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CLASSIFICATION OF SOLUTIONS OF SECOND ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract: The oscillatory and nonoscillatory behavior of solutions of the nonlinear neutral differential equation

$$\Big[r(t)[x(t)+c(t)x(t-\tau)]'\Big]'+p(t)f(x(t-\delta))-q(t)g(x(t-\sigma))=0,$$

where c, r, p, $q \in C([t_0,\infty),\mathbb{R})$, f, $g \in C(\mathbb{R},\mathbb{R})$, and τ, δ, $\sigma \in \mathbb{R}_+$, are studied. Examples illustrating the relevance of the results are given.

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1. Introduction

We consider the second order nonlinear neutral delay differential equation

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$$\big[r(t)[x(t)+c(t)x(t-\tau)]'\big]' + p(t)f(x(t-\delta)) - q(t)g(x(t-\sigma)) = 0, \qquad (1)$$

for $t \geq t_0$ subject to the conditions:

(H_1) r, c, p, $q \in C([t_0,\infty),\mathbb{R})$ with r positive and continuously differentiable;

(H_2) τ, δ, $\sigma \in \mathbb{R}_+ = [0,\infty)$;

(H_3) f, $g \in C(\mathbb{R},\mathbb{R})$ satisfy $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$.

Let $\phi \in C([t_0 - m, t_0],\mathbb{R})$, where $m = \max\{\tau,\, \delta,\, \sigma\}$. By a *solution* of equation (1) with initial function ϕ, we mean a function $x \in C([t_0 - m,\infty),\mathbb{R})$ such that $x(t) = \phi(t)$ for $t - m \leq t \leq t_0$ and $x(t)$ satisfies equation (1) for $t \geq t_0$.

The study of the oscillation and other asymptotic properties of solutions of neutral differential equations with positive and negative coefficients has attracted a good bit of attention in the last several years. However, many of the results in the literature are for first order linear equations. For results on second order nonlinear equations, we refer the reader to the recent papers by Dix, Ghose, and Rath [3], Padhi [12, 13], Yang, Zhang, and Ge, [16], and Yildiz, Karpuz, and Öcalan [17]. In papers [12, 13, 17], the functions f and g are the same, and in the paper [3] the function g is bounded. None of those restrictions apply here.

In [15], Thandapani and Manuel considered equation (1) in the case where $q(t) \equiv 0$ and $c(t) \equiv 0$ for all $t \geq t_0$; they classified all solutions of (1) into four classes and obtained criteria for the existence/nonexistence of solutions in these classes. We will take a similar approach for solutions of equation (1). For $p(t) = 0$ or $q(t) = 0$ for all $t \geq t_0$, the oscillatory and asymptotic behavior of solutions of equation (1) is discussed in [4].

In [1, 10, 11, 14, 17, 18], the authors considered equation (1) and obtained conditions for the existence of nonoscillatory solutions and established sufficient conditions for the oscillation of all/bounded solutions of equation (1). In this paper, we consider two cases, namely, $p \geq 0$ and p changes sign, and give sufficient conditions for every solution of equation (1) to oscillate; we also investigate the asymptotic behavior of the nonoscillatory solutions. With respect to their asymptotic behavior, all nontrivial solutions of equation (1) may be *a priori* divided into the following classes:

$$M^+ = \{x : \text{ there exists } t_x \geq t_0 \text{ such that } x(t)x'(t) \geq 0 \text{ for } t \geq t_x\};$$
$$M^- = \{x : \text{ there exists } t_x \geq t_0 \text{ such that } x(t)x'(t) \leq 0 \text{ for } t \geq t_x\};$$
$$OS = \{x : \text{ there exists } \{t_n\} \to \infty \text{ such that } x(t_n) = 0\};$$
$$WOS = \{x : x(t) \neq 0 \text{ for all large } t \text{ but } x'(t) \text{ oscillates}\}.$$

A solution in the class WOS is often referred to as *weakly oscillatory*. Other authors have used this type of classification of solutions as well, for example, see Cecchi and Marini [2].

2. Existence of Solutions in the Classes M^+, M^-, and WOS

First, we examine the question of existence of solutions of equation (1) in the class M^+.

Theorem 2.1. *Assume that:*

(i) $c(t) \geq 0$ *and is nondecreasing for all* $t \geq t_0$*;*

(ii) $q(t) \geq 0$ *for all* $t \geq t_0$*;*

(iii) $\sigma \geq \delta$*;*

(iv) there exists $M > 0$ *such that* $\frac{g(u)}{f(u)} \leq M$ *for* $u \neq 0$*;*

(v) f *is nondecreasing.*

If

$$\limsup_{t\to\infty} \int_{t_0}^{t} (p(s) - Mq(s))\, ds = \infty, \tag{2}$$

then $M^+ = \emptyset$.

Proof. Suppose that equation (1) has a solution $x \in M^+$. Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x'(t) \geq 0$, $x(t-m) > 0$, and $x'(t-m) \geq 0$ for all $t \geq t_1$. (The proof if $x(t) < 0$ and $x'(t) \leq 0$ is similar and will be omitted here and in future proofs.) Let

$$z(t) = x(t) + c(t)x(t-\tau) \quad \text{for} \quad t \geq t_1.$$

Then, by condition (i), we have $z(t) > 0$ and $z'(t) \geq 0$ for all $t \geq t_1$. Now

$$\begin{aligned}\left(\frac{r(t)z'(t)}{f(x(t-\delta))}\right)' &= \frac{(r(t)z'(t))'}{f(x(t-\delta))} - \frac{r(t)z'(t)f'(x(t-\delta))x'(t-\delta)}{f^2(x(t-\delta))} \\ &\leq -p(t) + \frac{q(t)g(x(t-\sigma))}{f(x(t-\delta))} \\ &\leq -p(t) + \frac{q(t)g(x(t-\sigma))}{f(x(t-\sigma))} \\ &\leq -(p(t) - Mq(t)),\end{aligned}$$

or

$$\frac{r(t)z'(t)}{f(x(t-\delta))} - \frac{r(t_1)z'(t_1)}{f(x(t_1-\delta))} \le -\int_{t_1}^{t} (p(s) - Mq(s))ds.$$

From (2), we obtain

$$\lim_{t\to\infty} \inf \frac{r(t)z'(t)}{f(x(t-\delta))} = -\infty,$$

which contradicts the fact that $z'(t) \ge 0$ for all large t. This completes the proof of the theorem. □

The following example shows that condition (2) is needed in Theorem 2.1

Example 2.2. Consider the neutral differential equation

$$\left[\frac{1}{t}\left(x(t) + 2x(t-1)\right)'\right]' + \frac{8(t-1)^2+1}{t^3(t-1)^2(t^2+1)} x(t)[x^2(t)+1] - \frac{5t^2-10t+6}{t^2(t-1)^5} x^3(t-1) = 0, \quad t \ge 2. \quad (3)$$

It is easy to see that all the hypotheses of Theorem 2.1 are satisfied except for condition (2). We see that equation (3) has a solution $x(t) = t \in M^+ \neq \emptyset$.

Theorem 2.3. *Assume that:*

(i) $-1 < c(t) \le 0$ *for all* $t \ge t_0$*;*

(ii) there exists $M > 0$ *such that* $\frac{g(u)}{f(u)} \le M$ *for* $u \neq 0$*;*

(iii) $p(t) \ge Mq(t) \ge 0$ *for all* $t \ge t_0$*;*

(iv) $\sigma \ge \delta$*;*

(v) f *is nondecreasing.*

If

$$\int_{t_0}^{\infty} \frac{1}{r(s)} ds = \infty, \quad \text{and} \quad \int_{t_0}^{\infty} (p(s) - Mq(s))\, ds = \infty, \quad (4)$$

then $M^+ = \emptyset$*.*

Proof. Suppose that equation (1) has a solution $x(t) \in M^+$, say $x(t) > 0$, $x'(t) \ge 0$, $x(t-m) > 0$, and $x'(t-m) \ge 0$ for all $t \ge t_1$, for some $t_1 \ge t_0$. (The proof for $x(t) < 0$ is similar). Note that

$$z(t) = x(t) + c(t)x(t-\tau) \ge (1 + c(t))x(t-\tau) > 0 \quad \text{for} \quad t \ge t_1.$$

From equation (1), we have

$$(r(t)z'(t))' = \Big[-p(t) + q(t)\frac{g(x(t-\sigma))}{f(x(t-\delta))}\Big] f(x(t-\delta))$$
$$\le -(p(t) - Mq(t)) f(x(t-\delta)) \le 0$$

for $t \ge t_1$. Hence, $r(t)z'(t)$ is nonincreasing for $t \ge t_1$, and we claim that $r(t)z'(t) \ge 0$ for $t \ge t_1$. If $r(t)z'(t) < 0$ for $t \ge t_2$ for some $t_2 \ge t_1$, then

$$r(t)z'(t) \le r(t_2)z'(t_2) < 0 \quad \text{for} \quad t \ge t_2,$$

so

$$z'(t) \le \frac{r(t_2)z'(t_2)}{r(t)} < 0.$$

Integrating the last inequality from t_2 to t, we obtain

$$z(t) - z(t_2) \le \int_{t_2}^{t} \frac{r(t_2)z'(t_2)}{r(s)} ds.$$

This implies that $z(t) \to -\infty$ as $t \to \infty$, which is a contradiction. Thus, $r(t)z'(t) \ge 0$. Now proceeding as in the proof of Theorem 2.1 and using (4), we have

$$\lim_{t\to\infty} \frac{r(t)z'(t)}{f(x(t-\delta))} = -\infty.$$

This contradicts the fact that $z'(t) \ge 0$ for all large t and completes the proof of the theorem. □

Next we examine the problem of existence of solutions of equation (1) in the class M^-.

Theorem 2.4. *Assume that:*

(i) $\tau \le \sigma \le \delta$;

(ii) $\int_0^\alpha \frac{du}{f(u)} < \infty$ *and* $\int_{-\alpha}^0 \frac{du}{f(u)} > -\infty$ *for some* $\alpha > 0$;

(iii) f *is submultiplicative, i.e.,* $f(uv) \le f(u)f(v)$ *for* $uv > 0$;

(iv) there exists $M > 0$ *such that* $\frac{g(u)}{f(u)} \le M$ *for* $u \ne 0$;

(v) f *is nondecreasing;*

(vi) $c(t) \ge 0$ *and in nonincreasing for all* $t \ge t_0$;

(vii) $q(t) \ge 0$ *for all* $t \ge t_0$.

If

$$\limsup_{t\to\infty}\int_T^t \frac{1}{r(s)f(1+c(s))}\left(\int_T^s (p(\xi)-Mq(\xi))d\xi\right)ds=\infty, \tag{5}$$

for all $T \geq t_0$, then $M^- = \emptyset$.

Proof. Suppose that equation (1) has a solution $x \in M^-$. Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x'(t) \leq 0$, $x(t-m) > 0$, and $x'(t-m) \leq 0$ for $t \geq t_1$. As in the proof of Theorem 2.1, let $z(t) = x(t) + c(t)x(t-\tau)$ and then observe from (vi) that $z(t) > 0$ and $z'(t) \leq 0$ for all $t \geq t_1$. Since here we have $\sigma \leq \delta$ and $x'(t) \leq 0$, we again obtain the inequality

$$\frac{r(t)z'(t)}{f(x(t-\delta))}-\frac{r(t_1)z'(t_1)}{f(x(t_1-\delta))}\leq -\int_{t_1}^t (p(s)-Mq(s))ds, \tag{6}$$

or

$$\frac{z'(t)}{f(x(t-\delta))}\leq -\frac{1}{r(t)}\int_{t_1}^t (p(s)-Mq(s))ds, \quad t\geq t_1. \tag{7}$$

Since x is nonincreasing and $\tau \leq \delta$, we see that

$$z(t) \leq (1+c(t))x(t-\delta)$$

and

$$f(z(t)) \leq f(1+c(t))f(x(t-\delta))$$

since f is submultiplicative. From (7), we have

$$\begin{aligned}\frac{z'(t)}{f(z(t))} &\leq \frac{z'(t)}{f(1+c(t))f(x(t-\delta))}\\ &\leq -\frac{1}{r(t)f(1+c(t))}\int_{t_1}^t (p(s)-Mq(s))ds,\end{aligned}$$

for $t \geq t_1$. Integrating the last inequality from t_1 to t yields

$$\int_{z(t_1)}^{z(t)}\frac{ds}{f(s)}\leq -\int_{t_1}^t \frac{1}{r(s)f(1+c(s))}\int_{t_1}^s (p(\xi)-Mq(\xi))d\xi\, ds,$$

or

$$\int_{z(t)}^{z(t_1)}\frac{ds}{f(s)}\geq \int_{t_1}^t \frac{1}{r(s)f(1+c(s))}\int_{t_1}^s (p(\xi)-Mq(\xi))d\xi ds.$$

By condition (5),

$$\limsup_{t\to\infty}\int_{z(t)}^{z(t_1)}\frac{ds}{f(s)}=\infty, \tag{8}$$

which contradicts condition (ii). This completes the proof. □

The following two examples show that conditions (ii) and (5) in Theorem

2.4 are essential.

Example 2.5. Consider the differential equation

$$\left[t^2(t-1)^2\left(x(t)+x(t-1)\right)'\right]' + 4t(t-2)^3x^3(t-2) - 2(t-1)^3x^3(t-1) = 0, \quad t \geq 3. \quad (9)$$

All conditions of Theorem 2.4 are satisfied except for condition (ii). Note that $x(t) = \frac{1}{t}$ is a solution of (9) belonging to M^-.

Example 2.6. Consider the differential equation

$$\left[t^2(t-1)^2\left(x(t)+\frac{1}{t}x(t-1)\right)'\right]' + 4t\left(\frac{t-2}{t-1}\right)^{\frac{1}{3}} x^{\frac{1}{3}}(t-2) - 2\left(\frac{t-1}{t}\right)^{\frac{1}{3}} x^{\frac{1}{3}}(t-1) = 0, \quad t \geq 3. \quad (10)$$

All assumptions of Theorem 2.4 are satisfied except for condition (5). Here, $x(t) = \frac{t+1}{t}$ is a solution of (10), so $M^- \neq \emptyset$.

Next, we establish sufficient conditions under which any solution of equation (1) is either oscillatory or weakly oscillatory.

Theorem 2.7. *Assume that conditions (ii) and (v) of Theorem 2.1 and (ii) of Theorem 2.3 hold and that $\sigma = \delta$. If $c(t) \equiv c \geq 0$ for all $t \geq t_0$ and* (4) *holds, then every solution of equation* (1) *is either oscillatory or weakly oscillatory.*

Proof. From Theorem 2.1, it follows that $M^+ = \emptyset$. To complete the proof, it suffices to show that $M^- = \emptyset$. Let x be a solution of equation (1) belonging to M^-, say $x(t) > 0$, $x'(t) \leq 0$, $x(t-m) > 0$, and $x'(t-m) \leq 0$ for $t \geq t_1 \geq t_0$. Let $z(t) = x(t) + cx(t-\tau)$; then $z(t) > 0$ and $z'(t) \leq 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 2.4, we obtain

$$\frac{r(t)z'(t)}{f(x(t-\delta))} - \frac{r(t_1)z'(t_1)}{f(x(t_1-\delta))} + \int_{t_1}^{t} \frac{r(s)z'(s)f'(x(s-\delta))x'(s-\delta)}{f^2(x(s-\delta))}ds \leq -\int_{t_1}^{t}(p(s)-Mq(s))ds.$$

Set

$$w(t) = \frac{r(t)z'(t)}{f(x(t-\delta))}.$$

Then

$$w(t_1) = \frac{r(t_1)z'(t_1)}{f(x(t_1-\delta))} \leq 0,$$

and in view of (4), we have

$$w(t) \le w(t_1) + \int_{t_1}^{t} w(s)\Big(- \frac{f'(x(s-\delta))x'(s-\delta)}{f(x(s-\delta))}\Big) ds$$

for $t \ge t_2$ for a sufficiently large $t_2 \ge t_1$. From Gronwall's inequality, we obtain

$$w(t) \le w(t_1)\frac{f(x(t_1-\delta))}{f(x(t-\delta))},$$

and so

$$r(t)z'(t) \le w(t_1)f(x(t_1-\delta)) = M_1 < 0,$$

or

$$z'(t) \le \frac{M_1}{r(t)}$$

for $t \ge t_2$. An integration yields

$$z(t) \le z(t_2) + M_1 \int_{t_2}^{t} \frac{1}{r(s)} ds \to -\infty$$

as $t \to \infty$ by (4). This contradiction completes the proof of the theorem. □

3. Behavior of Solutions in the Classes M^+ and M^-

First we study the asymptotic behavior of solutions in the class M^-.

Theorem 3.1. *Assume that conditions (i) and (iii)-(vii) of Theorem 2.4 hold. If* (5) *holds, then for every solution* $x(t) \in M^-$ *we have* $\lim_{t\to\infty} x(t) = 0$.

Proof. The argument used in the proof of Theorem 2.4 again leads to (8). This now implies that $\lim_{t\to\infty} z(t) = 0$ and $z(t) \ge x(t)$ for all $t \ge t_1$. Thus, $\lim_{t\to\infty} x(t) = 0$ which completes the proof. □

Finally, we examine the asymptotic behavior of solutions in the class M^+.

Theorem 3.2. *In addition to conditions (i)–(v) of Theorem 2.1, assume that* $c(t)$ *bounded and*

$$\limsup_{t\to\infty} \int_{T}^{t} (p(s) - Mq(s)) \left(\int_{T}^{s} \frac{1}{r(\xi)} d\xi \right) ds = \infty \quad \text{for every } T \ge t_0. \tag{11}$$

Then every solution in the class M^+ *is unbounded.*

Proof. Let x be a solution of equation (1) such that $x \in M^-$. Without loss of generality, we assume $x(t) > 0$, $x'(t) \ge 0$, $x(t-m) > 0$, and $x'(t-m) \ge 0$ for all $t \ge t_1$ for some $t_1 \ge t_0$. Let $z(t) = x(t) + c(t)x(t-\tau)$; then $z(t) > 0$ and

$z'(t) \geq 0$ for all $t \geq t_1$. Consider the function

$$w(t) = -\frac{r(t)z'(t)}{f(x(t-\delta))} \int_{t_1}^{t} \frac{1}{r(s)} ds.$$

Then, for $t \geq t_1$, we have

$$\begin{aligned} w'(t) = & -\frac{\Big(r(t)z'(t)\Big)'}{f(x(t-\delta))} \int_{t_1}^{t} \frac{1}{r(s)} ds - \frac{z'(t)}{f(x(t-\delta))} \\ & + \frac{r(t)z'(t)f'(x(t-\delta))x'(t-\delta)}{f^2(x(t-\delta))} \int_{t_1}^{t} \frac{1}{r(s)} ds \\ \geq & \ (p(t) - Mq(t)) \int_{t_1}^{t} \frac{1}{r(s)} ds - \frac{z'(t)}{f(x(t-\delta))}. \end{aligned}$$

Integrating the last inequality, we obtain

$$w(t) \geq w(t_1) + \int_{t_1}^{t} (p(s) - Mq(s)) \Big(\int_{t_1}^{s} \frac{1}{r(\xi)} d\xi \Big) ds - \int_{t_1}^{t} \frac{z'(s)}{f(x(s-\delta))} ds. \quad (12)$$

Since the function $\frac{z'(t)}{f(x(t-\delta))}$ is positive for $t > t_1$, the limit as $t \to \infty$ of the last integral on the right hand side of (12) exists. Assume that

$$\lim_{t\to\infty} \int_{t_1}^{t} \frac{z'(s)}{f(x(s-\delta))} ds = M_2 < \infty.$$

In view of (11), (12) implies $\lim_{t\to\infty} w(t) = \infty$ which contradicts $w(t)$ being negative for all large values of t. Thus,

$$\lim_{t\to\infty} \int_{t_1}^{t} \frac{z'(s)}{f(x(s-\delta))} ds = \infty. \quad (13)$$

Now for $t \geq t_1$, we have

$$f(x(t-\delta)) \geq f(x(t_1-\delta)) \text{ or } \frac{1}{f(x(t-\delta))} \leq \frac{1}{f(x(t_1-\delta))} = M_3,$$

and consequently

$$\int_{t_1}^{t} \frac{z'(s)}{f(x(s-\delta))} ds \leq M_3 \int_{t_1}^{t} z'(s) ds = M_3(z(t) - z(t_1)).$$

From (13), we obtain

$$\lim_{t\to\infty} z(t) = \infty. \quad (14)$$

Since $z(t) = x(t) + c(t)x(t-\tau)$ and $x'(t) \geq 0$, we have

$$z(t) \leq (1 + c(t))x(t).$$

In view of (14) and the fact that $c(t)$ is bounded, this implies $\lim_{t\to\infty} x(t) = \infty$. This completes the proof. □

It is possible to obtain some variations of Theorems 2.1–3.2 above. Rather than present these as separate theorems, we will collect them into the following remark.

Remark 3.3. Theorems 2.1, 2.3, 2.7, and 3.2 remain true if f is replaced by g in condition (v) in all four theorems. Theorem 2.4 remains true if f is replaced by g in conditions (ii), (iii), and (v), and Theorem 3.1 remains true if f is replaced by g in conditions (iii) and (v). The proofs make use of the quotient

$$\left(\frac{r(t)z'(t)}{g(x(t-\sigma))}\right)'.$$

We conclude this paper with some examples.

Example 3.4. Consider the differential equation (9). It is easy to see that all conditions of Theorem 3.1 are satisfied. In fact, $x(t) = \frac{1}{t} \in M^-$ is a solution of equation (9) such that $x(t) = \frac{1}{t} \to 0$ as $t \to \infty$.

Example 3.5. Consider the neutral differential equation

$$\left[\frac{1}{t}\left(x(t)+x(t-1)\right)'\right]' + \frac{1}{t^2}x(t) - \frac{t-1}{t^3}x(t-1) = 0, \quad t \geq 2. \tag{15}$$

Here $r(t) = 1/t$, $c(t) = 1$, $\sigma = 1$, $\delta = 0$, $p(t) = 1/t^2$, $q(t) = (t-1)/t^3$, $f(u) = g(u) = u$, and $M = 1$. It can easily be seen that all conditions of Theorem 3.2 are satisfied. Here, $x(t) = t+1 \in M^+$ is a solution of equation (15) such that $x(t) \to \infty$ as $t \to \infty$. Moreover, all conditions of Theorem 2.1 are satisfied except condition (2) and $\in M^+ \neq \emptyset$.

Final remarks and suggestions for further research. It is known that the values of the function $c(t)$ significantly affect the nature of solutions of neutral differential equations. For example, with either $p(t) \equiv 0$ or $q(t) \equiv 0$, for certain values of $c(t)$ the nonoscillatory solutions may converge to zero while for other values they may become unbounded. In fact, the value -1 behaves like a bifurcation point in this respect. For a classification of these kinds of behaviors of solutions, we refer the reader to the papers by Graef et al. [5, 6, 7, 8]. It would be of interest to further explore these various asymptotic properties of solutions for equations with positive and negative coefficients of the type studied in this paper. In the papers [3, 12, 13] mentioned above, the authors did explore some of these ideas for the equations with the restrictions mentioned in the Introduction. In addition, extending such results to higher order equations would also be of interest.

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