International Journal of Pure and Applied Mathematics
Volume 64 No. 1 2010, 9-19

HARMONIC ANALYSIS ON HYPERGROUPS

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Abstract: The main task in this article is to give the necessary and sufficient conditions which guarantee that the product of two positive definite functions defined on a hypergroup X is also positive definite on X. Also, we prove that a continuous function with compact support ψ is negative definite if and only if $\exp(-t\psi)$ is positive definite for each $t > 0$. Moreover, we will give some relations between the class of completely monotonic functions on a hypergroup and the set of τ-positive functions.

AMS Subject Classification: 43A62, 43A22, 43A10
Key Words: hypergroup, positive definite, completely monotone

1. Introduction

A central idea in harmonic analysis in various settings is the existence of a product, usually called convolution, for functions and measures. In particular, in the study of the harmonic analysis of orthogonal expansions, a convolution arises in a natural way and plays the same role as ordinary convolution in Fourier analysis. In some cases, an investigation begins with a convolution

Received: September 14, 2009

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algebra of measures as the primitive object upon which to build a theory; this is the case of the analysis of the objects called hypergroups which are the generalizations of the convolution algebra of Borel measures on a group. A hypergroup (see [3], [5] or [7]) is a locally compact Hausdorff space X with a certain convolution structure $*$ on the space of complex Radon measures on X, $M(X)$. One important reason that explains why the harmonic analysts did not attracted to study Fourier algebra over hypergroups is that, the product of two continuous positive definite functions on a hypergroup is not necessarily positive definite in general. Moreover, one should observe that, a function ψ is negative definite if and only if $\exp(-t\psi)$ is positive definite for each $t > 0$. While this result holds for all semigroups it is not clear how to prove the 'only if' part for hypergroups, since the usual technique cannot be applied (the 'if' part always holds provided that Reψ is locally lower bounded). The problem is that except when x or y belong to the maximal subgroup of the hypergroup, $\exp(-t\psi(x*y))$ and $\exp(-t\psi)(x*y)$ are usually not equal so that other methods have to be used to overcome this. Let δ_x be the Dirac measure at a point $x \in X$. Then the convolution $\delta_x * \delta_y$ of the two point measures δ_x and δ_y is a probability Radon measure on X with compact support and such that $(x, y) \mapsto$supp$(\delta_x * \delta_y)$ is a continuous mapping from $X \times X$ into the space of compact subsets of X. Unlike the case of groups, this convolution is not necessarily the point measure $\delta_{x.y}$ for a composition $x.y$ in X. The rule is played by the generalized (left) translation, defined on a hypergroup by

$$T_x f(y) = \int_X f(t) d(\delta_x * \delta_y)(t)$$

for all $y \in X$. As pointed in [3], every commutative hypergroup possesses a Haar measure η, which is unique modulo, a positive multiplicative constant and has support equal to the whole space X. Only compact hypergroups admit bounded Haar measure η. For a commutative hypergroup (i.e. $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in X$) the convolution of two functions k and f is given by

$$k * f(x) = \int_X k(y) T_x f(y) d\eta(y),$$

where η is a translation-invariant measure, called Haar measure of the hypergroup. The notion of an abstract algebraic hypergroup has its origins in the studies of E. Marty and H.S. Wall in the 1930s, and harmonic analysis on hypergroups dates back to J. Delsarte and B.M. Levitan work during the 1930s and 1940s, but the substantial development had to wait till the 1970s when Dunkl [5], Jewett [7] and Spector [9] put hypergroups in the right setting for harmonic analysis.

This paper contains 4 sections. In Section 2, we give the necessary and sufficient conditions which guarantee that the product of two positive definite functions defined on a hypergroup X is also positive definite on X, then we resuming some properties of the set of τ-positive definite functions on hypergroups. Section 3 is devoted to give some properties of the set of negative definite functions on hypergroups which help us to prove that a continuous function with compact support ψ is negative definite if and only if $\exp(-t\psi)$ is positive definite for each $t > 0$. In Section 4, we give some relations between the set of τ-positive definite functions, the set of the completely monotone and completely alternating functions.

2. Positive Definite Functions on Hypergroups

A hypergroup $(X, *)$ is called commutative if $(M(X), +, *)$ is a commutative algebra, and Hermitian if the involution $^-$ is the identity map. It is easy to prove that every Hermitian hypergroup is commutative. A locally bounded measurable function $\chi : X \to \mathbb{C}$ is called a semicharacter if $\chi(e) = 1$ and $\chi(x * y^-) = \chi(x)\overline{\chi(y)}$ for all $x, y \in X$. Every bounded semicharacter is called a character. If the character is not locally null then (see [3, Proposition 1.4.33]) it must be continuous. The dual X^* of X is just the set of continuous characters with the compact-open topology in which case X^* must be locally compact. In this paper we will be concerned with continuous characters on hypergroups. A locally bounded measurable function $\phi : X \to \mathbb{C}$ is said to be positive definite if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \phi(x_i * x_j^-) \geq 0$$

for all choice of $x_1, x_2, ..., x_n \in X, c_1, c_2, ..., c_n \in \mathbb{C}$ and $n \in \mathbb{N}$. The following two lemmas are in fact, an adaption of whatever done for semigroups in Berg et al [1]. We will not repeat the proof, wherever the proof for semigroups can be applied to the hypergroups with necessary modification.

Lemma 2.1. *(i) The sum and the point-wise limit of positive definite functions on hypergroups are also positive definite.*

(ii) Let ϕ be a continuous positive definite function on X and define $\Phi : M_c^1(X) \to \mathbb{C}$ by $\Phi(s) := \int \phi(s) d\mu(s)$. Then Φ is positive definite on $M_c^1(X)$.

Lemma 2.2. *A bounded measurable function $\phi \in C_c(X)$ is positive definite if and only if there exists a ψ in $L^2(X)$ such that $\phi = \psi \bullet \tilde{\psi}$, where*

$$f \bullet \tilde{g}(x) = \int_X f(x * y)\overline{g(y)}d\eta(y).$$

for all $f, g \in C_c(X)$.

Proof. The proof is as in Pederson [8, Lemma 7.2.4]. □

Theorem 2.3. *Let ϕ_1 and ϕ_2 belong to $C_c(X)$, then the product $\phi_1.\phi_2$ is positive definite on X if and only if ϕ_1 and ϕ_2 are positive definite on X.*

Proof. From the above lemma there exists $\psi_1, \psi_2 \in L^2(X)$ such that $\phi_1 = \psi_1 \bullet \tilde{\psi_1}$ and $\phi_2 = \psi_2 \bullet \tilde{\psi_2}$, so

$$\begin{aligned}
\phi_1.\phi_2(x) &= (\psi_1 \bullet \tilde{\psi_1}(x)).(\psi_2 \bullet \tilde{\psi_2}(x)) \\
&= \int_X \psi_1(x * y)\overline{\psi_1(y)}d\eta(y) \int_X \psi_2(x * z)\overline{\psi_2(z)}d\eta(z) \\
&= \int_X \int_X \psi_1(x * y)\psi_2(x * z)\overline{\psi_1(y)\psi_2(z)}d\eta(y)d\eta(z) \\
&= \int_X \int_X \psi_1.\psi_2(x * y, x * z)\overline{\psi_1.\psi_2(y, z)}d\eta(y)d\eta(z) \\
&= \int_X \int_X \psi_1.\psi_2(x * (y, z))\overline{\psi_1.\psi_2(y, z)}d\eta(y)d\eta(z).
\end{aligned}$$

Applying Fubini's Theorem to the right hand side we get

$$\phi_1.\phi_2(x) = \int_{X \times X} \psi_1.\psi_2(x * (y, z))\overline{\psi_1.\psi_2(y, z)}d\nu(y, z).$$

This implies

$$\phi_1.\phi_2(x) = \psi_1.\psi_2 \bullet \widetilde{\psi_1.\psi_2}(x).$$ □

Corollary 2.4. *Let $\phi \in C_c(X)$ be positive definite such that $|\phi(x * x^-)| < \chi$ for all $x \in X$. Then if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in $\{z \in \mathbb{C}; z < \chi\}$ and $a_n \geq 0$ for all $n \geq 0$, the composed kernel $f \circ \phi$ is a gain positive definite. In particular, if $\phi \in C_c(X)$ is positive definite, then so is $\exp(\phi)$.*

Let B be a maximal algebra in X. A linear functional $L : B \to \mathbb{C}$ is called τ-positive, where $\tau \subset B$ is admissible, if

$$L(a) \geq 0 \qquad \text{for all} \quad a \in \text{alg span}^+(\tau).$$

This holds if and only if

$$L(a_1 * ... * a_n) \geq 0 \quad \text{for all finite set} \quad \{a_1, ..., a_n\} \subseteq \tau.$$

Let $\alpha : X \to \mathbb{R}_+$ be an absolute value such that $\alpha(a) \geq 0$ for all $a \in X$. For

$\sigma \in \mathbb{C}, a \in X$, We define

$$\Omega_{\sigma,a} = \frac{1}{2}(I + \frac{\sigma}{2\alpha(a)}E_a + \frac{\bar{\sigma}}{2\alpha(a^-)}E_{a^-}).$$

Theorem 2.5. *Every τ-positive function $\phi : X \to \mathbb{C}$, where $\tau = \{\Omega_{\sigma,a}; \sigma \in \{\pm 1, \pm i\}, a \in X\}$ is positive definite and has integral representation*

$$\phi(x) = \int_{X^*} \chi(x) d\mu(\chi),$$

where $\mu \in M_+^b(X^*)$ is concentrated on the compact set of τ-positive characters.

Proof. Let Γ denote the set of τ-positive multiplicative linear functionals on $\mathbb{A}$, which are not identically zero. Clearly Γ is a compact subset of the set of τ-positive linear functionals on X. By [1,Theorem 4.5.4] the linear functional L corresponding to τ-positive function ϕ has a representation

$$L(T) = \int_{\Gamma} \delta(T) d\tilde{\mu}(\delta), \quad T \in \mathbb{A},$$

where $\tilde{\mu} \in M_+(\Gamma)$. For $\delta \in \Gamma$ the function $x \to \delta(E_x)$ is a τ-positive character, and the mapping $j : \Gamma \to X^*$ given by $j(\delta)(x) = \delta(E_x)$ is a homeomorphism of Γ onto the compact set $j(\Gamma)$ of τ-positive characters. The image measure $\mu := \tilde{\mu}^j$ of $\tilde{\mu}$ under j is a Radon measure on X^* with compact support contained in $j(\Gamma)$, and replacing T by E_x we get

$$\phi(x) = \int_{X^*} \chi(x) d\mu(\chi), \quad x \in X.$$

□

3. Negative Definite Functions on Hypergroups

One should be observe that, a function ψ is negative definite if and only if $\exp(-t\psi)$ is positive definite for each $t > 0$. While this result holds for all semigroups it is not clear how to prove the 'only if' part for hypergroups since the usual technique cannot be applied (the 'if' part always holds provided that $\mathrm{Re}\psi$ is locally lower bounded). The problem is that, except when x or y belong to the maximal subgroup of the hypergroup, $\exp(-t\psi(x*y))$ and $\exp(-t\psi)(x*y)$ are usually not equal so that other methods have to be used to overcome this. A locally bounded measurable function q is called a quadratic form if

$$q(x * y) + q(x * y^-) = 2q(x) + 2q(y)$$

for all $x, y \in X$ and additive if $q(x * y) = 2q(x) + 2q(y)$ for all $x, y \in X$. In the case X is Hermitian, that when X carries the identity involution, then every quadratic form is an additive function and every negative definite function is real. A locally bounded measurable function $\psi : X \to \mathbb{C}$ is said to be negative definite if $\psi(x^-) = \overline{\psi(x)}$ and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}\phi(x_i * x_j^-) \leq 0$$

for all choice of $x_1, x_2, ..., x_n \in X$, $n \in \mathbb{N}$ and all $c_1, c_2, ..., c_n \in \mathbb{C}$ that satisfy $\sum_{i=1}^{n} c_i = 0$.

A key result in the study of negative definite functions on hypergroups is the following Levy-Khinchin representation (see [3], Theorem 4.5.2)

$$\psi(x) = \psi(e) + q(x) + \int_{\hat{X}\backslash\{1\}} (1 - Re(\chi(x)))d\eta(\chi)$$

for all $x \in X$ where q is a nonnegative quadratic form on X and $\eta \in M_+(\hat{X}\backslash\{1\})$. Both q and the integral part $\psi(x) - \psi(e) - q(x)$ belong to the set of negative definite function on X and the pair (q, η) is uniquely determined by ψ with q being given by

$$q(x) = \lim\{\frac{\psi(x^{*n})}{n^2} + \frac{\psi((x * x)^{*n})}{2n}\}.$$

Lemma 3.1. (see [3]) *If ϕ is positive definite then $\phi(e) - \phi$ is negative definite.*

For hypergroups we have that if $\exp(-t\psi)$ is positive definite for all $t > 0$ then

$$\exp(-t\psi(e)) - \exp(-t\psi)$$

is negative definite and provided $Re\psi$ is locally lower bounded

$$\lim \frac{\exp(-t\psi(e)) - \exp(-t\psi)}{t} = \psi - \psi(e)$$

is also negative definite in which case so is ψ. In spite of the converse statement also holds for commutative semigroups, this result is not available for hypergroups which hinges on deciding whether $\exp(-\psi)$ is positive definite.

Lemma 3.2. *Let $\psi : X \times X \to \mathbb{C}$. Put*

$$\phi(x, y) := \psi(x, x_0) + \overline{\psi(y, x_0)} - \psi(x, y) - \psi(x_0, x_0)$$

for fixed $x_0 \in X$. Then ϕ is positive definite if and only if ψ is negative definite.

As a consequence of the above discussion and Theorem 2.3 we will prove the following corollary:

Corollary 3.3. *A function $\psi \in C_c(X)$ is negative definite if and only if $\exp(-t\psi)$ is positive definite for each $t > 0$.*

Proof. Suppose that ψ is negative definite. For obvious reasons we need only to show that $\exp(-t\psi)$ is positive definite for $t = 1$. We choose $x_0 \in X$ and with ϕ as in the above lemma we have

$$-\psi(x, y) := \phi(x, y) - \overline{\psi(y, x_0)} - \psi(x, x_0) + \psi(x_0, x_0),$$

where ϕ is positive definite. Hence

$$\exp(-\psi(x, y)) = \exp(\phi(x, y)).\overline{\exp(-\psi(y, x_0))}.\exp(-\psi(x, x_0)).\exp(\psi(x_0, x_0)).$$

Since, $\overline{\exp(-\psi(y, x_0))}.\exp(-\psi(x, x_0))$, is positive definite. From Theorem 2.3 we conclude that $\exp(-t\psi)$ is positive definite. □

4. Completely Monotone and Completely Alternating Functions on Hypergroups

Suppose that $L^\infty_{loc}(X)$ denotes the set of locally bounded measurable functions on X, and $L^1_c(X)$ the space of integrable functions on X with compact support. As pointed out in Bloom and Heyer [2]

$$L^1_c(X) * L^\infty_{loc}(X) \subset C(X).$$

For each $x \in X$, we define the shift operator E_y by $E_y\phi(x) = \phi(x * y)$ for all $x, y \in X$ and $\phi \in \mathbb{C}^X$. The complex span $\mathbb{A}$ of all such operators is a commutative algebra with identity $E_1 = I$ and involution $(\sum \alpha_i E_{x_i})^- = \sum \overline{\alpha_i} E_{x_i^-}$. For real valued $\phi \in L^\infty_{loc}(X)$ and $x \in X$ we define $\nabla_x\phi : X \to \mathbb{R}$ by

$$(\nabla_x\phi)(y) := (I - E_x)(\phi)(y).$$

We call ϕ completely monotone if $\phi \geq 0$ and

$$\nabla_{x_1}\nabla_{x_2}...\nabla_{x_n}\phi \geq 0$$

for all $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n \subset X$. The function ϕ is said to be completely alternating if

$$\nabla_{x_1}\nabla_{x_2}...\nabla_{x_n}\phi \leq 0$$

for all $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n \subset X$. With $\Delta_x\psi := -\nabla_x\psi$ we see from

$$\nabla_{x_1}\nabla_{x_2}...\nabla_{x_n}(\Delta_x\psi) = -\nabla_{x_1}\nabla_{x_2}...\nabla_{x_n}\nabla_x\psi$$

that $\psi \in L^\infty_{loc}(X)$ is completely alternating if and only if $\Delta_x\psi$ is completely monotone for each $x \in X$.

Theorem 4.1. *For a continuous function $\phi : X \longrightarrow \mathbb{R}$ the following conditions are equivalent:*

(i) ϕ is completely monotone.

(ii) ϕ is τ-positive.

(iii) There exists a measure $\mu \in M_+^b(\hat{X}_+)$ such that for all $x \in X$, $\phi(x) = \int_{\hat{X}_+} \chi(x) d\mu(\chi)$.

Proof. For $a_1, ..., a_n, x_1, ..., x_m \in X$ we have

$$(I - E_{a_1})...(I - E_{a_n}) E_{x_1} ... E_{x_m} \phi(e) = \nabla_{a_1} ... \nabla_{a_n} \phi(x_1 * ... * x_m)$$

so (i)⇒(ii). The implication "(ii)⇒(iii)" follows from the above theorem since $0 \leq \chi(x) \leq 1$ for all $x \in X$. Finally, if (iii) holds, then $\phi(x) \geq 0$ and since

$$\nabla_{a_1} ... \nabla_{a_n} \chi(x) = \chi(x) \prod_{i=1}^{n} [1 - \chi(a_i)]$$

so,

$$\nabla_{a_1} ... \nabla_{a_n} \phi(x) = \int_{\hat{X}_+} \chi(x) \prod_{i=1}^{n} [1 - \chi(a_i)] d\mu(\chi) \geq 0,$$

hence (i). □

Theorem 4.2. *A continuous function $\psi : X \to \mathbb{R}$ is completely alternating if and only if there exists an additive continuous function $h : X \to \mathbb{R}_+$ and a unique measure $\mu \in M_+(\hat{X}_+ \backslash \{1\})$ such that for all $x \in X$*

$$\psi(x) = \psi(e) + h(x) + \int_{\hat{X}_+ \backslash \{1\}} (1 - \chi(x)) d\mu(\chi).$$

Proof. It follows from the definition that ψ is lower bounded by $\psi(e)$, firstly, we assume that $\psi(e) = 0$. Let $S \supseteq X$ be a minimal semigroup containing the hypergroup X. Introducing

$$\Delta_y \psi(x) := \frac{1}{2} [\psi(x * y) + \psi(x * y^-)] - \psi(x); \quad x, y \in X,$$

and as pointed in [1, Proposition 4.3.11] $\Delta_y \psi$ is bounded and positive definite on S. Therefore appealing to Bochner's Theorem for hypergroups (see [7], Theorem 12.3B)

$$\Delta_y \psi(x) = \int_{\hat{X}} \rho_\chi(x) d\sigma_y(\chi)$$

for some $\sigma_y \in M_+^b(\hat{X})$, where we denote the canonical extension of $\chi \in \hat{X}$ to a function on S by ρ_χ. A simple calculation implies

$$-\Delta_z \Delta_y \psi(x) = \int_{\hat{X}} \rho_\chi(x) [1 - Re \rho_\chi(z)] d\sigma_y(\chi)$$
$$= \int_{\hat{X}} \rho_\chi(x) [1 - Re \rho_\chi(y)] d\sigma_z(\chi)$$

for $x, y, z \in X$, implying

$$[1 - Re\rho_\chi(z)]d\sigma_y(\chi) = [1 - Re\rho_\chi(y)]d\sigma_z(\chi)$$

by the uniqueness of the Fourier transform ([7], Theorem 12.2A). Noting that the $\{\chi \in \hat{X}; Re\chi(y) \leq 1\}$ are open sets in $\hat{X}$ with union (over y) given by $\hat{X}\backslash\{1\}$, we can find a unique Radon measure μ on $\hat{X}\backslash\{1\}$ such that for every $y \in S$

$$[1 - Re\rho_\chi(y)]d\mu(\chi) = d\sigma_y(\chi), \quad \text{on} \quad \hat{X}.$$

The set $\hat{S}$ of all bounded semigroup characters on the semigroup S is a compact Hausdorff space with respect to the topology of point wise convergence. The canonical mapping $\zeta : \hat{X} \to \hat{S}, \zeta(\chi) := \rho_\chi$ is continuous, and obviously $\Delta_y\psi$ is the Laplace transform of σ_y^ζ (the image measure of σ_y under ζ) for each $y \in S$ hence from [1, Lemma 4.3.12, Definition 4.3.13 and Theorem 4.6.7] there exists an additive continuous function $h : X \to \mathbb{R}_+$ on S such that for all $x \in S$

$$\psi(x) = h(x) + \int_{\hat{S}_+\backslash\{1\}} (1 - \rho(x))d\mu^\zeta(\rho) = h(x) + \int_{\hat{X}_+\backslash\{1\}} (1 - \chi(x))d\mu(\chi),$$

where $\mu \in M_+(\hat{X}_+\backslash\{1\})$.

Let $\tilde{X}$ denote the equivalence classes of all representations of X, and $C^*(X)$ denote the enveloping $C^*(X)$-algebra of $L^1(X)$. If (π, X) is a representation of X, let the associated representations of $L^1(X)$ and $C^*(X)$ be also denoted by π itself. If ζ, η are in X then the matrix coefficient $\pi_{\zeta,\eta}$ associated to π given by

$$\pi_{\zeta,\eta}(\delta_x) = \langle \pi(\delta_x)(\zeta), \eta \rangle, \quad \forall x \in X,$$

is continuous and bounded by $||\zeta||||\eta||$. If S is a subset of $\tilde{X}$ let

$$N_S := \{f \in L^1(X) : \pi(f) = 0 \ \forall \pi \in S\}.$$

Define an operator norm on $L^1(X)\backslash N_S$ by

$$||f^o|| := \sup\{||\pi(f)||; \quad \pi \in S\}.$$

We complete this getting a C^*-algebra and denote it by $C^*_S(X)$. As pointed out in Dixmier [4], Theorem 3.4.4, and by appropriate modification in Eymard [6], Proposition 1.21, we can state the following theorem.

Theorem 2.3. *Suppose that S is a subset of $\widetilde{X}$ and ϕ belongs to the set of all bounded positive definite functions on X. Then the following are equivalent:*

(i) ϕ is the limit of sums of positive definite functions associated to the representations belonging to S.

(ii)

$$\ker\pi_\phi \supseteq \bigcap_{\pi\in S} \ker(\pi).$$

(iii) There exists a completely monotonic form Φ *on* $C^*_S(X)$ *satisfying*

$$\Phi(f^o) = \int_X f(x)\phi(x)dx$$

for all f^o *belonging to* $L^1(X)/N_S$.

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