S_{j,n-1}, \quad j = 1,2,\ldots,k.$$

Lemma 2.1. *If service and arrival times in the MQS satisfy Lindeberg conditions, then the sojourn time of a customer in the MQS can be approximated by a recurrent functional, consisting of sums of service and arrival times in the MQS.*

Proof. It suffices to prove that for $\varepsilon > 0$ (see [12])

$$\lim_{n\to\infty} P\left(\frac{\sup_{0\le t\le 1}|T_{j,[nt]} - \hat{x}_{j,[nt]}|}{\sqrt{n}} > \varepsilon\right) = 0,\ j = 1,2,\ldots,k. \tag{4}$$

Let us investigate case of heavy traffic, where

$$(\alpha_j - \alpha_{j-1})\cdot\sqrt{n} \to +\infty,\ \ \alpha_0 \equiv 0,\ \ j = 1,2,\ldots,k. \tag{5}$$

We prove such a theorem.

Theorem 2.1. *If conditions (1), (2) and (5) are satisfied, then*

$$\left(\frac{T_{1,[nt]} - \alpha_1\cdot[nt]}{\beta_1\cdot\sqrt{n}}; \frac{T_{2,[nt]} - \alpha_2\cdot[nt]}{\beta_2\cdot\sqrt{n}}; \ldots; \frac{T_{k,[nt]} - \alpha_k\cdot[nt]}{\beta_k\cdot\sqrt{n}}\right) \Rightarrow (z_1(t), z_2(t), \ldots, z_k(t)),$$

where $z_j(t),\ \ j = 1,2,\ldots,k,\ \ 0 \le t \le 1$ *are independent standard Wiener processes.*

Proof. First let us estimate $\max_{0\le l\le n}(\hat{x}_{j-1,l} - S_{j,l}),\ \ j = 1,2,\ldots,k,\ \ \hat{x}_{0,n} \equiv 0.$

We obtain that

$$\begin{aligned}\max_{0\le l\le n}(\hat{x}_{j-1,n} - S_{j,l}) &= \max_{0\le l\le n}(\hat{S}_{j,l} + \max_{0\le m\le l}(\hat{x}_{j-2,m} - S_{j-1,m}))\\ &\le \max_{0\le l\le n}\hat{S}_{j,l} + \max_{0\le l\le n}\max_{0\le m\le l}(\hat{x}_{j-2,m} - S_{j-1,m})\\ &\le \max_{0\le l\le n}\hat{S}_{j,l} + \max_{0\le l\le n}(\hat{x}_{j-2,l} - S_{j-1,l}) \le \cdots\end{aligned}$$

$$\leq \sum_{i=2}^{j} \max_{0\leq l\leq n} \hat{S}_{i,l} + \max_{0\leq l\leq n}(-S_{1,l}) = \sum_{i=1}^{j} \max_{0\leq l\leq n} \hat{S}_{i,l}, \quad j = 1, 2, \ldots, k. \quad (6)$$

From (3) we note that $x_{j,n} \geq S_{j,n}, \ \ j = 1, 2, \ldots, k$. Therefore,

$$\max_{0\leq l\leq n}(x_{j-1,l} - S_{j-1,l}) \geq \max_{0\leq l\leq n}(S_{j-1,l} - S_{j-1,l}) = \max_{0\leq l\leq n} \hat{S}_{j-1,l},$$
$$j = 1, 2, \ldots, k. \quad (7)$$

From (6) and (7) it follows

$$|\max_{0\leq l\leq n}(x_{j-1,l} - S_{j-1,l})| \leq \sum_{i=1}^{j} |\max_{0\leq l\leq n} \hat{S}_{i,l}| \leq \sum_{i=1}^{k} |\max_{0\leq l\leq n} \hat{S}_{i,l}|, \quad j = 1, 2, \ldots, k.$$

Thus, from this we obtain that

$$|\hat{x}_{j,n} - S_{j,l}| = |\max_{0\leq l\leq n}(\hat{x}_{j-1,l} - S_{j,l}| \leq \sum_{i=1}^{k} |\max_{0\leq l\leq n} \hat{S}_{i,l}|, \quad j = 1, 2, \ldots, k. \quad (8)$$

It is proved (see [1]), that, if conditions (1) and (2) are fulfilled, then

$$\frac{S_{j,[nt]} - \alpha_j \cdot [nt]}{\beta_j \cdot \sqrt{n}} \Rightarrow z_j(t), \quad (9)$$

where $z_j(t), j = 1, 2, \ldots, k, \ 0 \leq t \leq 1$ are independent standard Wiener processes.

We note (see also [12]) that, if conditions (5) are fulfilled, then

$$\frac{\max_{0\leq l\leq n} \hat{S}_{j,l}}{\sqrt{n}} \Rightarrow 0, \quad j = 1, 2, \ldots, k. \quad (10)$$

Applying the theorem on continuous mapping (see [1]) for $|\cdot|$, we also obtain that

$$\frac{|\max_{0\leq l\leq n} \hat{S}_{j,l}|}{\sqrt{n}} \Rightarrow 0, \quad j = 1, 2, \ldots, k. \quad (11)$$

Therefore, from (8) and (11) it follows

$$\mathbf{P}\left(\frac{\sup_{0\leq t\leq 1} |\hat{x}j, [nt] - S_{j,[nt]}|}{\sqrt{n}} > \varepsilon\right)$$
$$\leq \sum_{i=1}^{k} \mathbf{P}\left(\frac{\max_{0\leq m\leq n} |\max_{0\leq l\leq m} \hat{S}_{i,l}|}{\sqrt{n}} \geq \frac{\varepsilon}{k}\right) \to 0, \quad (12)$$

$j = 1, 2, \ldots, k$ (see again the theorem on continous mapping).

Finally, from (4) and (12) we can derive that

$$\frac{T_{j,[nt]} - \alpha_j \cdot [nt]}{\beta_j \cdot \sqrt{n}} \Rightarrow z_j(t), \tag{13}$$

where $z_j(t), j = 1, 2, \ldots, k, \quad 0 \le t \le 1$ are independent standard Wiener processes.

Therefore, the proof of the theorem is complete. □

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