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THE ASYMPTOTIC REPRESENTATION FOR THE BEST APPROXIMATION OF SOME CLASSES NONPERIODIC CONTINUOUS FUNCTIONS

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Abstract: In this paper we will give the asymptotic representation of the best approximation for continuous 2π periodic functions, using the module of smoothness, like as the one given in [7], for module of continuity. Then considering that result we will give the asymptotic representation of the best approximation for some classes of nonperiodic continuous functions.

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1. Introduction

In this paper we will show the asymptotic representation of the best approximation for some classes of nonperiodic functions. In [7], the class of continuous 2π periodic functions $W^r H_\omega$ is given by

$$W^r H_\omega = \left\{ f : \omega(f^{(r)}, \delta) \le \omega(\delta) \right\}, \tag{1}$$

where $\omega(\delta) \neq 0$ is any module of continuity, f' denotes the first derivative of the function f and $f^{(r)}$ denotes the r-th derivative of f and is denoted by the relation $f^{(r)} = (f^{(r-1)})'$. Let us denote by $E_n(f)$ the best approximation of

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a function f from the class of 2π periodic, continuous functions $C[0,2\pi]$, with trigonometric polynomials of degree not greater then n (see [8], p. 40), i.e.,

$$E_n(f) = \inf_{a_k,b_k} vrai\sup_x \left| f(x) - \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) \right|,$$

where $T_n(x) = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) \in \Pi_n(x)$ and $\Pi_n(x)$ denotes the class of trigonometric polynomials of degree n. In $W^r H_\omega$, the best approximation is defined by

$$E_n(W^r H_\omega) = \sup_{f \in W^r H_\omega} E_n(f).$$

In the case $\omega(t) = t$ the following result appears in [4].

Theorem 1.1. (see [4]) *Let $\omega(t) = t$, module of continuity, then there exists a function $f \in W^r H_t$, $r = 0, 1, \ldots,$ such that*

$$\overline{\lim_{n\to\infty}} \frac{E_n(f)}{E_n(W^r H_t)} = 1.$$

In [7], it was shown that the above theorem holds for any module of continuity $\omega(t) \neq 0$.

Theorem 1.2. (see [7]) *Let $\omega(t) \neq 0$ be any module of continuity, then there is a function $f \in W^r H_\omega$, $r = 0, 1, \ldots,$ with the property that*

$$\overline{\lim_{n\to\infty}} \frac{E_n(f)}{E_n(W^r H_\omega)} = 1$$

Remark 1.3. The above Theorems remain true and they are “stronger” then their first versions, if the module of continuity is a convex function, then instead of superior limits we can use random limits (see [5], [6]).

Lemma 1.4. (see [7]) *Let χ_n be a positive number and $f(x)$ function such that*

$$f^{(r)}(x) = C + {f_n}^{(r)}(x + x_0(\chi_n)),$$

for $|x| \leq 2\delta_m$ and $|f(x)| \leq C_1$, where C and C_1 are absolute constants, δ_m positive number. Then the following inequality is true

$$E_{n-m(p+1)}(f) \geq (1 - C_r \cdot \chi_n) E_n(f_n) + O\left({\chi_n}^{-1} \cdot M_{m,p}\left(\frac{\delta_m}{r+1}\right)\right),$$

where C_r depends only from r and $M_{m,p}\left(\frac{\delta_m}{r+1}\right) = \max_{|x|>\delta} |U_{m,p}(x)|$, $U_{m,p}(x) = A_{m,p} \cdot \frac{K_m(x)^{p+1}}{m^p}$, $K_m(x) = \frac{2}{m} \cdot \left(\frac{\sin(\frac{mx}{2})}{2\sin(\frac{x}{2})}\right)^2$, $\frac{1}{\pi} \cdot \int_{-\pi}^{\pi} U_{m,p}(x)dx = 1$.

2. Results

In the sequel we will show the asymptotic representation for some classes of nonperiodic, continuous functions defined in $[-1,1]$. Let us denote by $\Omega_k(f,\delta)$ module of smoothness of order k by relation

$$\Omega_k(f,\delta) = \sup_{x\in[a,b],x+kh\in[a,b],|h|<\delta} |\Delta_h{}^k f(x)|,$$

$$|\Delta_h{}^k f(x)| = \sum_{v=0}^{k} (-1)^{k-v} \binom{k}{v} f(x+vh).$$

By $\widehat{W}^r H$, we will denote the class of 2π periodic, continuous functions such that

$$\widehat{W}^r H = \{f : \Omega_k(f^{(r)},\delta) \le a_1 \cdot \Omega_k(\delta)\}, \tag{2}$$

where $\Omega_k(\delta) \neq 0$ is any module of smoothness, of order k. $f^{(r)}$ denotes the derivate of order r and a_1 is a constant which does not depend from f. In $\widehat{W}^r H$, the best approximation is defined by

$$E_n(\widehat{W}^r H) = \sup_{f\in\widehat{W}^r H} E_n(f),$$

where $E_n(f)$, is defined as above. Then we have the following

Theorem 2.1. *Let $\Omega_k(t) \neq 0, k \in \mathbb{N}$ be module of smoothness of order k. Then there is a function $f \in \widehat{W}^r H$, $r = 0,1,\ldots,$ such that*

$$\overline{\lim_{n\to\infty}} \frac{E_n(f)}{E_n(\widehat{W}^r H)} = 1.$$

Proof. Let $\Omega_k(t)$ be module of smoothness of order k, then from $\Omega_k(t) \neq 0$ it follows that the module of continuity $\Omega_1(t)$ is different from 0 (which is obtained from the module of smoothness for $k = 1$). Then, from Theorem 1.2, there exists a function $f \in W^r H_\omega$, such that

$$\Omega_1(f^{(r)},t) \le \Omega_1(t) \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{E_n(f)}{E_n(W^r H_\omega)} = 1. \tag{3}$$

On the other hand it is well-known that (see [8])

$$\Omega_k\left(f^{(r)},t\right) \le 2^{k-1}\Omega_1\left(f^{(r)},t\right). \tag{4}$$

From relations (3) and (4) it follows that

$$\Omega_k\left(f^{(r)},t\right) \le 2^{k-1}\Omega_1\left(f^{(r)},t\right) \le 2^{k-1}\Omega_1(t). \tag{5}$$

Using Marchaud Theorem we will have the following relation (see [8])

$$\Omega_k\left(f,t\right) \leq c_k \cdot t^k \cdot \left\{ \int_t^c \frac{\Omega_{k+1}(f,u)}{u^{k+1}} du + \sup_x |f(x)| \right\}, \tag{6}$$

for t small enough, where c_k and c are positive constants not depending from f. Relation (6) implies

$$\Omega_1(f,t^s) \leq K \cdot \Omega_s(f,t), \tag{7}$$

for every $1 < s$ and for some constant K (see [8]). Finally from relations (5) and (7) it follows that

$$\Omega_k\left(f^{(r)},t\right) \leq 2^{k-1}\Omega_1\left(t\right) \leq 2^{k-1} \cdot K \cdot \Omega_k(t^{\frac{1}{k}}),$$

where $k > 1$. Now we divide the following cases:

1) If $t \geq 1$ then we will have the following estimation

$$\Omega_k\left(f^{(r)},t\right) \leq 2^{k-1} \cdot K \cdot \Omega_k(t^{\frac{1}{k}}) \leq 2^{k-1} \cdot K \cdot \Omega_k(t).$$

Note that the second inequality holds (because the module of smoothness $\Omega_k(t)$ is a nondecreasing function of t (see [2]) and for $t \geq 1$ we have $t \geq t^{\frac{1}{k}}$).

2) If $0 < t < 1$ then there exists a $\lambda \in \mathbb{R}, \lambda > 1$ such that $t^{\frac{1}{k}} \leq \lambda \cdot t$ and the following is true

$$\Omega_k\left(f^{(r)},t\right) \leq 2^{k-1} \cdot K \cdot \Omega_k(t^{\frac{1}{k}}) \leq 2^{k-1} \cdot K \cdot \Omega_k(\lambda \cdot t) \leq 2^{k-1} \cdot K \cdot B \cdot \lambda^k \cdot \Omega_k(t),$$

for some constant B (see [2]).

Let us denote by $a_1 = \max\{2^{k-1} \cdot K, 2^{k-1} \cdot K \cdot B \cdot \lambda^k\}$, then from relations (1) and (2) it follows that

$$\Omega_k\left(f^{(r)},t\right) \leq a_1 \cdot \Omega_k(t), \tag{A}$$

and from last relation it follows that $f \in \widehat{W}^r H$. From definition of the class $\widehat{W}^r H$, there exists a function $f_n \in \widehat{W}^r H$, such that

$$E_n(f_n) \geq \left(1 - \frac{1}{n}\right) E_n\left(\widehat{W}^r H\right), \quad \forall n \in \mathbb{N}.$$

For $2^{k-1} \leq n < 2^k$, let us denote by $m_n = [2^{\frac{2k}{3}}]$. The following relation

$$\overline{\lim_{n\to\infty}} \frac{E_n(\widehat{W}^r H)}{E_{n-(p+1)m_n}(\widehat{W}^r H)} = 1, \tag{8}$$

holds, for all natural numbers p (the proof is similar like to that of Theorem 1 in [7]). Under conditions of Lemma 1.4, for $n = n_k, \quad m = m_k = m_{n_k}, \quad \delta_m = k^{-2}, \quad \chi_n \equiv \chi_k \equiv \chi_{n_k} = k^{-\frac{1}{2}}, \quad p = 2r+1$ we will have estimation

$E_{n_k - m_k(p+1)}(f)$

$$\geq \left(1+O(k^{-\frac{1}{2}})\right) E_{n_k}\left(\widehat{W}^r H\right) + O\left(k^{\frac{1}{2}} \cdot M_{m_k,2r+1}\left(\frac{1}{k^2(r+1)}\right)\right).$$

Further more

$$M_{m_k,2r+1}\left(\frac{1}{k^2(r+1)}\right) = \frac{1}{{n_k}^{r+1}} \cdot O\left(\frac{1}{k}\right)$$

and

$$E_{n_k-m_k(p+1)}(f) \geq \left(1+O(k^{-\frac{1}{2}})\right) E_{n_k}\left(\widehat{W}^r H\right).$$

From the above relations it follows

$$\begin{aligned} \overline{\lim_{n\to\infty}} \frac{E_n(f)}{E_n(\widehat{W}^r H)} &\geq \overline{\lim_{k\to\infty}} \frac{E_{n_k-(p+1)m_k}(f)}{E_{n_k-(p+1)m_k}(\widehat{W}^r H)} \\ &\geq \lim_{k\to\infty} (1+O(k^{-\frac{1}{2}})) \frac{E_{n_k}(\widehat{W}^r H)}{E_{n_k-(p+1)m_k}(\widehat{W}^r H)} = 1. \end{aligned} \tag{9}$$

On the other hand, from the definition of $E_n(\widehat{W}^r H)$, it follows that

$$\overline{\lim_{n\to\infty}} \frac{E_n(f)}{E_n(\widehat{W}^r H)} \leq 1. \tag{10}$$

Now proof of the theorem follows from relations (9) and (10). □

Let ${W_\phi}^r H$ be the class of nonperiodic continuous functions $f \in C[-1,1]$ such that

$${W_\phi}^r H = \{f : {\Omega_\phi}^r(D^r f, \delta) \leq a_1 \cdot \Omega_r(\delta)\}, \tag{11}$$

where $\Omega_r(\delta) \neq 0$ is any module of smoothness of order r. $Df = f'\sqrt{1-x^2}$, $D^r f = \underbrace{Df \cdots Df}_{\text{r}-times}$, a_1 is a constant which does not depend from function f and ${\Omega_\phi}^r(f,\delta)$ is the module of smoothness defined as follows (see [3]):

$${\Omega_\phi}^r(f,\delta) = \sup_{|h|<\delta} ||{\Delta_h}^r f||.$$

Here $(\Delta_h f)(x) = f(x \oplus h) - f(x)$, $x \oplus h = x\sqrt{1-h^2} + h\sqrt{1-x^2}$, ${\Delta_h}^r = \underbrace{\Delta_h \cdots \Delta_h}_{r-\text{times}}$, $x, h \in [-1,1]$. Let ${E^*}_n(f)$ be the best approximation of the function f with algebraic polynomials in $[-1,1]$. Further, let

$${E^*}_n({W_\phi}^r H) = \sup_{f\in {W_\phi}^r H} {E_n}^*(f).$$

Proposition 2.2. *Let $\Omega_{2r}(\delta) \neq 0$ be a module of smoothness, $r = 0, 1, \ldots,$ then there is a function $f \in W_\phi^{2r} H$, such that*

$$\overline{\lim_{n\to\infty}} \frac{{E_n}^*(f)}{{E^*}_n({W_\phi}^{2r} H)} = 1.$$

Proof. Let f be any function from $C[-1,1]$. If we use the substitution $x = \cos t$ then we have $f(x) = f(\cos t) = g(t)$, $g(t)$ is a trigonometric 2π periodic function and

$$(Df)(x) = (Df)(\cos t) = (Dg)(t) = g^{'}(t)\sqrt{1-\cos^2 t} = g^{'}(t)\sin t$$
$$= f^{'}(\cos t)\sin t = -(f(\cos t))^{'} = -(g(t))^{'}.$$

Proceeding in this way, we observe that

$$(D^r f)(\cos t) = (-1)^k (f\cos t)^{(k)} = (-1)^k (g(t))^{(k)},$$

which means that

$$D^{2r} f(x) = (g(t))^{(2r)}. \tag{12}$$

Let us denote by $\Omega_{2r}(g^{(2r)},\delta) = {\Omega_\phi}^{2r}(D^{2r}f,\delta)$. Then from relation (12) it follows that

$${W_\phi}^{2r}H = \{f : {\Omega_\phi}^{2r}(D^{2r}f,\delta) \le a_1 \cdot \Omega_{2r}(\delta)\} = \{g : \Omega_{2r}(g^{(2r)},\delta) \le a_1 \cdot \Omega_{2r}(\delta)\}.$$

Now proof of the proposition follows from Theorem 2.1 and the last relation. □

Remark 2.3. Proposition 2.2 is still valid if we use the module ${\Omega_\phi}^r(\delta) \neq 0$ instead of ${\Omega_\phi}^{2r}(\delta) \neq 0$, under condition that f and $D^r f$ are positive for any $r \in \mathbb{N}$.

In the following we will show some other classes of nonperiodic, continuous functions for which we have the same asymptotic relation as the one given in Proposition 2.2. Let ${W_T}^r H$ denote the class of nonperiodic functions $f \in C[-1,1]$ such that

$${W_T}^r H = \{f : {\Omega_T}^r({D_T}^r f,\delta) \le a_1 \cdot \Omega_r(\delta)\},$$

where $\Omega_r(\delta) \neq 0$ is any module of smoothness, a_1 is a constant which does not depend from function f.

$$(D_T f)(x) = (1-x^2)f^{''}(x) - xf^{'}(x)$$

and ${\Omega_T}^r({D_T}^r f,\delta)$ is the module of smoothness defined as in [1]. Let us denote by

$${E^*}_n({W_T}^r H) = \sup_{f\in {W_T}^r H} {E_n}^*(f).$$

Proposition 2.4. *Let $\Omega_{2r}(\delta) \neq 0$ be a module of smoothness, $r = 0,1,\dots,$ then there is a function $f \in {W_T}^{2r}H$ such that*

$$\overline{\lim_{n\to\infty}} \frac{{E_n}^*(f)}{{E^*}_n({W_T}^{2r}H)} = 1.$$

Proof. From definition of $D_T f(x)$ we will have that

$$D_T{}^r f(x) = D^{2r} f(x). \tag{13}$$

Let us denote by

$$\Omega_T{}^r(D_T{}^r f, \delta) = \Omega_{2r}(D^{2r} f, \delta),$$

then

$$W_T{}^r H = \{f : \Omega_T{}^r(D_T{}^r f, \delta) \leq a_1 \cdot \Omega_r(\delta)\} = \{f : \Omega_{2r}(D^{2r} f, \delta) \leq a_1 \cdot \Omega_r(\delta)\}$$

On the other hand it is known that

$$\Omega_k(t) \leq K_1 \cdot \Omega_{2k}(t^{\frac{1}{k}}), \tag{B}$$

for some constant K_1. From this it follows that the following relation holds:

$$W_T{}^r H = \{f : \Omega_T{}^r(D_T{}^r f, \delta) \leq a_1 \cdot \Omega_r(\delta)\} = \{f : \Omega_{2r}(D^{2r} f, \delta) \leq a_1 \cdot \Omega_r(\delta)\}$$
$$= \{f : \Omega_{2r}(D^{2r} f, \delta) \leq a_2 \cdot \Omega_{2r}(\delta)\},$$

(the last relation we can prove like to that given by relation (A) in Theorem 2.1, taking into consideration relation (B)) where $a_2 = a_1 \cdot K_1$ is a constant which does not depend from function f. Now proof of proposition follows from Proposition 2.2 □

Now we denote by $\widetilde{W}^1 H_\omega$ the class of nonperiodic continuous functions from $C[-1, 1]$ such that

$$\widetilde{W}^1 H_\omega = \{f : \Omega_\phi{}^1(f' \phi, \delta)_\infty \leq a_1 \cdot \omega(\delta)\},$$

where $\omega(\delta) \neq 0$ is any module of continuity, f and $f'\phi$ are positive, and

$$\Omega_\phi{}^r(f, t)_\infty = \sup_{|h| \leq \delta} \left\| \overline{\Delta}_{\phi h}{}^r f \right\|_\infty.$$

$\phi = \sqrt{1 - x^2}$, a_1 is a constant which does not depend from function f (see [3]). Let $E^*{}_n(f)$ be the best approximation of the function f (from the class of nonperiodic continuous functions) with algebraic polynomials in $[-1, 1]$. Further, let

$$E^*{}_n(\widetilde{W}^1 H_\omega) = \sup_{f \in \widetilde{W}^1 H_\omega} E_n{}^*(f).$$

Proposition 2.5. *Let $\omega(\delta) \neq 0$ be the module of continuity, then there is a function $f \in \widetilde{W}^1 H_\omega$ such that*

$$\overline{\lim_{n \to \infty}} \frac{E_n{}^*(f)}{E_n{}^*(\widetilde{W}^1 H_\omega)} = 1.$$

Proof. For $r = 1$ we have $D^1 f = f'\phi$. Let us denote by $\Omega_\phi{}^1(f'\phi, \delta)_\infty =$

${\Omega_\phi}^1(D^1 f, \delta)$, then it follows that:

$$\widetilde{W}^1 H_\omega = \{f : {\Omega_\phi}^1(f^{'}\phi, \delta)_\infty \le a_1 \cdot \omega(\delta)\} = \{f : {\Omega_\phi}^1(D^1 f, \delta) \le a_1 \cdot \omega(\delta)\}.$$

Now the proof of the proposition follows from Proposition 2.2 and Remark 2.3 □

References

[1] P.L. Butzer, R.L. Stens, Chebyshev transform methods in the theory of best algebraic approximation, *Math. Seminar Hamburg*, **45** (1976), 165-190.

[2] Z. Ditzian, V. Totik, *Moduli of Smoothness*, Springer-Varlag, New York (1987).

[3] M. Felten, Characterization of best algebraic approximation by an algebraic modulus of smoothness, *J. Approx. Theory*, **89** (1997), 1-25.

[4] S.M. Nikolskii, Ob interpolirovani i nailuqshem priblizhenie deferenciruemih periodiceskih funkcii trigonometriceskim polinomami, *An SSSR Seria Mat.*, Moskva (1946), 393-410, In Russian.

[5] V.N. Temljakov, Ob asimtoticeskom povedenie nailuqshih priblizhenie neprerivnih funkci, *DAN SSSR*, **228** (1976), 318-321, In Russian.

[6] V.N. Temljakov, Asimtoticeskoj povedenie nailuqshih priblizhenie neprerivnih funkci, *An SSSR Seria Matem.*, Moskva, **41** (1977), 587-606, In Russian.

[7] V.N. Temljakov, K voprosu ob asimtoticeskom povedenie nailuqshih priblizhenie neprerivnih funkcii, *Mat. Sbornik*, Moskva, **110** (1979), 399-413, In Russian.

[8] A.F. Timan, *Teorija Priblizhenii Funkcii Deistvelnogo Peremenogo*, Moskva (1960), In Russian.