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ON NORM OF THE SUM OF OPERATORS EQUAL TO THE SUM OF THEIR NORMS

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Abstract: In this paper we first characterize the norm of the sum of m vectors in a normed linear space being equal to the sum of their norms. These results make it possible to characterize the norm of the sum of bounded linear operators in a normed linear space being equal to the sum of their norms. Many interesting results about approximate eigenvalues follows. In applications, these results are characterized in terms of numerical ranges for operators in Hilbert spaces.

Dedicated to Professor Robert M. MacGregor,
an esteemed colleague and friend.

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1. Introduction

Few years ago we generalized Daugavet equations on a uniformly convex Banach space, see [6]. It seems natural to consider and study the norm of the sum of bounded linear operators and the sum of their norms on a normed linear space, and on a Hilbert space in particular. This is precisely the purpose of this article

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which is organized as follows: In Section 2 we generalize and characterize in Theorem 2.1 the norm of the sum of two vectors in a normed space (as well as in Theorem 2.2 and Corollary 2.3 for a strictly convex space) being equal to the sum of their norms. Similar techniques and results are carried over in Section 3 to bounded linear operators in a normed space, and it is characterized in Theorem 3.2 and 3.3 in terms of approximate eigenvalue if the space is uniformly convex. For operators in Hilbert spaces in Section 4 many related results are obtained in Theorem 4.1 together with its consequences. Finally, the topic of this paper is also characterized in Theorem 4.4 in terms of numerical ranges.

Throughout this paper, unless otherwise stated explicitly, $(X, \|\cdot\|)$ denotes a normed linear space X with the norm $\|\cdot\|$, and $e \in X$ is a unit vector.

2. Generalizations and Characterizations of the Norm of the Sum of Two Vectors being Equal to the Sum of their Norms

In [1, Lemma 2.1] it was proved that for $x, y \in X$, $\| x+y \| = \| x \| + \| y \|$ if and only if $\| ax+by \| = a \| x \| + b \| y \|$ holds for all positive real numbers a and b. In this section various generalizations and characterizations of the result above are given in Theorem 2.1, and the theorem is an indispensable tool in the proofs of many subsequent theorems in Section 3 and Section 4. When X is strictly convex in particular, we obtained Theorem 2.2 and Corollary 2.3.

Theorem 2.1. *Let $\{x_i\}_{i=1}^m \subseteq X$. Then the following are equivalent.*

(1.1) $\| \sum_{i=1}^m x_i \| = \sum_{i=1}^m \| x_i \|$.

(1.2) $\| \sum_{i=1}^m a_i x_i \| = \sum_{i=1}^m a_i \| x_i \|$

for any sequence $\{a_i\}_{i=1}^m$ of positive real numbers.

(1.3) $\| (\sum_{i=2}^m \| x_i \|) x_1 + \| x_1 \| \sum_{i=2}^m x_i \| = 2 \| x_1 \| \sum_{i=2}^m \| x_i \|$.

(1.4) $\| \| \sum_{i=2}^m x_i \| x_1 + \| x_1 \| \sum_{i=2}^m x_i \| = \| x_1 \| [\| \sum_{i=2}^m x_i \| + \sum_{i=2}^m \| x_i \|]$.

Moreover, any one of the statements above implies

(1.5) $\| \| \sum_{i=2}^m x_i \| x_1 + \| x_1 \| \sum_{i=2}^m x_i \| = 2 \| x_1 \| \| \sum_{i=2}^m x_i \|$.

Proof. (1.1)$\Rightarrow$(1.2). We may assume without loss of generality that $a_1 \geq a_i$ for $i = 2, 3, ..., m$. Since

$$\sum_{i=1}^m a_i \| x_i \| \geq \| \sum_{i=1}^m a_i x_i \| = \| a_1 \sum_{i=1}^m x_i - \sum_{i=2}^m (a_1 - a_i) x_i \| \geq a_1 \| \sum_{i=1}^m x_i \|$$

$$- \sum_{i=2}^{m}(a_1 - a_i) \parallel x_i \parallel = a_1 \sum_{i=1}^{m} \parallel x_i \parallel - \sum_{i=2}^{m}(a_1 - a_i) \parallel x_i \parallel = \sum_{i=1}^{m} a_i \parallel x_i \parallel,$$

the second equality is due to (1.1), and the desired equality follows.

(1.2)⇒(1.3). Let $a_1 = \sum_{i=2}^{m} \parallel x_i \parallel$ and $a_i = \parallel x_1 \parallel$, $i = 2, 3, ..., m$, in (1.2).

(1.3)⇒(1.4). Rewrite the equality (1.3) as

$$\parallel (\sum_{i=2}^{m} \parallel x_i \parallel) x_1 + \parallel x_1 \parallel x_2 + \parallel x_1 \parallel x_3 + \cdots + \parallel x_1 \parallel x_m \parallel$$

$$= (\sum_{i=2}^{m} \parallel x_i \parallel) \parallel x_1 \parallel + \parallel x_1 \parallel \parallel x_2 \parallel + \cdots + \parallel x_1 \parallel \parallel x_m \parallel,$$

which is a similar form as (1.1). Hence, for a sequence $\{b_i\}_{i=1}^m$ of any positive real numbers the above equality implies that (use the similar method as in the proved of (1.1)⇒(1.2))

$$\parallel b_1(\sum_{i=2}^{m} \parallel x_i \parallel) x_1 + \parallel x_1 \parallel \sum_{i=2}^{m} b_i x_i \parallel$$

$$= b_1(\sum_{i=2}^{m} \parallel x_i \parallel) \parallel x_1 \parallel + \parallel x_1 \parallel \textstyle\sum_{i=2}^{m} b_i \parallel x_i \parallel .$$

The desired relation follows by letting $b_1 = \frac{\|\sum_{i=2}^{m} x_i\|}{\sum_{i=2}^{m} \|x_i\|}$ and $b_i = 1$, $i = 2, 3, ..., m$ in above.

(1.4)⇒(1.1) Similarly, rewrite the equality (1.4) and pick up a sequence $\{c_i\}_{i=1}^m$ of positive real numbers, then we have

$$\parallel c_1 \parallel \sum_{i=2}^{m} x_i \parallel x_1 + \parallel x_1 \parallel \sum_{i=2}^{m} c_i x_i \parallel$$

$$= c_1 \parallel \sum_{i=2}^{m} x_i \parallel \parallel x_1 \parallel + \parallel x_1 \parallel \sum_{i=2}^{m} c_i \parallel x_i \parallel .$$

Let $c_1 = \frac{1}{\|\sum_{i=2}^{m} x_i\|}$ and $c_i = \frac{1}{\|x_1\|}$ for $i = 2, 3, ..., m$ in above. Then (1.1) follows.

To prove (1.5) we see that

$$2 \parallel x_1 \parallel \parallel \sum_{i=2}^{m} x_i \parallel \geq \parallel \parallel \sum_{i=2}^{m} x_i \parallel x_1 + \parallel x_1 \parallel \sum_{i=2}^{m} x_i \parallel$$

$$= \parallel x_1 \parallel [\parallel \sum_{i=2}^{m} x_i \parallel + \sum_{i=2}^{m} \parallel x_i \parallel] \geq 2 \parallel x_1 \parallel \parallel \sum_{i=2}^{m} x_i \parallel .$$

The equality above is due to (1.4), and the proof of the theorem is completed.

Next, we require some well-known definitions. X is a uniformly convex space if for any sequences $\{x_n\}$ and $\{y_n\} \subseteq X$ satisfying $\| x_n \| \leq 1$ and $\| y_n \| \leq 1$ for all n, and $\lim_{n\to\infty} \| \frac{1}{2}(x_n + y_n) \| = 1$, we have $lim_{n\to\infty} \| x_n - y_n \| = 0$ [9; 26, p. 353]. It is well-known that an inner product space is uniformly convex, and a uniformly convex space is strictly convex. As well, a normed space is strictly convex if $\frac{1}{2} \| x + y \| = \| x \| = \| y \| = 1$ implies $y = x$. See [3] for such spaces and their applications. Incidently, it is a fact that a finite-dimensional space is uniformly convex if and only if it is strictly convex.

Theorem 2.2. *Let $\{x_i\}_{i=1}^m \subseteq X$, and consider the following three statements.*

(a) *Any equivalent statement in Theorem 1.1 (i.e., any equality (1.1) through (1.4)).*

(b) $(\sum_{i=2}^m \| x_i \|)x_1 = \| x_1 \| (\sum_{i=2}^m x_i)$.

(c) $\| \sum_{i=2}^m x_i \| x_1 = \| x_1 \| (\sum_{i=2}^m x_i)$.

Then we have

(1.6) *If X is strictly convex and* (a) *holds, then both* (b) *and* (c) *hold.*

(1.7) (b) *implies* (a) *(strict convexity of X is not assumed here).*

Proof. (1.6) Rewrite (1.3) of Theorem 2.1 as $\| \frac{x_1}{\|x_1\|} + \frac{\sum_{i=2}^m x_i}{\sum_{i=2}^m \|x_i\|} \| = 2$, and let $u = \frac{x_1}{\|x_1\|}$ and $v = \frac{\sum_{i=2}^m x_i}{\sum_{i=2}^m \|x_i\|}$. Then

$$2 = \| u + v \| \leq \| u \| + \| v \| \leq 1 + 1 = 2,$$

so that $\frac{1}{2} \| u + v \| = \| u \| = \| v \| = 1$. Hence $u = v$ as X is strictly convex, and we have (b).

Similarly, rewrite (1.5) of Theorem 2.1 as $\| \frac{x_1}{\|x_1\|} + \frac{\sum_{i=2}^m x_i}{\|\sum_{i=2}^m x_i\|} \| = 2$ and let $u = \frac{x_1}{\|x_1\|}$ as before and $w = \frac{\sum_{i=2}^m x_i}{\|\sum_{i=2}^m x_i\|}$. Then $\| u + w \| = 2$, and (c) follows likewise.

(1.7) Since

$$\begin{aligned} 2 \| x_1 \| \sum_{i=2}^m \| x_i \| &\geq \| (\sum_{i=2}^m \| x_i \|)x_1 + \| x_1 \| \sum_{i=2}^m x_i \| \\ &= \| 2(\sum_{i=2}^m \| x_i \|)x_1 - [\sum_{i=2}^m \| x_i \| x_1 - \| x_1 \| \sum_{i=2}^m x_i] \| \end{aligned}$$

$$\geq 2 \parallel x_1 \parallel \sum_{i=2}^{m} \parallel x_i \parallel -(\parallel \sum_{i=2}^{m} \parallel x_i \parallel) x_1 - \parallel x_1 \parallel (\sum_{i=2}^{m} x_i) \parallel = 2 \parallel x_1 \parallel \sum_{i=2}^{m} \parallel x_i \parallel,$$

the last equality is due to (b). Hence, we have (1.3) of Theorem 2.1, and this completes the proof of the theorem.

To conclude this section we have the next generalization and characterization if the space is strictly convex,. The proof is immediate due to Theorem 2.2 and should be omitted.

Corollary 2.3. *Let X be stricly convex and $\{x_i\}_{i=1}^m \subseteq X$. Then*

(1.8) $\parallel \sum_{i=1}^m x_i \parallel = \sum_{i=1}^m \parallel x_i \parallel$ *if and only if* $(\sum_{i=2}^m \parallel x_i \parallel) x_1 = \parallel x_1 \parallel (\sum_{i=2}^m x_i)$.

(1.9) *That* $\parallel \sum_{i=1}^m x_i \parallel = \sum_{i=1}^m \parallel x_i \parallel$ *implies* $\parallel \sum_{i=2}^m x_i \parallel x_1 = \parallel x_1 \parallel (\sum_{i=2}^m x_i) = (\sum_{i=2}^m \parallel x_i \parallel) x_1$.

3. Results for Operators in Normed Spaces

In this section let $B(X)$ denote the algebra of all bounded linear operators in X into stself, and I the identity operator. The well-known Daugavet equation is $\parallel I + T \parallel = 1 + \parallel T \parallel$ for $T \in B(X)$. A characterization in [1, Theorem 2.2] says that if X is uniformly convex Banach space, the Daugavet equation holds if and only if the norm $\parallel T \parallel$ is an approximate eigenvalue of T. The equation has been studied extensively by several authors, and the results obtained have been found useful in approximation theory and in other fields of mathematics, see [1], [4] and references therein. In this section we give various generalizations and characterizations of the Daugavet equation.

Recall that $T \in B(X)$ is an isometric isomorphism, or isometric operator, if $\parallel Tx \parallel = \parallel x \parallel$ for all x. Clearly in this case $\parallel T \parallel = 1$.

Lemma 3.1. *Let $\{T_i\}_{i=1}^m \subseteq B(X)$. Then Theorem 2.1 holds in terms of the operators T_i (instead of x_i).*

Proof. All statements and proofs should be omitted since $B(X)$ is a normed linear space.

Next, we prove, due to Lemma 3.1, that Theorem 2.2 may be naturally carried over to Theorem 3.2 below in terms of operators in $B(X)$.

Theorem 3.2. *Let $\{T_i\}_{i=1}^m \subseteq B(X)$, and consider the following three statements.*

(a) *Any one of the four equivalent statements in Theorem 2.1 in terms of* *operators* T_i *(instead of* x_i*).*

(b) *There exists a sequence* $\{e_i\}_{i=1}^{\infty} \subseteq X$ *of unit vectors such that*

$$\lim_{n\to\infty}[(\sum_{i=2}^{m} \| T_i \|)T_1 e_n - \| T_1 \| (\sum_{i=2}^{m} T_i)e_n] = 0.$$

(c) *There exists a sequence* $\{f_i\}_{i=1}^{\infty} \subseteq X$ *of unit vectors such that*

$$\lim_{n\to\infty}[\| \sum_{i=2}^{m} T_i \| T_1 f_n - \| T_1 \| (\sum_{i=2}^{m} T_i)f_n] = 0.$$

Then we have

(2.1) *If* X *is uniformly convex and* (a) *holds, then both* (b) *and* (c) *hold.*

(2.2) *In addition to the limit in* (b), *if* $\lim_{n\to\infty} \| T_1 e_n \| = \| T_1 \|$, *then* (b) *implies* (a).

(2.3) *If* T_1 *is isometric, then* (b) *implies* (a).

(Note that the uniform convexity of X is not assumed in both (2.2) and 2.3)).

Proof. (2.1) By Lemma 3.1 we may rewrite (1.3) of Theorem 2.1 in terms of operators as follows.

$$\| (\sum_{i=2}^{m} \| T_i \|)T_1 + \| T_1 \| \sum_{i=2}^{m} T_i \| = 2 \| T_1 \| \sum_{i=2}^{m} \| T_i \|,$$

or

$$\| \frac{T_1}{\| T_1 \|} + \frac{\sum_{i=2}^{m} T_i}{\sum_{i=2}^{m} \| T_i \|} \| = 2.$$

Put $U = \frac{T_1}{\|T_1\|}$ and $V = \frac{\sum_{i=2}^{m} T_i}{\sum_{i=2}^{m} \|T_i\|}$ in above, then $\frac{1}{2} \| U+V \| = \| U \| = \| V \| = 1$. So, by the definition of norm there exists a sequence $\{e_i\}_{i=1}^{\infty} \subseteq X$ of unit vectors such that $\lim_{n\to\infty} \| \frac{1}{2}(U+V)e_n \| = 0$, and clearly $\| Ue_n \| \leq 1$ and $\| Ve_n \| \leq 1$ for all n. Since X is uniformly convex we have $\lim_{n\to\infty} \| Ue_n - Ve_n \| = 0$, and hence (b) holds.

Next, the equality

$$\| \| \sum_{i=2}^{m} T_i \| T_1 + \| T_1 \| \sum_{i=2}^{m} T_i \| = 2 \| T_1 \| \| \sum_{i=2}^{m} T_i \|$$

holds by (1.5) of Theorem 2.1 in terms of operators, or

$$\| \frac{T_1}{\| T_1 \|} + \frac{\sum_{i=2}^{m} T_i}{\| \sum_{i=2}^{m} T_i \|} \| = 2.$$

From U above and put $W = \frac{\sum_{i=2}^m T_i}{\|\sum_{i=2}^m T_i\|}$, we get $\frac{1}{2} \| U+W \| = \| U \| = \| W \| = 1$; the remainder of the proof goes through as above and (c) is obtained.

(2.2) By (b) there exists a sequence $\{e_i\}_{i=1}^\infty \subseteq X$ of unit vectors, and since

$$2 \| T_1 \| \sum_{i=2}^m \| T_i \| \geq \| (\sum_{i=2}^m \| T_i \|)T_1 + \| T_1 \| \sum_{i=2}^m T_i \|$$

$$\geq \| 2(\sum_{i=2} \| T_i \|)T_1 e_n - [(\sum_{i=2}^m \| T_i \|)T_1 e_n - \| T_1 \| (\sum_{i=2}^m T_i)e_n] \| \text{ for all } n$$

$$\geq 2 \| T_1 e_n \| \sum_{i=2}^m \| T_i \| - \| (\sum_{i=2}^m \| T_i \|)T_1 e_n - \| T_1 \| (\sum_{i=2}^m T_i)e_n \| \text{ for all } n$$

$$\to 2 \| T_1 \| \sum_{i=2}^m \| T_i \| \text{ as } n \to \infty$$

by (b) and the assumption that $\lim_{n\to\infty} \| T_1 e_n \| = \| T_1 \|$. So that

$$\| (\sum_{i=2}^m \| T_i \|)T_1 + \| T_1 \| \sum_{i=2}^m T_i \| = 2 \| T_1 \| \sum_{i=2}^m \| T_i \|,$$

which is precisely (1.3) of Theorem 2.1 in terms of operators.

(2.3) If T_1 is isometric, then $\| T_1 e_n \| = \| e_n \| = 1 = \| T_1 \|$ for all n. The required result follows by the proof of (2.2) above, which yields

$$\| (\sum_{i=2}^m \| T_i \|)T_1 + \sum_{i=2}^m T_i \| = 2 \sum_{i=2}^m \| T_i \| .$$

The equality is precisely (1.3) of Theorem 2.1 in terms of operators and we have (a), and the proof of the theorem is finished.

We recall that a scalar α is said to be an approximate eigenvalue of the operator T if there exists a sequence $\{e_i\}_{i=1}^\infty \subseteq X$ of unit vectors such that $\lim_{n\to\infty} (Te_n - \alpha e_n) = 0$. The next is the main result in this section, which characterizes the topic of this paper in terms of approximate eigenvalue, and is consequences of Theorem 3.2.

Theorem 3.3. *Let X be uniformly convex and $\{T_i\}_{i=1}^m \subseteq B(X)$. Then the following assertions are valid.*

(2.4) *If $\| \sum_{i=1}^m T_i \| = \sum_{i=1}^m \| T_i \|$, then 0 is an approximate eigenvalue of both operators*

$$(\sum_{i=2}^m \| T_i \|)T_1 - \| T_1 \| (\sum_{i=2}^m T_i) \text{ and } \| \sum_{i=2}^m T_i \| T_1 - \| T_1 \| (\sum_{i=2}^m T_i).$$

(2.5) *If* T_1 *is isometric, then* $\| \sum_{i=1}^m T_i \| = \sum_{i=1}^m \| T_i \|$ *if and only if* 0 *is an approximate eigenvalue of the operator* $(\sum_{i=2}^m \| T_i \|)T_1 - (\sum_{i=2}^m T_i)$.

(2.6) $\| I + \sum_{i=2}^m T_i \| = 1 + \sum_{i=2}^m \| T_i \|$ *if and only if* 0 *is an approximate eigenvalue of the operator* $(\sum_{i=2}^m \| T_i \|)I - (\sum_{i=2}^m T_i)$.

Proof. (2.4) This is due to the statement (2.1).

(2.5) By statements (2.1) and (2.3).

(2.6) Let $T_1 = I$ in (2.5).

Remark that statement (2.5) gives rise to generalizations of many results in [1,4,6,7,8]. A characterization of the Daugavet equation mentioned at the beginning of this section is clearly a special case of (2.6).

4. Results for Operators in Hilbert Spaces with Applications

In this final section let $(H, (\cdot,\cdot))$ be a Hilbert space with the inner product $(\cdot,\cdot)$. For $x, y \in H$ we denote by Re (x, y) and Im (x, y) the real and imaginary parts, respectively, of the inner product (x, y). Also let $B(H)$ denote the algebra of all bounded linear operators in H into itself, and T^* denotes the adjoint operator of $T \in B(H)$. With the aid of the inner product and the Cauchy-Schwarz inequality in this space, our consideration of the norm of the sum of operators and the sum of their norms is getting more interesting and more results.

Remark that the uniform convexity of X in Theorem 3.2 and Theorem 3.3 in section two may be replaced by H without affecting results there, since an inner product space is uniforly convex as noted in section one.

Theorem 4.1. *Let* $\{T_i\}_{i=1}^m \subseteq B(H)$. *Then the following are equivalent.*

(3.1) $(\sum_{i=1}^m T_i)e = (\sum_{i=1}^m \| T_i \|)e$.

(3.2) $1 + \sum_{i=1}^m \| T_i \| = \| (I + \sum_{i=1}^m T_i)e \|$.

(3.3) $\sum_{i=1}^m \| T_i \| = (\sum_{i=1}^m T_i e, e)$.

(3.4) $\sum_{i=1}^m \| T_i \| = (\sum_{i=1}^m T_i e, e) = \sum_{i=1}^m \| T_i e \|$.

(3.5) $(\sum_{i=1}^m T_i)e = (\sum_{i=1}^m \| T_i \|)e = (\sum_{i=1}^m \| T_i e \|)e$.

Proof. (3.1)⇒(3.2) Because

$$\| (I + \sum_{i=1}^m T_i)e \| = \| e + (\sum_{i=1}^m \| T_i \|)e \| = 1 + \sum_{i=1}^m \| T_i \| .$$

(3.2)⇒(3.1) Since

$$\begin{aligned}
&\| (\sum_{i=1}^{m} T_i)e - (\sum_{i=1}^{m} \| T_i \|)e \|^2 \\
&\qquad =\| (\sum_{i=1}^{m} T_i)e \|^2 +(\sum_{i=1}^{m} \| T_i \|)^2 - (\sum_{i=1}^{m} \| T_i \|)2Re(\sum_{i=1}^{m} T_i e, e) \\
&\qquad\qquad \leq 2(\sum_{i=1}^{m} \| T_i \|)[\sum_{i=1}^{m} \| T_i \| -Re(\sum_{i=1}^{m} T_i e, e)]
\end{aligned}$$

(as $\| (\sum_{i=1}^{m} T_i)e \| \leq \sum_{i=1}^{m} \| T_i \|$)

$$\begin{aligned}
&\leq (\sum_{i=1}^{m} \| T_i \|)[(\sum_{i=1}^{m} \| T_i \|)^2 - \| (\sum_{i=1}^{m} T_i)e \|^2] \\
&\qquad + 2(\sum_{i=1}^{m} \| T_i \|)[\sum_{i=1}^{m} \| T_i \| -Re(\sum_{i=1}^{m} T_i e, e)] \\
&\qquad = (\sum_{i=1}^{m} \| T_i \|)[(1 + (\sum_{i=1}^{m} \| T_i \|)^2 - \| (I + \sum_{i=1}^{m} T_i)e \|^2] = 0.
\end{aligned}$$

The last equality is due to (3.2), and (3.1) follows.

(3.1)⇒(3.4) Since, by (3.1)

$$((\sum_{i=1}^{m} T_i)e, e) = ((\sum_{i=1}^{m} \| T_i \|)e, e) = \sum_{i=1}^{m} \| T_i \|,$$

and

$$\sum_{i=1}^{m} \| T_i \| = \| (\sum_{i=1}^{m} T_i)e \| \leq \sum_{i=1}^{m} \| T_i e \| \leq \sum_{i=1}^{m} \| T_i \| .$$

Next, the proofs of the implications (3.4)⇒(3.3) and (3.5)⇒(3.1) are trivial and should be omitted. It remains to show that (3.3)⇒(3.1)⇒(3.5).

(3.3)⇒(3.1). Since, as in the proof above,

$$\begin{aligned}
&\| (\sum_{i=1}^{m} T_i)e - (\sum_{i=1}^{m} \| T_i \|)e \|^2 \\
&\qquad \leq 2(\sum_{i=1}^{m} \| T_i \|)[\sum_{i=1}^{m} \| T_i \| -\mathrm{Re}\ (\textstyle\sum_{i=1}^{m} T_i e, e)] = 0.
\end{aligned}$$

The equality is by (3.3), and we have (3.1).

Finally, since $\sum_{i=1}^{m} \| T_i \| = \sum_{i=1}^{m} \| T_i e \|$ by (3.1) (see the proof (3.1)⇒(3.4) above), and applying (3.1) again yields (3.5). This completes the proof of the theorem.

Before proceeding further we require a few familiar definitions and recall the following ones: A unit vector $e \in X$ is called a complete vector for the operator S if (Se, e) is a real number such that $\| S \|= (Se, e) =\| Se \|$ [8, Definition], and e is called a norm attaining vector for S if $\| S \|=\| Se \|$ [5]. Note that a compact operator enjoys the last definition and useful properties in [5, p. 85]. The numerical range of S is the convex set $W(S) = \{(Sx, x) : x \in X \text{ and } \| x \|= 1\}$, and $\overline{W}(S)$ denotes its closure [5].

Corollary 4.2. *Let $\{T_i\}_{i=1}^m \subseteq B(H)$. Then any one of the five equivalent statement in Theorem 3.1 implies the following.*

(3.6) $\| I + \sum_{i=1}^m T_i \|= 1 + \sum_{i=1}^m \| T_i \|$.

(3.7) *e is the norm attaining vector for the operator $\sum_{i=1}^m T_i$.*

(3.8) *e is the norm attaining vector for the operator $I + \sum_{i=1}^m T_i$.*

(3.9) *$\sum_{i=1}^m \| T_i \|$ is in the numerical range of the operator $\sum_{i=1}^m T_i$.*

(3.10) $\| \sum_{i=1}^m T_i \|= \sum_{i=1}^m \| T_i \|$.

(3.11) *e is the unit eigenvector corresponding to the eigenvalue $\sum_{i=1}^m \| T_i \|$ of the operator $\sum_{i=1}^m T_i$.*

(3.12) *e is the complete vector for the operator $\sum_{i=1}^m T_i$.*

Proof. Above results are consequences of Theorem 4.1, of course, and we shall omit the proofs. As the matter of fact it is easy to show the following implications: (3.2)⇒(3.6), (3.1)⇒(3.7), (3.2)⇒(3.8), (3.3)⇒(3.9), (3.3) and the Cauchy-Schwarz inequality imply (3.10), and (3.1)⇒(3.11). Finally, (3.4) and the Cauchy-Schwarz inequality imply (3.12).

The next result is special cases of Theorem 4.1 and Corollary 4.2.

Corollary 4.3. *Let $T \in B(H)$. Then the following are equivalent.*

(3.13) $Te =\| T \| e$

($\| T \|$ is an eigenvalue corresponding to a unit eigenvector e of T).

(3.14) $1+ \| T \|=\| (I + T)e \|$.

(3.15) $\| T \|= (Te, e)$ *($\| T \|$ is in the numerical range of T).*

(3.16) $\| T \|= (Te, e) =\| Te \|$ (e is the *complete vector for T*).

(3.17) $Te =\| T \| e =\| Te \| e$.

Moreover, any statement above implies the following.

(3.18) *The Daugavet equation $\| I + T \|= 1+ \| T \|$ holds.*

(3.19) *e is the norm attending vector for T.*

(3.20) *e is the norm attending vector for $I+T$.*

Proof. Let $m=1$ and $T_1=T$ in both Theorem 4.1 and Corollary 4.2.

In despite of (2.5) in Theorem 3.3 of section three, where T_1 is restricted to an isometric operator, we shall give next result without such restriction a characterization of the the norm of the sum of operators being equal to the sum of their norms. Actually, the characterization is different in nature from Theorem 3.3 and is a generalization of the main result in [2], which says that for $T_1,T_2\in B(H)$, $\| T_1+T_2 \|=\| T_1 \| + \| T_2 \|$ if and only if $\| T_1 \|\| T_2 \|\in \overline{W}(T_1^*T_2)$ So, in particular, the Daugavet equalition $\| I+T \|= 1+\| T \|$ holds if and only if $\| T \|\in \overline{W}(T)$.

Theorem 4.4. *Let $\{T_i\}_{i=1}^m \subseteq B(H)$. Then the following are equivalent.*

(3.21) $\| \sum_{i=1}^m T_i \|=\sum_{i=1}^m \| T_i \|$.

(3.22) $\sum_{i=1,i<j\le m}^{m-1} \| T_i \|\| T_j \| \in \overline{W}(\sum_{i=1,i<j\le m}^{m-1} T_j^*T_i)$.

Proof. (3.21)⇒(3.22). By the definition of norm there exists a sequence $\{e_i\}_{i=1}^\infty \subseteq H$ of unit vectors such that

$$\lim_{n\to\infty} \| (\sum_{i=1}^m T_i)e_n \|=\| \sum_{i=1}^m T_i \|=\sum_{i=1}^m \| T_i \| . \tag{$*$}$$

We claim first that $\lim_{n\to\infty} \| T_ie_n \| =\| T_i \|$, $i=1,2,...,m$.

To this end, since, by (*),

$$\sum_{i=1}^m \| T_i \|= lim_{n\to\infty} \| (\sum_{i=1}^m T_i)e_n \|\le lim_{n\to\infty}\sum_{i=1}^m \| T_ie_n \|$$
$$\le lim_{n\to\infty} \| T_ie_n \| + \sum_{k=1,k\ne i}^m \| T_k \|\le \sum_{i=1}^m \| T_i \|,$$

and the first claim is proved. Next, since

$$\| (\sum_{i=1}^m T_i)e_n \|^2=\sum_{i=1}^m \| T_ie_n \|^2 +2\sum_{i=1,i<j\le m}^{m-1} Re(T_j^*T_ie_n,e_n).$$

Passing to the limit as $n\to\infty$ yields

$$(\sum_{i=1}^m \| T_i \|)^2=\sum_{i=1}^m \| T_i \|^2 +2lim_{n\to\infty}\sum_{i=1,i<j\le m}^{m-1} Re(T_j^*T_ie_n,e_n)$$

by (*) and the first claim, which also yields

$$\sum_{i=1,i<j\le m}^{m-1} \| T_i \|\| T_j \|= \lim_{n\to\infty}\sum_{i=1,i<j\le m}^{m-1} Re(T_j^*T_ie_n,e_n).$$

Next, we claim second that $\lim_{n\to\infty}$ Re $(T_j^*T_ie_n, e_n) =\| T_i \|\| T_j \|$ for $i = 1, 2, ..., m-1$ and $i < j \leq m$.

To this end, by the equality above and the fact that Re $z \leq| z |$ for any complex number z, we have

$$\sum_{i=1,i<j\leq m}^{m-1} \| T_i \|\| T_j \| = \lim_{n\to\infty} \sum_{i=1,i<j\leq m}^{m-1} Re(T_j^*T_ie_n, e_n)$$
$$\leq \lim_{n\to\infty} Re(T_j^*T_ie_n, e_n) + \sum_{i=1,i<j\leq m}^{m-1} \| T_i \|\| T_j \| - \| T_j \|\| T_i \|$$
$$\leq \sum_{i=1,i<j\leq m}^{m-1} \| T_i \|\| T_j \|,$$

and the second claim is proved. Next, as $| z |^2 = (\text{Re } z)^2 + (\text{Im } z)^2$,

$$\{ \sum_{i=1,i<j\leq m}^{m-1} [Re(T_j^*T_ie_n, e_n)]^2 + \sum_{i=1,i<j\leq m}^{m-1} [Im(T_j^*T_ie_n, e_n)]^2 \}^{1/2}$$
$$= \{ \sum_{i=1,i<j\leq m}^{m-1} | (T_j^*T_ie_n, e_n) |^2 \}^{1/2} \leq \{ \sum_{i=1,i<j\leq m}^{m-1} (\| T_i \|\| T_j \|)^2 \}^{1/2}.$$

Passing to the limit as $n \to \infty$ and by the second claim yield

$$\sum_{i=1,i<j\leq m}^{m-1} \| T_i \|^2 \| T_j \|^2$$
$$\leq \sum_{i=1,i<j\leq m}^{m-1} \| T_i \|^2 \| T_j \|^2 + \lim_{n\to\infty} \sum_{i=1,i<j\leq m}^{m-1} [Im(T_j^*T_ie_n, e_n)]^2$$
$$\leq \sum_{i=1,i<j\leq m}^{m-1} \| T_i \|^2 \| T_j \|^2,$$

which shows that $\lim_{n\to\infty} \sum_{i=1,i<j\leq m}^{m-1}$ [Im $(T_j^*T_ie_n, e_n)]^2 = 0$. Consequently, by the second claim again,

$$\sum_{i=1,i<j\leq m}^{m-1} \| T_i \|\| T_j \| = \lim_{n\to\infty} (\sum_{i=1,i<j\leq m}^{m-1} T_j^*T_ie_n, e_n),$$

which is precisely (3.22).

(3.22)⇒(3.21). This can be proved in a similar manner as above. Let us pick a sequence $\{e_i\}_{i=1}^{\infty} \subseteq H$ of unit vectors such that

$$\sum_{i=1,i<j\leq m}^{m-1} \| T_i \|\| T_j \| = \lim_{n\to\infty} (\sum_{i=1,i<j\leq m}^{m-1} T_j^* T_i e_n, e_n),$$

and so,

$$\sum_{i=1,i<j\leq m}^{m-1} \| T_i \|\| T_j \| = \lim_{n\to\infty} \sum_{i=1,i<j\leq m}^{m-1} Re(T_j^* T_i e_n, e_n)$$

$$\leq \lim_{n\to\infty} \| T_1 e_n \|\| T_2 \| + \sum_{i=1,i<j\leq m}^{m-1} \| T_i \|\| T_j \| - \| T_2 \|\| T_1 \|$$

$$\leq \sum_{i=1,i<j\leq m}^{m-1} \| T_i \|\| T_j \|,$$

from which $\lim_{n\to\infty} \| T_1 e_n \| = \| T_1 \|$, and $\lim_{n\to\infty} \| T_i e_n \| = \| T_i \|$ for $i = 1, 2, ..., m$, similarly. Thus we have

$$lim_{n\to\infty} \sum_{i=1}^{m} \| T_i e_n \| = \sum_{i=1}^{m} \| T_i \| .$$

Finally, since

$$\| (\sum_{i=1}^{m} T_i) e_n \|^2 = \sum_{i=1}^{m} \| T_i e_n \|^2 + 2 \sum_{i=1,i<j\leq m}^{m-1} Re(T_j^* T_i e_n, e_n).$$

Passing to the limit as $n \to \infty$ yields

$$lim_{n\to\infty} \| (\sum_{i=1}^{m} T_i) e_n \|^2 = \sum_{i=1}^{m} \| T_i \|^2 + 2 \sum_{i=1,i<j\leq m}^{m-1} \| T_i \|\| T_j \| = (\sum_{i=1}^{m} \| T_i \|)^2,$$

or

$$lim_{n\to\infty} \| (\sum_{i=1}^{m} T_i) e_n \| = \sum_{i=1}^{m} \| T_i \|$$

and so (3.21) follows. The proof of the theorem is now completed.

Our proof above looks long, but only because we are writing out every little step in detail. Now, combination of Theorem 3.3 with Theorem 4.4 yield the following two results.

Corollary 4.5. *Let $\{T_i\}_{i=1}^{m} \subseteq B(H)$. Then any equivalent statement in Theorem 4.4 implies that* 0 *is an approximate eigenvalue of both operators*

$$(\sum_{i=2}^{m} \| T_i \|) T_1 - \| T_1 \| (\sum_{i=2}^{m} T_i) \quad and \quad \| \sum_{i=2}^{m} T_i \| T_1 - \| T_1 \| (\sum_{i=2}^{m} T_i).$$

Corollary 4.6. *Let* $\{T_i\}_{i=1}^m \subseteq B(H)$ *and let* T_1 *be isometric. Then any equivalent statement in Theorem 4.4 holds if and only if* 0 *is an approximate eigenvalue of the operator* $(\sum_{i=2}^m \| T_i \|)T_1 - (\sum_{i=2}^m T_i)$.

In conclusion we notice that it might be interesting to convert the results we obtained above to the differrent spaces such as in a unital Banach algebra and in a unital C^*-algebra.

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