International Journal of Pure and Applied Mathematics

Volume 64 No. 1 2010, 31-41

ON THE OPERATOR $\circledast_B^k$ RELATED TO THE BESSEL HEAT EQUATION

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Abstract: In this paper, we study the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \circledast_B^k u(x,t)$$

with the initial condition $u(x,0) = f(x)$ for $x \in \mathbb{R}_n^+$, where the operator $\circledast_B^k$ is defined by

$$\circledast_B^k = \left[\left(B_{x_1} + \cdots + B_{x_p}\right)^3 + \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^3\right]^k,$$

$p+q = n$ is the dimension of the space $\mathbb{R}_n^+ = \{x = (x_1, x_2, \ldots, x_n) : x_1 > 0, x_2 > 0, \ldots, x_n > 0\}$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $i = 1, 2, \ldots, n$, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}_n^+ \times (0,\infty)$, $f(x)$ is a generalized function, k is a positive integer and c is a positive constant. We obtain the solution of such equation which is related to the spectrum and the heat kernel. Moreover, such Bessel heat kernel has interesting properties and also related to the kernel of an extension of the heat equation.

AMS Subject Classification: 46F10, 35K05

Key Words: heat kernel, Dirac-delta distribution, spectrum, Bessel operator

Received: May 12, 2010

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1. Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \triangle u(x,t) \tag{1.1}$$

with the initial condition $u(x,0) = f(x)$, where $\triangle$ is the Laplace operator which is defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, we obtain the solution in the convolution form $u(x,t) = E(x,t) * f(x)$, where $E(x,t)$ is the heat kernel defined by

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} e^{-\frac{|x|^2}{4c^2 t}}, \tag{1.2}$$

$|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [1, p. 208-209]. We can extend (1.1) to the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \Box u(x,t), \tag{1.3}$$

where $\Box$ is the ultra-hyperbolic operator which is defined by

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2},$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$. We obtain the heat kernel in the form

$$E(x,t) = \frac{(i)^q}{(4c^2\pi t)^{n/2}} e^{\frac{-1}{4c^2 t}\left(\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2\right)}, \tag{1.4}$$

where $i = \sqrt{-1}$ and $\sum_{i=1}^{p} x_i^2 > \sum_{j=p+1}^{p+q} x_j^2$, see [4].

On the other hand, the diamond heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \Diamond u(x,t), \tag{1.5}$$

where $\Diamond$ is the diamond operator which is defined by

$$\Diamond = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2,$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, we obtain the heat kernel in the form

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega f(x-y) e^{c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right) + i(\xi,y)} d\xi$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of the kernel $E(x,t)$ for any fixed $t > 0$, see [2].

Furthermore, Yildirim et al [8] first introduced the Bessel diamond operator $\Diamond_B$ which is defined by

$$\Diamond_B = \left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^2 - \left(B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}}\right)^2,$$

where $p+q = n$ is the dimension of the space $\mathbb{R}_n^+ = \{x = (x_1, x_2, \ldots, x_n) : x_1 > 0, x_2 > 0, \ldots, x_n > 0\}$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$ for $i = 1, 2, \ldots, n$. The Bessel diamond operator can also be expressed in the form $\Diamond_B = \triangle_B \Box_B = \Box_B \triangle_B$, where $\triangle_B$ is the Laplace-Bessel operator which is defined by

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_n}, \tag{1.6}$$

and $\Box_B$ is the Bessel ultra-hyperbolic operator which is defined by

$$\Box_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \cdots - B_{x_{p+q}}. \tag{1.7}$$

Now, the purpose of this article is to study the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \circledast_B^k u(x,t) \tag{1.8}$$

with the initial condition $u(x,0) = f(x)$ for $x \in \mathbb{R}_n^+$, where the operator $\circledast_B^k$ is defined by

$$\begin{aligned} \circledast_B^k &= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^3 + \left(\sum_{i=p+1}^{p+q} B_{x_i}\right)^3\right]^k \\ &= \triangle_B^k \left[\triangle_B^2 - \frac{3}{4}\left(\triangle_B + \Box_B\right)\left(\triangle_B - \Box_B\right)\right]^k \\ &= \left[\frac{3}{4}\triangle_B \Box_B^2 + \frac{1}{4}\triangle_B^3\right]^k = \left[\frac{3}{4}\Diamond_B \Box_B + \frac{1}{4}\triangle_B^3\right]^k, \end{aligned} \tag{1.9}$$

$p+q = n$ is the dimension of the space $\mathbb{R}_n^+$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \ldots, n$, $u(x,t)$ is an unknown function, $f(x)$ is a generalized function, k is a positive integer and c is a positive constant. Moreover, such Bessel heat kernel has interesting properties and also related to the kernel of an extension of the heat equation. We obtain $u(x,t) = E(x,t) * f(x)$, the symbol $*$ is the B-convolution which is defined (2.1), as a solution of the heat equation (1.8) which satisfies the initial condition $u(x,0) = f(x)$, where

$E(x,t)$

$$= C_v \int_{\Omega^+} e^{(-1)^k c^2 t\left[(y_1^2+\cdots+y_p^2)^3+(y_{p+1}^2+\cdots+y_{p+q}^2)^3\right]^k} \prod_{i=1}^{n} j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy, \quad (1.10)$$

$\Omega^+ \subset \mathbb{R}_n^+$ is the spectrum of the kernel $E(x,t)$ for any fixed $t > 0$, and the function $j_{v_i-\frac{1}{2}}(x_i y_i)$ is the normalized Bessel function.

2. Preliminaries

The generalized shift operator T_x^y has the following form (see [3]),

$$T_x^y = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi(s_1, \ldots, s_n) \left(\prod_{i=1}^{n} \sin^{2v_i-1}\theta_i\right) d\theta_1 \cdots d\theta_n,$$

where $s_i^2 = x_i^2 + y_i^2 - 2x_i y_i \cos\theta_i$, $x, y \in \mathbb{R}_n^+$ and $C_v^* = \prod_{i=1}^{n} \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator (see [3]),

$$\begin{aligned} \frac{d^2\varphi}{dx_i^2} + \frac{2v_i}{x_i}\frac{d\varphi}{dx_i} &= \frac{d^2\varphi}{dy_i^2} + \frac{2v_i}{y_i}\frac{d\varphi}{dy_i}, \\ \varphi(x_i, 0) &= f(x), \\ \varphi_{y_i}(x_i, 0) &= 0, \end{aligned}$$

for $x_i, y_i \in \mathbb{R}^+$. The convolution operator determined by T_x^y is as follows

$$(f * \varphi)(x) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy. \quad (2.1)$$

Convolution in (2.1) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator,

(a) $T_x^y \cdot 1 = 1$.

(b) $T_x^0 \cdot f(x) = f(x)$.

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function for $x \in \mathbb{R}_n^+$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1$,

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy.$$

(e) $(f * g)(x) = (g * f)(x)$.

The Fourier-Bessel transformation and its inverse transformation are defined as follows (see [7]),

$$(F_B f)(x) = C_v \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^n j_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} \right) dy,$$

and

$$(F_B^{-1} f)(x) = (F_B f)(-x),\ C_v = \left(\prod_{i=1}^n 2^{v_i - \frac{1}{2}} \Gamma \left(v_i + \frac{1}{2} \right) \right)^{-1},$$

where $j_{v_i - \frac{1}{2}}(x_i y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. There are the following equalities for Fourier-Bessel transformation (see [7]),

$$F_B \delta(x) = 1 \ \text{ and } \ F_B(f * g)(x) = F_B f(x) \cdot F_B g(x).$$

Definition 2.1. The spectrum of the kernel $E(x,t)$ defined by (1.10) is the bounded support of the Fourier-Bessel transform $F_B E(x,t)$ for any fixed $t > 0$.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$ and denote

$$\Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 > 0\}$$

to be the set of an interior of the forward cone and $\overline{\Gamma}_+$ denoted the closure of Γ_+. Let Ω^+ be the spectrum of $E(x,t)$ defined by *(1.10)* and $\Omega^+ \subset \overline{\Gamma}_+$. Let $F_B E(y,t)$ be the Fourier-Bessel transform of $E(x,t)$ which is defined by

$$F_B E(y,t) = \begin{cases} e^{(-1)^k c^2 t \left[(y_1^2 + \cdots + y_p^2)^3 + (y_{p+1}^2 + \cdots + y_{p+q}^2)^3 \right]^k} & \text{for } x \in \Omega^+; \\ 0 & \text{for } x \notin \Omega^+. \end{cases} \tag{2.2}$$

Lemma 2.1. *For $t, v > 0$ and $x, y \in \mathbb{R}^+$, we have*

$$\int_0^\infty e^{-c^2 x^2 t} x^{2v} dx = \frac{\Gamma(v)}{2 c^{2v+1} t^{v + \frac{1}{2}}}$$

and

$$\int_0^\infty e^{-c^2 x^2 t} j_{v - \frac{1}{2}}(xy) x^{2v} dx = \frac{\Gamma(v + \frac{1}{2}) e^{\frac{-y^2}{4c^2 t}}}{2 (c^2 t)^{v + \frac{1}{2}}},$$

where c is a positive constant.

Lemma 2.2. (Fourier-Bessel Transform of the Operator $\Box_B^k$)

$$F_B\Box_B^k u(x) = (-1)^k V_1^k(x) F_B u(x),$$

where $V_1^k(x) = (x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2)^k$.

The proof of this lemma is given in [5].

Lemma 2.3. (Fourier-Bessel Transform of the Operator $\triangle_B^k$)

$$F_B\triangle_B^k u(x) = (-1)^k V_2^k(x) F_B u(x),$$

where $V_2^k(x) = (x_1^2 + x_2^2 + \cdots + x_n^2)^k$.

The proof of this lemma can be obtained by Lemma 2.2.

Lemma 2.4. (Fourier-Bessel Transform of the Operator $\Diamond_B^k$)

$$F_B\Diamond_B^k u(x) = V_3^k(x) F_B u(x),$$

where $V_3^k(x) = \Big((\sum_{i=1}^p x_i^2)^2 - (\sum_{j=p+1}^{p+q} x_j^2)^2\Big)^k$.

The proof of this lemma is given in [5].

Lemma 2.5. (Fourier-Bessel Transform of the Operator $\circledast_B^k$)

$$F_B \circledast_B^k u(x) = (-1)^k V^k(x) F_B u(x),$$

where $V^k(x) = \Big((\sum_{i=1}^p x_i^2)^3 + (\sum_{j=p+1}^{p+q} x_j^2)^3\Big)^k$.

Proof. Following Lemmas 2.2, 2.3 and 2.4, we have

$$F_B(\circledast_B^k u)(x)$$

$$= C_v \int_{\mathbb{R}_n^+} \circledast_B^k u(y) \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy$$

$$= C_v \int_{\mathbb{R}_n^+} \left(\frac{3}{4}\Diamond_B\Box_B + \frac{1}{4}\triangle_B^3\right)^k u(y) \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy$$

$$= C_v \int_{\mathbb{R}_n^+} \sum_{r=0}^k \binom{k}{r} \left(\frac{3}{4}\Diamond_B\Box_B\right)^{k-r} \left(\frac{1}{4}\triangle_B^3\right)^r u(y) \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy$$

$$= \sum_{r=0}^k \binom{k}{r} \left(\frac{3}{4}\right)^{k-r} \left(\frac{1}{4}\right)^r C_v \int_{\mathbb{R}_n^+} \Diamond_B^{k-r} g(y) \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy;$$

where $g(x) = \Box_B^{k-r}\triangle_B^{3r} u(x)$

$$= \sum_{r=0}^k \binom{k}{r} \left(\frac{3}{4}\right)^{k-r} \left(\frac{1}{4}\right)^r F_B(\Diamond_B^{k-r} g)(x)$$

$$= \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \left(\frac{3}{4}\right)^{k-r} \left(\frac{1}{4}\right)^{r} V_3^{k-r}(x) F_B g(x),$$

$$= \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \left(\frac{3}{4}\right)^{k-r} \left(\frac{1}{4}\right)^{r} V_3^{k-r}(x) F_B (\Box_B^{k-r} h)(x);$$

where $h(x) = \triangle_B^{3r} u(x)$,

$$= \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \left(\frac{3}{4}\right)^{k-r} \left(\frac{1}{4}\right)^{r} (-1)^{k-r} V_3^{k-r}(x) V_1^{k-r}(x) F_B h(x)$$

$$= \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \left(\frac{3}{4}\right)^{k-r} \left(\frac{1}{4}\right)^{r} (-V_3(x)V_1(x))^{k-r} F_B (\triangle_B^{3r} u)(x)$$

$$= \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \left(\frac{3}{4}\right)^{k-r} \left(\frac{1}{4}\right)^{r} (-V_3(x)V_1(x))^{k-r} (-1)^{3r} V_2^{3r}(x) F_B u(x)$$

$$= \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \left(\frac{-3}{4} V_3(x) V_1(x)\right)^{k-r} \left(\frac{-1}{4} V_2^3(x)\right)^{r} F_B u(x)$$

$$= \left(-\frac{3}{4} V_3(x) V_1(x) - \frac{1}{4} V_2^3(x)\right)^{k} F_B u(x)$$

$$= (-1)^k \left(\frac{3}{4} V_3(x) V_1(x) + \frac{1}{4} V_2^3(x)\right)^{k} F_B u(x)$$

$$= (-1)^k \left[\left(\sum_{i=1}^{p} x_i^2 \right)^3 + \left(\sum_{j=p+1}^{p+q} x_j^2 \right)^3 \right]^k F_B u(x).$$

This completes the proof. □

Lemma 2.6. *Let the operator L be defined by*

$$L = \frac{\partial}{\partial t} - c^2 \circledast_B^k, \tag{2.3}$$

where $\circledast_B^k$ is defined by

$$\circledast_B^k = [(B_{x_1} + B_{x_2} + \cdots + B_{x_p})^3 + (B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}})^3]^k,$$

$$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i},$$

$p + q = n$ is the dimension of the space $\mathbb{R}_n^+$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $i = 1, 2, \ldots, n$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$, k is a positive integer and c is a positive constant. Then

$$E(x,t) = C_v \int_{\Omega^+} e^{(-1)^k c^2 t\left[(y_1^2+\cdots+y_p^2)^3+(y_{p+1}^2+\cdots+y_{p+q}^2)^3\right]^k} \prod_{i=1}^{n} j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy \quad (2.4)$$

is the elementary solution of the operator L *for the spectrum* $\Omega^+ \subset \mathbb{R}_n^+$, $t > 0$.

Proof. Let $E(x,t)$ be the kernel or elementary solution of L operator and let δ be the Dirac-delta distribution. Then we obtain

$$\frac{\partial}{\partial t} E(x,t) - c^2 \circledast_B^k E(x,t) = \delta(x,t).$$

Applying the Fourier-Bessel transform to the both sides of the above equation and using Lemma 2.5, we have

$$\frac{\partial}{\partial t} F_B E(x,t) - (-1)^k c^2 \left[(x_1^2+\cdots+x_p^2)^3+(x_{p+1}^2+\cdots+x_{p+q}^2)^3\right]^k F_B E(x,t) = \delta(t).$$

Hence, we obtain

$$F_B E(x,t) = H(t) e^{(-1)^k c^2 t\left[(x_1^2+\cdots+x_p^2)^3+(x_{p+1}^2+\cdots+x_{p+q}^2)^3\right]^k},$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ holds for $t \geq 0$,

$$F_B E(x,t) = e^{(-1)^k c^2 t\left[(x_1^2+\cdots+x_p^2)^3+(x_{p+1}^2+\cdots+x_{p+q}^2)^3\right]^k}$$

which has been shown already by (2.2). By inverse Fourier-Bessel transform, we have

$$E(x,t) = C_v \int_{\mathbb{R}_n^+} e^{(-1)^k c^2 t\left[(y_1^2+\cdots+y_p^2)^3+(y_{p+1}^2+\cdots+y_{p+q}^2)^3\right]^k} \prod_{i=1}^{n} j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy.$$

Since Ω^+ is the spectrum of $E(x,t)$, we obtain

$$E(x,t) = C_v \int_{\Omega^+} e^{(-1)^k c^2 t\left[(y_1^2+\cdots+y_p^2)^3+(y_{p+1}^2+\cdots+y_{p+q}^2)^3\right]^k} \prod_{i=1}^{n} j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy.$$

This completes the proof. □

3. Main Results

Theorem 3.1. *Given the equation*

$$\frac{\partial}{\partial t} u(x,t) = c^2 \circledast_B^k u(x,t) \quad (3.1)$$

with initial condition

$$u(x,0) = f(x), \quad (3.2)$$

where $\circledast_B^k$ *is defined by*

$$\circledast_B^k = [(B_{x_1} + B_{x_2} + \cdots + B_{x_p})^3 + (B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}})^3]^k,$$

$$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i},$$

$2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $i = 1, 2, \ldots, n$, $p + q = n$ *is the dimension of the space* $\mathbb{R}_n^+ = \{x = (x_1, x_2, \ldots, x_n) : x_1 > 0, x_2 > 0, \ldots, x_n > 0\}$, $u(x,t)$ *is an unknown function for* $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}_n^+ \times (0, \infty)$, $f(x)$ *is a generalized function,* k *is a positive integer and* c *is a positive constant. Then*

$$u(x,t) = E(x,t) * f(x) \tag{3.3}$$

is a solution of the equation (3.1) *which satisfies initial condition* (3.2), *where* $E(x,t)$ *is given by* (2.4).

Proof. Taking the Fourier-Bessel transform to the both sides of the equation (3.1) and using Lemma 2.5, we obtain

$$\frac{\partial}{\partial t}F_B u(x,t) = (-1)^k c^2 \left[(x_1^2 + \cdots + x_p^2)^3 + (x_{p+1}^2 + \cdots + x_{p+q}^2)^3\right]^k F_B u(x,t). \tag{3.4}$$

Next, we consider the initial condition (3.2), then we have the following equality for (3.4)

$$u(x,t) = f(x) * F_B^{-1} e^{(-1)^k c^2 t\left[(x_1^2+\cdots+x_p^2)^3+(x_{p+1}^2+\cdots+x_{p+q}^2)^3\right]^k}. \tag{3.5}$$

Now, if we use the B-convolution and the inverse Fourier-Bessel transformation, then we have

$$\begin{aligned} &u(x,t) \\ &= f(x) * F_B^{-1} e^{(-1)^k c^2 t\left[((x_1^2+\cdots+x_p^2)^3+x_{p+1}^2+\cdots+x_{p+q}^2)^3\right]^k} \\ &= \int_{\mathbb{R}_n^+} F_B^{-1} e^{(-1)^k c^2 t\left[(y_1^2+\cdots+y_p^2)^3+(y_{p+1}^2+\cdots+y_{p+q}^2)^3\right]^k} T_x^y f(x) \prod_{i=1}^n y_i^{2v_i}\, dy \\ &= \int_{\mathbb{R}_n^+} \left[C_v \int_{\mathbb{R}_n^+} e^{c^2 t(-1)^k V^k(z)} \prod_{i=1}^n j_{v_i-\frac{1}{2}}(y_i z_i) z_i^{2v_i}\, dz\right] T_x^y f(x) \prod_{i=1}^n y_i^{2v_i}\, dy, \end{aligned} \tag{3.6}$$

where $V(z) = (z_1^2 + \cdots + z_p^2)^3 + (z_{p+1}^2 + \cdots + z_{p+q}^2)^3$. Set

$$\begin{aligned} &E(x,t) \\ &\quad = C_v \int_{\mathbb{R}_n^+} e^{(-1)^k c^2 t\left[(y_1^2+\cdots+y_p^2)^3+(y_{p+1}^2+\cdots+y_{p+q}^2)^3\right]^k} \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i}\, dy. \end{aligned} \tag{3.7}$$

Since the integral in (3.7) is divergent, therefore we choose $\Omega^+ \subset \mathbb{R}_n^+$ be the spectrum of the kernel $E(x,t)$ and by (2.4), we have

$$E(x,t) = C_v \int_{\mathbb{R}_n^+} e^{(-1)^k c^2 t\left[(y_1^2+\cdots+y_p^2)^3+(y_{p+1}^2+\cdots+y_{p+q}^2)^3\right]^k} \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy$$

$$= C_v \int_{\Omega^+} e^{(-1)^k c^2 t\left[(y_1^2+\cdots+y_p^2)^3+(y_{p+1}^2+\cdots+y_{p+q}^2)^3\right]^k} \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy. \quad (3.8)$$

Thus (3.6) can be written in the convolution form $u(x,t) = E(x,t) * f(x)$. Moreover, since $E(x,t)$ exists, we see that

$$\lim_{t\to 0} E(x,t) = C_v \int_{\Omega^+} \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy$$

$$= C_v \int_{\mathbb{R}_n^+} \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy = \delta(x), \quad \text{for } x \in \mathbb{R}_n^+, \quad (3.9)$$

holds (see [6]). Thus the solution of the equation (3.1) is $u(x,t) = E(x,t) * f(x)$, then we have

$$u(x,0) = \lim_{t\to 0} u(x,t) = \lim_{t\to 0} E(x,t) * f(x) = \delta(x) * f(x) = f(x)$$

which satisfies the equation (3.2). This complete the proof. □

Theorem 3.2. *The kernel $E(x,t)$ is defined by (3.7) has the following properties:*

(i) $E(x,t) \in C^\infty(\mathbb{R}_n^+ \times (0,\infty))$ *the space of continuous with infinitely differentiable,*

(ii) $(\frac{\partial}{\partial t} - c^2 \circledast_B^k) E(x,t) = 0$ *for* $x \in \mathbb{R}_n^+$ *and* $t > 0$,

(iii) $\lim_{t\to 0} E(x,t) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$.

Proof. (i) From (3.8) and

$$\frac{\partial^n}{\partial t^n} E(x,t) = C_v \int_{\Omega^+} \frac{\partial^n}{\partial t^n} e^{(-1)^k c^2 t\left[(y_1^2+\cdots+y_p^2)^3+(y_{p+1}^2+\cdots+y_{p+q}^2)^3\right]^k} \times \prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i) y_i^{2v_i} dy, \quad (3.10)$$

we have $E(x,t) \in C^\infty$ for $x \in \mathbb{R}_n^+$ and $t > 0$.

(ii) We have $u(x,t) = E(x,t)$ because $u(x,t) = E(x,t) * f(x)$ holds. We use the fact $f(x) = \delta(x)$ by the Fourier-Bessel transform. Then we obtain $\left(\frac{\partial}{\partial t} - c^2 \circledast_B^k\right) E(x,t) = 0$ by direct computation.

(iii) This case is obvious by (3.9). □

Acknowledgments

This work was granted by Office of the Higher Education Commission. Mr. Somboon Niyom was supported by CHE Ph.D. scholarship and Graduate School, Chiang Mai University, Thailand.

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