International Journal of Pure and Applied Mathematics
Volume 64 No. 1 2010, 21-29

NEARMEDIA AND THEIR REPRESENTATIONS

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Abstract: Let $\mathscr{S}$ be a nonempty set and $\mathscr{T}$ a nonempty set of functions on $\mathscr{S}$. Elements of $\mathscr{S}$ are called *states* and elements of $\mathscr{T}$ are called *token*. A token system $(\mathscr{S}, \mathscr{T})$ is called a *medium* if it satisfies:

[M_1] for any two distinct states S and T in $\mathscr{S}$, there is a concise message transforming S into T, and

[M_2] a message which is closed for some state is vacuous.

A *nearmedium* is a token system which satisfies the condition [M_1]. In this work, we present some families and elementary properties of nearmedia and their representations.

AMS Subject Classification: 05C62, 68R10
Key Words: media, nearmedia, token systems, graded families

1. Preliminaries

Let $\mathscr{S}$ be a nonempty set and $\mathscr{T}$ a nonempty set of functions on $\mathscr{S}$. The elements of $\mathscr{S}$ are called *states* and the elements of $\mathscr{T}$ are called *token*. We write $\tau(S) = S\tau$ and we assume that the identity function τ_0 is not a token. The pair $(\mathscr{S}, \mathscr{T})$ is said to be a *token system*. μ is a *reverse* of a token τ if $S\tau = T \Leftrightarrow T\mu = S$ for all distinct states S and T. Note that $\tilde{\tau}$ is unique but may not be in $\mathscr{T}$.

Received: September 18, 2009

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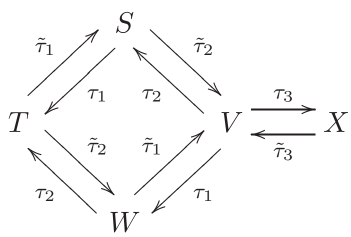
Let S and T be two states of a token system $(\mathscr{S}, \mathscr{T})$. T is *adjacent* to S if $T \neq S$ and $T = S\tau$ for some $\tau \in \mathscr{T}$. A finite composition of tokens $m = \tau_1 \ldots \tau_n$ is called a *message*. Its *content* is the set $\mathscr{C}(m) = \{\tau_1, \ldots, \tau_n\}$. For two distinct states S and T, if $Sm = T$ for some message m, then we say that m *produces* T *from* S. A message m is said to be *effective* (resp. *ineffective*) for a state S if $Sm \neq S$ (resp. $Sm = S$). It is called *stepwise effective* if $S\tau_1 \ldots \tau_i \neq S\tau_1 \ldots \tau_{i+1}$ for all $i = 1, \ldots, n-1$. A message is *inconsistent* if $\tau, \tilde{\tau} \in \mathscr{C}(m)$ and *consistent* otherwise. A *concise message for a state* S is a message which is stepwise effective for S, consistent, and any token occurs at most once in the message. A message is *closed* for a state S if it is stepwise effective and ineffective for S. A message $m = \tau_1 \ldots \tau_n$ is *vacuous* if the set of indices $\{1, \ldots, n\}$ can be partitioned into pairs $\{i, j\}$, such τ_i and τ_j are mutual reverses.

A token system $(\mathscr{S}, \mathscr{T})$ is called a *medium* if the following two axioms are satisfied.

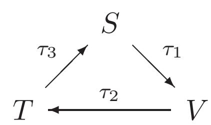
$[M_1]$ For any two distinct states S and T in $\mathscr{S}$, there is a concise message transforming S into T.

$[M_2]$ A message which is closed for some state is vacuous.

Example 1.1. The token system $(\mathscr{S}, \mathscr{T})$ with $\mathscr{S} = \{S, T, V, W, X\}$ and $\mathscr{T} = \{\tau_1, \tau_2, \tau_3, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3\}$ as shown satisfies $[M_1]$ and $[M_2]$.



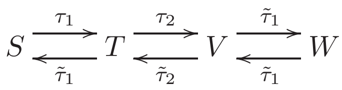
Example 1.2. The token system $(\mathscr{S}, \mathscr{T})$ with $\mathscr{S} = \{S, T, V\}$ and $\mathscr{T} = \{\tau_1, \tau_2, \tau_3\}$ as shown satisfies $[M_1]$ but not satisfies $[M_2]$.



Note. The token system $(\mathscr{S}, \mathscr{T})$ which satisfies $[M_1']$ may not has a reverse in $\mathscr{T}$.

Example 1.3. The token system $(\mathscr{S}, \mathscr{T})$ with $\mathscr{S} = \{S, T, V, W\}$ and

$\mathscr{T} = \{\tau_1, \tau_2, \tilde{\tau}_1, \tilde{\tau}_2\}$ as shown satisfies $[M_2]$ but not satisfies $[M_1]$.



Hence we obtain

Theorem 1.4. *Axioms $[M_1]$ and $[M_2]$ are independent.*

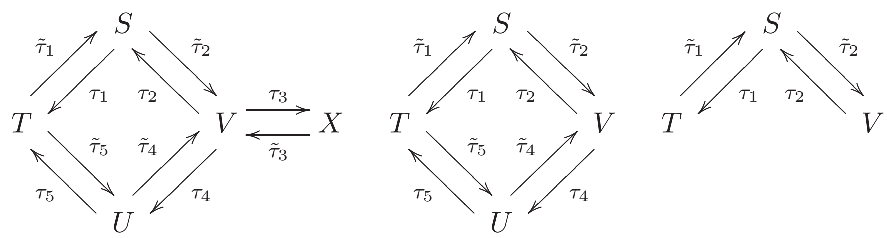
A representational approach to media theory was developed in [2] by S. Ovchinnikov. He constructed representations of media by well-graded families of sets and partial cubes and established the uniqueness of these representations. He is also responsible for a lot of work on this topic, e.g., [1], [2] and [3]. In this work, we soften the definition of a medium to a *nearmedium* defined as a token system $(\mathscr{S}, \mathscr{T})$ which satisfies the condition $[M_1]$.

Let $(\mathscr{S}, \mathscr{T})$ be a token system and $\emptyset \neq \mathscr{S}' \subseteq \mathscr{S}$. Write

$$\mathscr{T}_{\mathscr{S}'} = \{\tau \in \mathscr{T} \mid S\tau \neq S,\ S\tau \in \mathscr{S}' \text{ for some } S \text{ in } \mathscr{S}\}.$$

We call $(\mathscr{S}', \mathscr{T}_{\mathscr{S}'})$ an *induced subtoken system of* $(\mathscr{S}, \mathscr{T})$.

Example 1.5. Consider the token system $(\mathscr{S}, \mathscr{T})$ with $\mathscr{S} = \{S, T, U, V, X\}$ and $\mathscr{T} = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4, \tilde{\tau}_5\}$ as shown. For $\mathscr{S}' = \{S, T, U, V\}$, we have $\mathscr{T}_{\mathscr{S}'} = \{\tau_2, \tau_3, \tau_4, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4\}$ and for $\mathscr{S}'' = \{S, T, V\}$, we have $\mathscr{T}_{\mathscr{S}''} = \{\tau_2, \tau_3, \tilde{\tau}_2, \tilde{\tau}_3\}$. The token system $(\mathscr{S}, \mathscr{T})$, $(\mathscr{S}', \mathscr{T}')$ and $(\mathscr{S}'', \mathscr{T}'')$ are respectively shown below.



A graph is said to be *complete* if all vertices are adjacent and it is said to be *r-regular* if all vertices have degree r.

Let $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ be a nearmedia. The graph $G = (V, E)$ is said to *represent* $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ if $V = \mathscr{F}$ and two vertices of the graph are adjacent if and only if the corresponding states are adjacent in the token system.

The *cube* on a set X is the graph $\mathscr{H}(X) = (\mathscr{P}(X), E)$ and two vertices P and Q are adjacent if the symmetric difference $P\Delta Q = (P \setminus Q) \cup (Q \setminus P)$ is a singleton. A graph G is called a *partial cube* if it is isometrically embeddable into the cube $\mathscr{H}(X)$ for some set X. The following representations are due to

S. Ovchinnikov.

Theorem 1.6. (see [2]) *An induced subgraph $G = (V, E)$ of the cube $\mathscr{H}(X)$ is a partial cube if and only if V is a well-graded family of finite subsets of X. Then a shortest path in G is a line segment in $\mathscr{H}(X)$ and the graph distance function d on both $\mathscr{H}(X)$ and G is given by $d(P, Q) = |P\Delta Q|$.*

Theorem 1.7. (see [2]) *A graph G represents a medium $(\mathscr{S}, \mathscr{T})$ if and only if G is a partial cube.*

In the following two sections, we construct some token systems which are nearmedia and give elementary properties and their representations as graded families and graphs.

2. $P_n(X)$-Families

Let X be a set. For $n \geq 1$, we define

$$P_n(X) = \{S \subseteq X : S \text{ is finite and } n \text{ divides } |S|\}.$$

Let $\mathscr{G} = \{\gamma_{a_1\ldots a_n}, \tilde{\gamma}_{a_1\ldots a_n} : a_1, \ldots, a_n \text{ are distinct elements in } X\}$ be the family of functions on $P_n(X)$ given by

$$\gamma_{a_1\ldots a_n} : S \mapsto S\gamma_{a_1\ldots a_n} = \begin{cases} S \cup \{a_1 \ldots a_n\}, & \text{if } \{a_1, \ldots, a_n\} \subseteq S^c; \\ S, & \text{otherwise,} \end{cases}$$

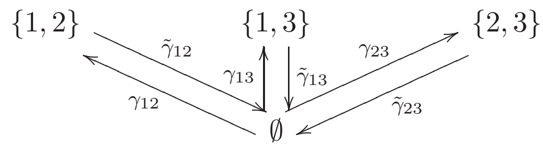
and

$$\tilde{\gamma}_{a_1\ldots a_n} : S \mapsto S\tilde{\gamma}_{a_1\ldots a_n} = \begin{cases} S \smallsetminus \{a_1 \ldots a_n\}, & \text{if } \{a_1, \ldots, a_n\} \subseteq S; \\ S, & \text{otherwise.} \end{cases}$$

It is clear that $(P_n(X), \mathscr{G})$ is a token system with $\gamma_{a_1\ldots a_n}$ and $\tilde{\gamma}_{a_1\ldots a_n}$ are mutual reverses.

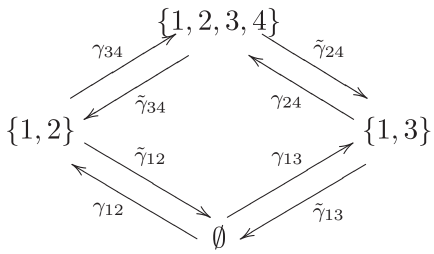
Let $\mathscr{F} \subseteq P_n(X)$. We say that $\mathscr{F}$ is *connected* if for any two sets $P, Q \in \mathscr{F}$, there is a sequence $P = R_0, R_1, \ldots, R_m = Q$ in $\mathscr{F}$ such that R_{i-1}, R_i are adjacent for $i = 1, \ldots, m$.

Example 2.1. Let $X = \{1, 2, 3\}$. Then $P_3(X) = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and $\mathscr{G} = \left\{\gamma_{12}, \gamma_{13}, \gamma_{23}, \tilde{\gamma}_{12}, \tilde{\gamma}_{13}, \tilde{\gamma}_{23}\right\}$.



Example 2.2. Let $X = \{1, 2, 3, 4\}$. Then $\mathscr{F} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 2, 3, 4\}\}$

$\subseteq P_2(X)$. Thus $\mathscr{G}_{\mathscr{F}} = \{\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{24}, \tilde{\gamma}_{12}, \tilde{\gamma}_{13}, \tilde{\gamma}_{14}, \tilde{\gamma}_{24}\}$.



Lemma 2.3. *If $\mathscr{F}$ is connected, then $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ is a token system.*

Proof. Let $P, Q \in \mathscr{F}$. Then there is a sequence $P = R_0, R_1, \ldots, R_m = Q$ such that R_{i-1}, R_i are adjacent for $i = 1, \ldots, m$. Thus $\gamma_{R_i \smallsetminus R_{i-1}} \in \mathscr{G}_{\mathscr{F}}$ for $R_{i-1} \subseteq R_i$ and $i = 1, \ldots m$ and $\tilde{\gamma}_{R_{i-1} \smallsetminus R_i} \in \mathscr{G}_{\mathscr{F}}$ for $R_i \subseteq R_{i-1}$ and $i = 1, \ldots m$. Then $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ is a token system. □

Remark 2.4. The converse of the above lemma does not hold. For $X = \{1, 2, 3, 4, 5, 6\}$, let $\mathscr{F} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\} \subseteq P_2(X)$. The family $\mathscr{F}$ is not connected, but $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ is a token system.

Lemma 2.5. *Let $\mathscr{F}$ be a graded family of subsets of some set X. If $a_1, \ldots, a_n \in X$ defines token $\tilde{\gamma}_{a_1 \ldots a_n}, \gamma_{a_1 \ldots a_n} \in \mathscr{G}_{\mathscr{F}}$, then $a_1, \ldots, a_n \in \bigcup \mathscr{F} \smallsetminus \bigcap \mathscr{F}$.*

Proof. Let $P, Q \in \mathscr{F}$ and $P\gamma_{a_1 \ldots a_n} = Q, P = Q\tilde{\gamma}_{a_1 \ldots a_n}$. Then $a_1, \ldots, a_n \notin \bigcap \mathscr{F}$, so $a_1, \ldots, a_n$ are in $\bigcup \mathscr{F} \smallsetminus \bigcap \mathscr{F}$. □

Remark 2.6. The converse of the lemma does not hold. For instance, let $X = \{1, 2, 3, 4\}$ and $\mathscr{F} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 2, 3, 4\}\} \subseteq P_2(X)$. Then $\bigcup \mathscr{F} \smallsetminus \bigcap \mathscr{F} = \{1, 2, 3, 4\}$ but $\tilde{\gamma}_{23}, \gamma_{23} \notin \mathscr{G}_{\mathscr{F}}$

Theorem 2.7. *Let X be a set. Then:*

1. $(P_n(X), \mathscr{G})$ satisfies $[M_1]$ for all $n \geq 1$.

2. $(P_1(X), \mathscr{G})$ satisfies $[M_2]$.

3. For $n \geq 2$, $(P_n(X), \mathscr{G})$ satisfies $[M_2]$ if and only if $|X| < 2n$.

Proof. 1. Let $P, Q \in P_n(X)$. Assume that $|P \cap Q| \equiv k \mod n$ for some $k = 0, \ldots, n-1$. Then $|P \cap Q| = nh + k$, $|P \smallsetminus Q| = nj - k$ and $|Q \smallsetminus P| = nl - k$ for some $h, j, k, l \in \mathbb{Z}^+, k < n$. Let $P \smallsetminus Q = \{p_1, \ldots, p_{nj-k}\}, Q \smallsetminus P = \{q_1, \ldots, q_{nl-k}\}, P \cap Q = \{a_1, \ldots, a_{ni+k}\}$ and let

$$m = \tilde{\gamma}_{p_1 \ldots p_n} \cdots \tilde{\gamma}_{p_{n(j-2)+1} \ldots p_{n(j-1)}} \tilde{\gamma}_{p_{n(j-1)+1} \ldots p_{nj-k} a_1 \ldots a_k}$$
$$\gamma_{q_1 \ldots q_n} \cdots \gamma_{q_{n(l-2)+1} \ldots q_{n(l-1)}} \gamma_{q_{n(l-1)+1} \ldots q_{(nl-k)} a_1 \ldots a_k}.$$

It is clear from the construction that m is a concise message and $Pm = Q$.

2. Let m be a closed message for a state P. That is, m is stepwise effective and ineffective for P, so $Pm = P$. Let $\gamma_x \in m$ and $\tilde{\gamma}_x \notin m$. Then $x \in Pm$ and $x \notin P$. We have a contradiction since $Pm = P$. Thus, for each token γ_x in m, there is an appearance of the reverse token $\tilde{\gamma}_x$ in m. Because m is stepwise effective, the appearances of token γ_x and $\tilde{\gamma}_x$ in m must alternate. Suppose that the sequence of appearances of γ_x and $\tilde{\gamma}_x$ begins and ends with γ_x. Since the message m is stepwise effective for state P and ineffective for this state, we must have $x \notin P$ and $x \in Pm = P$, a contradiction. Hence m is vacuous.

3. Assume that $n \geq 2$. Let $a_1, \ldots, a_{2n} \in X$. Notice that $\emptyset$, $\{a_1, \ldots, a_{2n}\}$, $\{a_1, \ldots, a_n\}$ and $\{a_n, \ldots, a_{2n-1}\}$ are in $P_n(X)$. We may choose the message $m = \gamma_{a_1 \ldots a_n} \gamma_{a_{n+1} \ldots a_{2n}} \tilde{\gamma}_{a_1 \ldots a_{n-1} a_{2n}} \tilde{\gamma}_{a_n \ldots a_{2n-1}}$. Then $\emptyset m = \emptyset$. It is clearly that m is stepwise effective and ineffective but m is not vacuous. Thus $(P_n(X), \mathscr{G})$ is not satisfy $[M_2]$. Conversely, suppose that $|X| < 2n$. The token system $(P_n(X), \mathscr{G})$. Let $S \in P_n(X)$. Then $S = \emptyset$ or $|S| = n$. Therefore a message which is stepwise effective for some state and ineffective for this state must be vacuous. □

Corollary 2.8. *$(P_n(X), \mathscr{G})$ is a nearmedium for all $n \geq 1$. In particular, $(P_1(X), \mathscr{G})$ is a medium and $(P_n(X), \mathscr{G})$ is a medium whenever $n \geq 2$ and $|X| < 2n$.*

The *distance between P and Q* is defined by

$$d(P, Q) = \begin{cases} |P \Delta Q| = |(P \smallsetminus Q) \cup (Q \smallsetminus P)|, & \text{if } P\Delta Q \text{ is finite;} \\ \infty, & \text{otherwise.} \end{cases}$$

A family $\mathscr{F} \subseteq P_1(X)$ is called *well-graded* if for any two distinct sets P and Q in $\mathscr{F}$, there is a sequence of sets $P = R_0, R_1, \ldots, R_n = Q$ such that $d(R_{i-1}, R_i) = 1$ for $i = 1, \ldots, n$ and $d(P, Q) = n$. S. Ovchinnikov established that

Theorem 2.9. (see [2]) *A token system $(\mathscr{F}, \mathscr{Q}_{\mathscr{F}})$ is a medium if and only if $\mathscr{F}$ is a well-graded family of subsets of X.*

A family $\mathscr{F} \subseteq P_n(X)$ is called *graded* if for any two distinct sets P and Q in $\mathscr{F}$, $|P \cap Q| = nh + k$, $|P \smallsetminus Q| = nj - k$, $|Q \smallsetminus P| = nl - k$ for some $h, k, j, l \in \mathbb{Z}^+, k < n$ there is a sequence of sets $P = R_0, R_1, \ldots, R_{j+l} = Q$ in $\mathscr{F}$ with, $R_{i-1} \subseteq R_i$ or $R_i \subseteq R_{i-1}$ such that $d(R_{i-1}, R_i) = n$ for $i = 1, \ldots, j + l$ and $d(P, Q) = n(j + l) - 2k$.

Theorem 2.10. *Let $\mathscr{F} \subseteq P_n(X)$. A token system $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ is a nearmedium if and only if $\mathscr{F}$ is a graded family of X.*

Proof. Assume that $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ satisfies $[M_1]$. Let $P, Q \in \mathscr{F}$. Then $Pm = Q$ for some concise message m. By the proof of Theorem 2.7 (1), we have $|P \cap Q| = nh + k,\ |P \smallsetminus Q| = nj - k,\ |Q \smallsetminus P| = nl - k$ for some $n, j, k, l \in \mathbb{Z}^+,\ k < n$. Let $P \smallsetminus Q = \{p_1, \ldots, p_{nj-k}\}, Q \smallsetminus P = \{q_1, \ldots, q_{nl-k}\}, P \cap Q = \{a_1, \ldots, a_{ni+k}\}$ and let

$$m = \tilde{\gamma}_{p_1 \ldots p_n} \cdots \tilde{\gamma}_{p_{n(j-2)+1} \cdots p_{n(j-1)}} \tilde{\gamma}_{p_{n(j-1)+1} \cdots p_{nj-k} a_1 \ldots a_k}$$
$$\gamma_{q_1 \ldots q_n} \cdots \gamma_{q_{n(l-2)+1} \cdots q_{n(l-1)}} \gamma_{q_{n(l-1)+1} \cdots q_{(nl-k)} a_1 \ldots a_k}.$$

Define a sequence of sets in $\mathscr{F}$ as follows:

$$R_0 = P, R_1 = R_0 \tilde{\gamma}_{p_1 \ldots p_n}, \ldots,$$
$$R_j = R_{j-1} \tilde{\gamma}_{p_{n(j-1)+1} \cdots p_{nj-k} a_1 \ldots a_k} \gamma_{q_1 \ldots q_n}, \ldots,$$
$$R_{j+l} = R_{j+l-1} \gamma_{q_{n(l-1)+1} \cdots q_{(nl-k)} a_1 \ldots a_k}.$$

It is clear that $d(R_{i-1}, R_i) = n$ for $i = 1, \ldots, j + l$ and $d(P, Q) = n(j + l) - 2k$. Since m is stepwise effective, $R_{i-1} \subseteq R_i$ or $R_i \subseteq R_{i-1}$.

Conversely, assume that $\mathscr{F}$ is a graded family. Let $P, Q \in \mathscr{F} \subseteq P_n(X)$. Then $|P \smallsetminus Q| = nj - k,\ |Q \smallsetminus P| = nl - k$, there is a sequence of sets $P = R_0, R_1, \ldots, R_{j+l} = Q,\ R_{i-1} \subseteq R_i$ or $R_i \subseteq R_{i-1}$ such that $d(R_{i-1}, R_i) = n$ for $i = 1, \ldots, j + l$ and $d(P, Q) = n(j + l) - 2k$. Define a message m as

$$m = \tau_1 \ldots \tau_{j+l},\ \tau_i = \begin{cases} \gamma_{R_i \smallsetminus R_{i-1}}, & R_{i-1} \subseteq R_i; \\ \tilde{\gamma}_{R_{i-1} \smallsetminus R_i}, & R_i \subseteq R_{i-1}. \end{cases}$$

Then $Pm = Q$. Since $R_{i-1} \subseteq R_i$ or $R_i \subseteq R_{i-1}$ for $i = 1, \ldots, j + l$, m is stepwise effective. Suppose that $\tau,\ \tilde{\tau} \in \mathscr{C}(m)$. Hence $d(P, Q) \leq n(j + l - 2) = n(j + l) - 2n \leq n(j + l) - 2k$, a contradicting to $d(P, Q) = n(j + l) - 2k$. Then m is consistent. Suppose that τ occurs twice in m. So $m = m_1 \tau m_2 \tau m_3$ with each m_1, m_3 is a stepwise effective and consistent message or empty. Suppose $\tau = \gamma_{a_1 \ldots a_n}$. Thus $\{a_1, \ldots, a_n\}$ is removed by the message m_2. Therefore $d(P, Q) \leq n(j + l - 2) = n(j + l) - 2n \leq n(j + l) - 2k$, a contradiction to $d(P, Q) = n(j + l) - 2k$. The proof of $\tau = \tilde{\gamma}_{a_1 \ldots a_n}$ is similar. Hence any token occurs at most once in the message m. □

3. $Q_n(X)$-Families

Let X be a set. For $n \geq 1$, we define

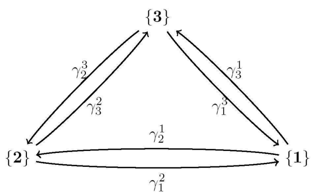
$$Q_n(X) = \{S \subseteq X : |S| = n\}.$$

Let $\mathscr{G} = \{\gamma_{a_i}^{a_j} : a_i, a_j \text{ are distinct elements in } X\}$ be the family of functions on $Q_n(X)$ given by

$$\gamma_{a_i}^{a_j} : S \mapsto S\gamma_{a_i}^{a_j} = \begin{cases} (S \cup \{a_i\}) \smallsetminus \{a_j\}, & \text{if } a_i \notin S \text{ and } a_j \in S; \\ S, & \text{otherwise.} \end{cases}$$

It is clear that $(Q_n(X), \mathscr{G})$ is a token system and $\gamma_{a_i}^{a_j}$ and $\gamma_{a_j}^{a_i}$ are mutual reverses.

Example 3.1. Let $X = \{1, 2, 3\}$. Then $Q_1(X) = \big\{\{1\}, \{2\}, \{3\}\big\}$ and $\mathscr{G} = \big\{\gamma_1^2, \gamma_1^3, \gamma_2^3, \gamma_2^1, \gamma_3^1, \gamma_3^2\big\}$. The media $(Q_1(X), \mathscr{G})$ is shown below.



Example 3.2. Let $X = \{1, 2, 3, 4\}$. Then $Q_2(X) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ and $\mathscr{G} = \big\{\gamma_1^2, \gamma_1^3, \gamma_1^4, \gamma_2^3, \gamma_2^4, \gamma_3^4, \gamma_2^1, \gamma_3^1, \gamma_4^1, \gamma_3^2, \gamma_4^2, \gamma_4^3\big\}$.

Let $\mathscr{F} \subseteq Q_n(X)$ and $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ be an induced token system of $(Q_n(X), \mathscr{F})$.

Lemma 3.3. *If $\mathscr{F}$ is connected, then $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ is a token system.*

Proof. The set of state is $\mathscr{F}$. Let $P, Q \in \mathscr{F}$, then there is a sequence $P = R_0, R_1, \ldots, R_m = Q$ such that R_{i-1}, R_i are adjacent for $i = 1, \ldots, m$. Then $\gamma_{R_i \smallsetminus R_{i-1}}^{R_{i-1} \smallsetminus R_i}$ and $\gamma_{R_{i-1} \smallsetminus R_i}^{R_i \smallsetminus R_{i-1}}$ are in $\mathscr{G}_{\mathscr{F}}$ for R_{i-1} and R_i are adjacent and $i = 1, \ldots, m$. Then $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ is a token system. □

Remark 3.4. The converse of the above lemma does not hold. For example, let $X = \{1, 2, 3, 4, 5, 6\}$ and $\mathscr{F} = \big\{\{1, 2\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\big\} \subseteq Q_2(X)$. The family $\mathscr{F}$ is not connected, but $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ is a token system.

Lemma 3.5. *Let X be a set. Then $(Q_n(X), \mathscr{G})$ satisfies $[M_1]$ for all $n \geq 1$.*

Proof. Let $P, Q \in Q_n(X)$. Assume that $|P \cap Q| = m$. Then $|P \smallsetminus Q| = n - m = |Q \smallsetminus P|$. Let $P \smallsetminus Q = \{p_1, \ldots, p_{n-m}\}, Q \smallsetminus P = \{q_1, \ldots, q_{n-m}\}$ and $m = \gamma_{q_1}^{p_1} \ldots \gamma_{q_{n-m}}^{p_{n-m}}$. It is clear from construction that m is a concise message and $Pm = Q$. □

Corollary 3.6. *Let $P, Q \in Q_n(X)$. If $|P \Delta Q| = 2n$, then there is a sequence of sets $P = R_0, R_1, \ldots, R_n = Q$ in $Q_n(X)$ such that $d(R_{i-1}, R_i) = 2$ for $i = 1, \ldots, n$ and $d(P, Q) = 2n$.*

Proof. By the above lemma, we have a message $m = \gamma_{q_1}^{p_1} \dots \gamma_{q_n}^{p_n}$. Define a sequence of sets in $Q_n(X)$ as follows:

$$R_0 = P, R_1 = R_0\gamma_{q_1}^{p_1}, \dots, R_n = R_{n-1}\gamma_{q_n}^{p_n} = Q.$$

It is clear that $d(R_{i-1}, R_i) = 2$ for $i = 1, \dots, n$. □

Theorem 3.7. *Let $|X| = m$. If G is the graph represented $(Q_n(X), \mathscr{G})$, then G is $n(m-n)$-regular.*

Proof. Let P, Q be adjacent vertices in G and let $P = \{p_1, \dots, p_n\} \in Q_n(X)$. Observe that if vertices P and Q are adjacent, then $|P \Delta Q| = 2, |P \smallsetminus Q| = 1 = |Q \smallsetminus P|$ and $|P \cap Q| = n - 1$. Thus all neighbours of P are of the form $(P \smallsetminus \{p_i\}) \cup \{x\}$ for some $i \in \{1, 2, \dots, n\}$ and $x \in X \smallsetminus P$. Hence we have $n(m-n)$ choices of neighbours of P, and so G is $n(m-n)$-regular. □

Remark 3.8. Every state in $(Q_1(X), \mathscr{G})$ is adjacent and so its graph is a complete graph.

Acknowledgments

This work grows out of the second author's master thesis at Chulalongkorn University written under the direction of the first author to which he expresses his gratitude.

References

[1] D. Eppstein, J.-Cl. Falmagne, S. Ovchinnikov, *Media Theory: Interdisciplinary Applied Mathematics*, Springer, Berlin (2008).

[2] S. Ovchinnikov, Media theory: Representations and examples, *Discrete Appl. Math.*, **156** (2008), 1197-1219.

[3] S. Ovchinnikov, Cubical token systems, *J. Math. Soc.*, **56** (2008), 149-165.