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DIFFERENTIAL GEOMETRY OF MICROLINEAR FRÖLICHER SPACES I

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Abstract: The central object of synthetic differential geometry is microlinear spaces. In our previous paper (Microlinearity in Frölicher spaces - beyond the regnant philosophy of manifolds, *International Journal of Pure and Applied Mathematics*, **60** (2010), 15-24) we have emancipated microlinearity from within well-adapted models to Frölicher spaces. Therein we have shown that Frölicher spaces which are microlinear as well as Weil exponentiable form a Cartesian closed category. To make sure that such Frölicher spaces are the central object of infinite-dimensional differential geometry, we develop the theory of vector fields on them in this paper. Our principal result is that all vector fields on such a Frölicher space form a Lie algebra.

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1. Introduction

It is often forgotten that Newton, Leibniz, Euler and many other mathematicians in the 17-th and 18-th centuries developed differential calculus and analysis by using nilpotent infinitesimals without any recourse to limits. It is in the 19-th century, in the midst of the Industrial Revolution in Europe, that nilpotent infinitesimals were overtaken by so-called $\varepsilon - \delta$ arguments. In the middle of the 20-th century moribund nilpotent infinitesimals were revived by

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Grothendieck in algebraic geometry and by Lawvere in differential geometry. Differential geomery using nilpotent infinitesimals consistently and coherently is now called *synthetic differential geometry*, since it prefers synthetic arguments to dull calculations, as was the case in ancient Euclidean geometry. The central object of study under synthetic differential geometry is microlinear spaces, while the central object of study under orthodox differential geometry has been finite-dimensional smooth manifolds. Since we can not expect to see nilpotent infinitesimals in our real world, model theory of synthetic differential geometry was developed vigorously by Dubuc and others around 1980 by using techniques of topos theory. For a standard reference on model theory of synthetic differential geometry, the reader is referred to Kock's well-written [11].

It is well known that the category of smooth manifolds, whether finite-dimensional or infinite-dimensional (modelled after Hilbert spaces, Banach spaces, Fréchet spaces or, most generally, convenient vector spaces) is by no means Cartesian closed, while the category of Frölicher spaces and smooth mappings among them is certainly so. Although nilpotent infinitesimals are not visible in our real world, which has harassed such philosophers of the 17-th and 18-th centuries as extremely skeptical Berkeley, the notion of Weil functor is applicable to both finite-dimensional and infinite-dimensional smooth manifolds. The notion of Weil functor has been generalized in our previous paper [20] so that it is applicable to Frölicher spaces. Therein we have delineated the class of Frölicher spaces which give credence to the illusion that Weil functors are the exponentiation by infinitesimal objects in the shade. Such Frölicher spaces, which were shown to form a Cartesian closed category, are called *Weil exponentiable*. In our subsequent paper [21] we have finally succeeded in externalizing the notion of microlinearity, so that it is applicable to Frölicher spaces in the real world. Therein we have shown that Frölicher spaces that are Weil exponentiable and microlinear form a Cartesian closed category. We would like to hold that such Frölicher spaces are the central object of study for infinite-dimensional differential geometry.

The aim in our series of papers, the first of which is the present paper under your eyes, is to develop differential geometry for Frölicher spaces that are Weil exponentiable and microlinear. The present paper is devoted to tangent spaces and vector fields. Its principal result is that vector fields on such a Frölicher space form a Lie algebra. The reader should note that our present mission for microlinear Frölicher spaces is not so easy as that for smooth manifolds in orthodox differential geometry, since coordinates are no longer available nor are we admitted to identity vector fields with their associated differential operators.

2. Preliminaries

2.1. Frölicher Spaces

Frölicher and his followers have vigorously and consistently developed a general theory of smooth spaces, often called *Frölicher spaces* for his celebrity, which were intended to be the *underlying set theory* for infinite-dimensional differential geometry in a sense. A Frölicher space is an underlying set endowed with a class of real-valued functions on it (simply called *structure functions*) and a class of mappings from the set $\mathbf{R}$ of real numbers to the underlying set (called *structure curves*) subject to the condition that structure curves and structure functions should compose so as to yield smooth mappings from $\mathbf{R}$ to itself. It is required that the class of structure functions and that of structure curves should determine each other so that each of the two classes is maximal with respect to the other as far as they abide by the above condition. What is most important among many nice properties about the category $\mathbf{FS}$ of Frölicher spaces and smooth mappings is that it is Cartesian closed, while neither the category of finite-dimensional smooth manifolds nor that of infinite-dimensional smooth manifolds modelled after any infinite-dimensional vector spaces such as Hilbert spaces, Banach spaces, Fréchet spaces or the like is so at all. For a standard reference on Frölicher spaces the reader is referred to [5].

2.2. Weil Algebras and Infinitesimal Objects

The notion of a *Weil algebra* was introduced by Weil himself in [26]. We denote by $\mathbf{W}$ the category of Weil algebras. Roughly speaking, each Weil algebra corresponds to an infinitesimal object in the shade. By way of example, the Weil algebra $\mathbf{R}[X]/(X^2)$ (=the quotient ring of the polynomial ring $\mathbf{R}[X]$ of an indeterminate X modulo the ideal (X^2) generated by X^2) corresponds to the infinitesimal object of first-order nilpotent infinitesimals, while the Weil algebra $\mathbf{R}[X]/(X^3)$ corresponds to the infinitesimal object of second-order nilpotent infinitesimals. Although an infinitesimal object is undoubtedly imaginary in the real world, as has harassed both mathematicians and philosophers of the 17th and the 18th centuries because mathematicians at that time preferred to talk infinitesimal objects as if they were real entities, each Weil algebra yields its corresponding *Weil functor* on the category of smooth manifolds of some kind to itself, which is no doubt a real entity. Intuitively speaking, the Weil functor corresponding to a Weil algebra stands for the exponentiation by the infinitesimal object corresponding to the Weil algebra at issue. For Weil functors on the category of finite-dimensional smooth manifolds, the reader is referred to §35 of [12], while the reader can find a readable treatment of Weil functors on the category of smooth manifolds modelled on convenient vector spaces in §31 of [13].

Synthetic differential geometry (usually abbreviated to SDG), which is a kind of differential geometry with a cornucopia of nilpotent infinitesimals, was forced to invent its models, in which nilpotent infinitesimals were visible. For a standard textbook on SDG, the reader is referred to [14], while he or she is referred to [11] for the model theory of SDG vigorously constructed by Dubuc [1] and others. Although we do not get involved in SDG herein, we will exploit locutions in terms of infinitesimal objects so as to make the paper highly readable. Thus we prefer to write W_D and W_{D_2} in place of $\mathbf{R}[X]/(X^2)$ and $\mathbf{R}[X]/(X^3)$ respectively, where D stands for the infinitesimal object of first-order nilpotent infinitesimals, and D_2 stands for the infinitesimal object of second-order nilpotent infinitesimals. To Newton and Leibniz, D stood for

$$\{d \in \mathbf{R} \mid d^2 = 0\}$$

while D_2 stood for

$$\{d \in \mathbf{R} \mid d^3 = 0\}$$

We will write $W_{d\in D_2 \mapsto d^2 \in D}$ for the homomorphim of Weil algebras $\mathbf{R}[X]/(X^2) \to \mathbf{R}[X]/(X^3)$ induced by the homomorphism $X \to X^2$ of the polynomial ring $\mathbf{R}[X]$ to itself. Such locutions are justifiable, because the category $\mathbf{W}$ of Weil algebras in the real world and the category of infinitesimal objects in the shade are dual to each other in a sense. To familiarize himself or herself with such locutions, the reader is strongly encouraged to read the first two chapters of [14], even if he or she is not interested in SDG at all.

We need to fix notation and terminology for simplicial objects, which form an important subclass of infinitesimal objects. *Simplicial objects* are infinitesimal objects of the form

$$D^n\{\mathfrak{p}\} = \{(d_1, ..., d_n) \in D^n \mid d_{i_1}...d_{i_k} = 0 \ (\forall (i_1, ..., i_k) \in \mathfrak{p})\}$$

where $\mathfrak{p}$ is a finite set of finite sequences $(i_1, ..., i_k)$ of natural numbers between 1 and n, including the endpoints, with $i_1 < ... < i_k$. If $\mathfrak{p}$ is empty, $D^n\{\mathfrak{p}\}$ is D^n itself. If $\mathfrak{p}$ consists of all the binary sequences, then $D^n\{\mathfrak{p}\}$ represents $D(n)$ in the standard terminology of SDG. Given two simplicial objects $D^m\{\mathfrak{p}\}$ and $D^n\{\mathfrak{q}\}$, we define a simplicial object $D^m\{\mathfrak{p}\} \oplus D^n\{\mathfrak{q}\}$ to be

$$D^{m+n}\{\mathfrak{p} \oplus \mathfrak{q}\}$$

where

$$\mathfrak{p} \oplus \mathfrak{q} = \mathfrak{p} \cup \{(j_1 + m, ..., j_k + m) \mid (j_1, ..., j_k) \in \mathfrak{q}\} \\ \cup \{(i, j + m) \mid 1 \leq i \leq m, \ 1 \leq j \leq n\}$$

Since the operation $\oplus$ is associative, we can combine any finite number of simplicial objects by $\oplus$ without bothering about how to insert parentheses. Given morphisms of simplicial objects $\Phi_i : D^{m_i}\{\mathfrak{p}_i\} \to D^m\{\mathfrak{p}\}$ $(1 \leq i \leq n)$, there exists a unique morphism of simplicial objects $\Phi : D^{m_1}\{\mathfrak{p}_1\} \oplus ... \oplus D^{m_n}\{\mathfrak{p}_n\} \to D^m\{\mathfrak{p}\}$ whose restriction to $D^{m_i}\{\mathfrak{p}_i\}$ coincides with Φ_i for each i. We denote this Φ by $\Phi_1 \oplus ... \oplus \Phi_n$. We write $D(n)$ for $\{(d, ..., d) \in D^n \mid d_i d_j = 0$ for any $i \neq j\}$.

2.3. Microlinearity

In [20] we have discussed how to assign, to each pair (X, W) of a Frölicher space X and a Weil algebra W, another Frölicher space $X \otimes W$ called the *Weil prolongation of* X *with respect to* W, which is naturally extended to a bifunctor $\mathbf{FS} \times \mathbf{W} \to \mathbf{FS}$, and then to show that the functor $\cdot \otimes W : \mathbf{FS} \to \mathbf{FS}$ is product-preserving for any Weil algebra W. Weil prolongations are well-known as *Weil functors* for finite-dimensional and infinite-dimensional smooth manifolds in orthodox differential geometry, as we have already touched upon in the preceding subsection.

The central object of study in SDG is *microlinear* spaces. Although the notion of a manifold (=a pasting of copies of a certain linear space) is defined on the local level, the notion of microlinearity is defined absolutely on the genuinely infinitesimal level. For the historical account of microlinearity, the reader is referred to §§2.4 of [14] or Appendix D of [11]. To get an adequately restricted Cartesian closed subcategory of Frölicher spaces, we have emancipated microlinearity from within a well-adapted model of SDG to Frölicher spaces in the real world in [21]. Recall that a Frölicher space X is called *microlinear* providing that any finite limit diagram $\mathcal{D}$ in $\mathbf{W}$ yields a limit diagram $X \otimes \mathcal{D}$ in $\mathbf{FS}$, where $X \otimes \mathcal{D}$ is obtained from $\mathcal{D}$ by putting $X\otimes$ to the left of every object and every morphism in $\mathcal{D}$. As we have discussed there, all convenient vector spaces are microlinear, so that all C^∞-manifolds in the sense of [13] (cf. Section 27) are also microlinear.

We have no reason to hold that all Frölicher spaces credit Weil prolongations as exponentiation by infinitesimal objects in the shade. Therefore we need a notion which distinguishes Frölicher spaces that do so from those that do not.

A Frölicher space X is called *Weil exponentiable* if

$$(X \otimes (W_1 \otimes_\infty W_2))^Y = (X \otimes W_1)^Y \otimes W_2 \tag{1}$$

holds naturally for any Frölicher space Y and any Weil algebras W_1 and W_2. If $Y = 1$, then (1) degenerates into

$$X \otimes (W_1 \otimes_\infty W_2) = (X \otimes W_1) \otimes W_2 \tag{2}$$

If $W_1 = \mathbf{R}$, then (1) degenerates into

$$(X \otimes W_2)^Y = X^Y \otimes W_2 \tag{3}$$

We have shown in [20] that all convenient vector spaces are Weil exponentiable, so that all C^∞-manifolds in the sense of [13] (cf. Section 27) are Weil exponentiable.

We have demonstrated in [21] that all Frölicher spaces that are microlinear and Weil exponentiable form a Cartesian closed category. In the sequel M is assumed to be such a Frölicher space.

2.4. Comma Categories

In the next section we will work on the category $\mathbf{FS}/M$ of $\mathbf{FS}$ over M, for which we need to know that the product of $N_1 \to M$ and $N_1 \to M$ in $\mathbf{FS}/M$ is no other than the fibred product $N_1 \times_M N_2 \to M$. For the general theory of comma categories, the reader is referred to [6], [7] and [8].

3. Tangent Bundles

Proposition 1. *If M is a microlinear Frölicher space, then we have*

$$M \otimes W_{D(2)} = (M \otimes W_D) \times_M (M \otimes W_D)$$

Proof. We have the following pullback diagram of Weil algebras:

$$\begin{array}{ccc} W_{D(2)} & \to & W_D \\ \downarrow & & \downarrow \\ W_D & \to & W_1 \end{array} \tag{4}$$

where the left vertical arrow is induced by the mapping

$$d \in D \mapsto (d, 0) \in D(2)$$

while the upper horizontal arrow is induced by the mapping

$$d \in D \mapsto (0, d) \in D(2)$$

The above pullback diagram naturally gives rise to the following pullback diagram because of the microlinearity of M:

$$\begin{array}{ccc} M \otimes W_{D(2)} & \rightarrow & M \otimes W_D \\ \downarrow & & \downarrow \\ M \otimes W_D & \rightarrow & M \otimes W_1 = M \end{array}$$

This completes the proof. □

Corollary 2. *The canonical projection $M \otimes W_{D(2)} \rightarrow M$ is a biproduct of two copies of $M \otimes W_D \rightarrow M$ in the comma category $\mathbf{FS}/M$.*

Now we are in a position to define basic operations on $M \otimes W_D \rightarrow M$ in the category $\mathbf{FS}/M$ so as to make it a vector space over the projection $\mathbb{R} \times M \rightarrow M$.

1. The addition is defined by
$$\mathrm{id}_M \otimes W_{+_D} : M \otimes W_{D(2)} \rightarrow M \otimes W_D$$
where the putative mapping $+_D : D \rightarrow D(2)$ is
$$+_D : d \in D \mapsto (d, d) \in D(2)$$

2. The identity with respect to the above addition is defined by
$$\mathrm{id}_M \otimes W_{\mathbf{0}_D} : M = M \otimes W_1 \rightarrow M \otimes W_D$$
where the putative mapping $\mathbf{0}_D : D \rightarrow 1$ is the unique mapping.

3. The inverse with respect to the above addition is defined by
$$\mathrm{id}_M \otimes W_{-_D} : M \otimes W_D \rightarrow M \otimes W_D$$
where the putative mapping $-_D : D \rightarrow D$ is
$$-_D : d \in D \mapsto -d \in D$$

4. The scalar multiplication by a scalar $\alpha \in \mathbf{R}$ is defined by
$$\mathrm{id}_M \otimes W_{\alpha_D} : M \otimes W_D \rightarrow M \otimes W_D$$
where the putative mapping $\alpha_D : D \rightarrow D$ is
$$\alpha_D : d \in D \mapsto \alpha d \in D$$

Theorem 3. *The canonical projection $M \otimes W_D \rightarrow M$ is a vector space over the projection $\mathbf{R} \times M \rightarrow M$ in the category $\mathbf{FS}/M$.*

Proof.

1. The associativity of the addition follows from the following commutative

diagram:

$$\begin{array}{ccccc} & & \mathrm{id}_M \otimes W_{\epsilon_{23}} & & \\ & M \otimes W_{D(3)} & \rightarrow & M \otimes W_{D(2)} & \\ \mathrm{id}_M \otimes W_{\epsilon_{12}} & \downarrow & & \downarrow & \mathrm{id}_M \otimes W_{+D} \\ & M \otimes W_{D(2)} & \rightarrow & M \otimes W_D & \\ & & \mathrm{id}_M \otimes W_{+D} & & \end{array}$$

where the putative mapping $\epsilon_{23} : D(2) \to D(3)$ is

$$(d_1, d_2) \in D(2) \mapsto (d_1, d_1, d_2) \in D(3)$$

while the putative mapping $\epsilon_{12} : D(2) \to D(3)$ is

$$(d_1, d_2) \in D(2) \mapsto (d_1, d_2, d_2) \in D(3)$$

2. The commutativity of the addition follows readily from the commutative diagram

$$\begin{array}{cccc} & W_{+D} & & \\ W_D & \leftarrow & W_{D(2)} & \\ & \nwarrow & \uparrow & W_\tau \\ W_{+D} & & W_{D(2)} & \end{array}$$

where the putative mapping $\tau : D(2) \to D(2)$ is

$$(d_1, d_2) \in D(2) \mapsto (d_2, d_1) \in D(2).$$

3. To see that the identity defined above really plays the identity with respect to the above addition, it suffices to note that the composition of the following two putative mappings

$$d \in D \mapsto (d, 0) \in D(2)$$

$$(d_1, d_2) \in D(2) \mapsto d_1 \in D$$

in order is the identity mapping of D, while the composition of the following two putative mappings

$$d \in D \mapsto (0, d) \in D(2)$$

$$(d_1, d_2) \in D(2) \mapsto d_1 \in D$$

in order is the constant mapping

$$d \in D \mapsto 0 \in D$$

4. To see that the addition of scalars distributes with respect to the scalar multiplication, it suffices to note that, for any $\alpha_1, \alpha_2 \in \mathbf{R}$, the composition of the following two putative mappings

$$d \in D \mapsto (d, 0) \in D(2)$$

$$(d_1, d_2) \in D(2) \mapsto \alpha_1 d_1 + \alpha_2 d_2 \in D$$

in order is the mapping

$$d \in D \mapsto \alpha_1 d_1 \in D$$

and the composition of the two putative mappings

$$d \in D \mapsto (0, d) \in D(2)$$

$$(d_1, d_2) \in D(2) \mapsto \alpha_1 d_1 + \alpha_2 d_2 \in D$$

in order is the mapping

$$d \in D \mapsto \alpha_2 d_2 \in D$$

while the composition of the two putative mappings

$$d \in D \mapsto (d, d) \in D(2)$$

$$(d_1, d_2) \in D(2) \mapsto \alpha_1 d_1 + \alpha_2 d_2 \in D$$

in order is no other than the mapping

$$d \in D \mapsto (\alpha_1 + \alpha_2) d \in D$$

5. To see that the addition of vectors distributes with respect to the scalar multiplication, it suffices to note that, for any $\alpha \in \mathbf{R}$, the composition of the two putative mappings

$$d \in D \mapsto (d, 0) \in D(2)$$

$$(d_1, d_2) \in D(2) \mapsto (\alpha d_1, \alpha d_2) \in D(2)$$

in order is the composition of the two putative mappings

$$d \in D \mapsto \alpha d \in D$$

$$d \in D \mapsto (d, 0) \in D(2)$$

in order, and the composition of the two putative mappings

$$d \in D \mapsto (0, d) \in D(2)$$

$$(d_1, d_2) \in D(2) \mapsto (\alpha d_1, \alpha d_2) \in D(2)$$

in order is the composition of the two putative mappings

$$d \in D \mapsto \alpha d \in D$$

$$d \in D \mapsto (0, d) \in D(2)$$

in order, while the composition of the two putative mappings

$$d \in D \mapsto (d, d) \in D(2)$$

$$(d_1, d_2) \in D(2) \mapsto (\alpha d_1, \alpha d_2) \in D(2)$$

in order is no other than the composition of the two putative mappings

$$d \in D \mapsto \alpha d \in D$$

$$d \in D \mapsto (d, d) \in D(2)$$

in order.

6. The verification of the other axioms for a vector space can safely be left to the reader.

□

Corollary 4. *For any $x \in M$, the inverse image $(M \otimes W_D)_x$ of x under the canonical projection $M \otimes W_D \to M$ is a vector space over $\mathbf{R}$.*

The proof of the following easy proposition is left to the reader.

Proposition 5. *For any $t_1, ..., t_n \in (M \otimes W_D)_x$, there exists a unique $l_{(t_1,...,t_n)} \in M \otimes W_{D(n)}$ with*

$$\mathrm{id}_M \otimes W_{i_j^{D(n)}}(l_{(t_1,...,t_n)}) = t_j \qquad (j = 1, ..., n)$$

where the putative mapping $i_j^{D(n)} : D \to D(n)$ is

$$d \in D \mapsto (0, ..., 0, \underset{i}{d}, 0, ..., 0) \in D(n)$$

4. Vector Fields

Vector fields can be delineated in two distinct ways.

Theorem 6. *The space of sections of the tangent bundle $M \otimes W_D \to M$ can naturally be identified with the space $(M^M \otimes W_D)_{\mathrm{id}_M}$.*

Proof. This follows simply from the following instance of the Weil exponentiability of M:

$$M^M \otimes W_D = (M \otimes W_D)^M$$

□

Remark 7. In this paper the viewpoint of a vector field on M as an element of $(M^M \otimes W_D)_{\mathrm{id}_M}$ is preferred to that as a section of the tangent bundle $M \otimes W_D \to M$.

Notation 8. *The totality of vector fields on M is denoted by $\aleph(M)$.*

Definition 9. For any $\gamma_1 \in M^M \otimes W_{D^m}$ and any $\gamma_2 \in M^M \otimes W_{D^n}$ we define $\gamma_2 * \gamma_1 \in M^M \otimes W_{D^{m+n}}$ to be

$$(\circ_{M^M} \otimes \mathrm{id}_{W_{D^{m+n}}})((\mathrm{id}_{M^M} \otimes W_{p^{D^{m+n}}_{D^m}})(\gamma_1), (\mathrm{id}_{M^M} \otimes W_{p^{D^{m+n}}_{D^n}})(\gamma_2))$$

where $\circ_{M^M}$ is the bifunctor assigning the composition $g \circ f$ to each pair $(f, g) \in M^M \times M^M$ while $p^{D^{m+n}}_{D^m}: D^{m+n} \to D^m$ and $p^{D^{m+n}}_{D^n}: D^{m+n} \to D^n$ are the canonical projections.

Proposition 10. *For any $\gamma_1 \in M^M \otimes W_{D^l}$, any $\gamma_2 \in M^M \otimes W_{D^m}$ and any $\gamma_3 \in M^M \otimes W_{D^n}$, we have*

$$\gamma_3 * (\gamma_2 * \gamma_1) = (\gamma_3 * \gamma_2) * \gamma_1$$

In other words, the operation $*$ is associative.

Proof. By dint of the bifunctionality of $\otimes$, it is easy to see that the diagram

$$\begin{array}{ccc} (M^M \otimes W_{D^{l+m}}) \times (M^M \otimes W_{D^{l+m}}) & \xrightarrow{\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m}}}} & M^M \otimes W_{D^{l+m}} \\ \downarrow & & \downarrow \\ (M^M \otimes W_{D^{l+m+n}}) \times (M^M \otimes W_{D^{l+m+n}}) & \xrightarrow[\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}}]{} & M^M \otimes W_{D^{l+m+n}} \end{array} \tag{5}$$

is commutative, where the left vertical arrow stands for $(\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m+n}}_{D^{l+m}}}) \times (\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m+n}}_{D^{l+m}}})$, and the right vertical arrow stands for $\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m+n}}_{D^{l+m}}}$. By dint of the bifunctionality of $\otimes$ again, it is also easy to see that the diagram

$$\begin{array}{ccc} (M^M \otimes W_{D^{m+n}}) \times (M^M \otimes W_{D^{m+n}}) & \xrightarrow{\circ_{M^M} \otimes \mathrm{id}_{W_{D^{m+n}}}} & M^M \otimes W_{D^{m+n}} \\ \downarrow & & \downarrow \\ (M^M \otimes W_{D^{l+m+n}}) \times (M^M \otimes W_{D^{l+m+n}}) & \xrightarrow[\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}}]{} & M^M \otimes W_{D^{l+m+n}} \end{array} \tag{6}$$

is commutative, where the left vertical arrow stands for $(\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m+n}}_{D^{m+n}}}) \times (\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m+n}}_{D^{m+n}}})$, and the right vertical arrow stands for $\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m+n}}_{D^{m+n}}}$. Therefore we have

$$\begin{aligned} &\gamma_3 * (\gamma_2 * \gamma_1) \\ &= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m+n}}_{D^{l+m}}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m}}}) \\ &((\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m}}_{D^l}})(\gamma_1), (\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m}}_{D^m}})(\gamma_2))), (\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m+n}}_{D^n}})(\gamma_3)) \\ &= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\mathrm{id}_{M^M} \otimes W_{p^{D^{l+m+n}}_{D^{l+m}}}) \circ (\mathrm{id}_{M^M} \end{aligned}$$

$\otimes W_{p_{D^l}^{D^{l+m}}})(\gamma_1),$

$(\mathrm{id}_{M^M} \otimes W_{p_{D^{l+m}}^{D^{l+m+n}}}) \circ (\mathrm{id}_{M^M} \otimes W_{p_{D^m}^{D^{l+m}}})(\gamma_2)), (\mathrm{id}_{M^M} \otimes W_{p_{D^n}^{D^{l+m+n}}})(\gamma_3))$

$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\mathrm{id}_{M^M} \otimes W_{p_{D^l}^{D^{l+m+n}}})(\gamma_1),$

$(\mathrm{id}_{M^M} \otimes W_{p_{D^m}^{D^{l+m+n}}})(\gamma_2)), (\mathrm{id}_{M^M} \otimes W_{p_{D^n}^{D^{l+m+n}}})(\gamma_3))$

[by dint of the commutativity of the diagram (5)]

$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\mathrm{id}_{M^M} \otimes W_{p_{D^l}^{D^{l+m+n}}})(\gamma_1),$

$(\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\mathrm{id}_{M^M} \otimes W_{p_{D^m}^{D^{l+m+n}}})(\gamma_2),$

$(\mathrm{id}_{M^M} \otimes W_{p_{D^n}^{D^{l+m+n}}})(\gamma_3)))$

[since the operation $\circ_{M^M}$ is associative]

$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\mathrm{id}_{M^M} \otimes W_{p_{D^l}^{D^{l+m+n}}})(\gamma_1),$

$(\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\mathrm{id}_{M^M} \otimes W_{p_{D^{m+n}}^{D^{l+m+n}}})\circ$

$(\mathrm{id}_{M^M} \otimes W_{p_{D^m}^{D^{m+n}}})(\gamma_2), (\mathrm{id}_{M^M} \otimes W_{p_{D^{m+n}}^{D^{l+m+n}}}) \circ (\mathrm{id}_{M^M} \otimes W_{p_{D^n}^{D^{m+n}}})(\gamma_3)))$

[by dint of the commutativity of the diagram (6)]

$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^{l+m+n}}})((\mathrm{id}_{M^M} \otimes W_{p_{D^l}^{D^{l+m+n}}})(\gamma_1),$

$(\mathrm{id}_{M^M} \otimes W_{p_{D^{m+n}}^{D^{l+m+n}}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D^{m+n}}})$

$((\mathrm{id}_{M^M} \otimes W_{p_{D^m}^{D^{m+n}}})(\gamma_2), (\mathrm{id}_{M^M} \otimes W_{p_{D^n}^{D^{m+n}}})(\gamma_3))))$

$= (\gamma_3 * \gamma_2) * \gamma_1$

□

It is easy to see that

Proposition 11. *For any $\gamma \in M^M \otimes W_{D^m}$ we have*

$$\gamma * I_l = (\mathrm{id}_{M^M} \otimes W_{p_{D^m}^{D^{l+m}}})(\gamma),$$

$$I_n * \gamma = (\mathrm{id}_{M^M} \otimes W_{p_{D^m}^{D^{m+n}}})(\gamma),$$

where $I_l : M^M \otimes W_{D^l} \to M^M \otimes W_{D^l}$ and $I_n : M^M \otimes W_{D^n} \to M^M \otimes W_{D^n}$ are the identity mappings.

The following proposition is essentially a variant of Proposition 3 in §§3.2 of [14].

Proposition 12. *For any $X \in \aleph(M)$, we have*

$$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D(2)\mapsto d_1+d_2\in D})(X) \\ = (\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))$$

where $p_1^{D(2)} : D(2) \to D$ and $p_2^{D(2)} : D(2) \to D$ are the canonical projections onto the first and second factors.

Proof. With due regard to the limit diagram (4) of Weil algebras and the microlinearity of M^M, it suffices to see that

$$(\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D(2)\mapsto d_1+d_2\in D})(X)) = (\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}}) \\ ((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))) \quad (7)$$

and that

$$(\mathrm{id}_{M^M} \otimes W_{i_2^{D(2)}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D(2)\mapsto d_1+d_2\in D})(X)) = (\mathrm{id}_{M^M} \otimes W_{i_2^{D(2)}}) \\ ((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))) \quad (8)$$

Here we deal only with (7), leaving the similar treatment of (8) safely to the reader. By dint of the bifunctionality of $\otimes$, it is easy to see that the diagram

$$\begin{array}{ccc} & \circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}} & \\ (M^M \otimes W_{D(2)}) \times (M^M \otimes W_{D(2)}) & \rightarrow & M^M \otimes W_{D(2)} \\ \downarrow & & \downarrow \\ (M^M \otimes W_D) \times (M^M \otimes W_D) & \rightarrow & M^M \otimes W_D \\ & \circ_{M^M} \otimes \mathrm{id}_{W_D} & \end{array} \quad (9)$$

commutes, where the left vertical arrow stands for $(\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}}) \times (\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}})$, and the right vertical arrow stands for $\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}}$ with $i_1^{D(2)} : D \to D(2)$ being the canonical injection $d \in D \mapsto (d, 0) \in D(2)$. Therefore we have

$(\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X)))$

$= (\circ_{M^M} \otimes \mathrm{id}_{W_D})((\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}}) \circ (\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X),$

$(\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}}) \circ (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X)))$

$= (\circ_{M^M} \otimes \mathrm{id}_{W_D})((\mathrm{id}_{M^M} \otimes (W_{i_1^{D(2)}} \circ W_{p_1^{D(2)}}))(X),$

$(\mathrm{id}_{M^M} \otimes (W_{i_1^{D(2)}} \circ W_{p_2^{D(2)}})(X))$

$= X$

while it is trivial to see that

$$
\begin{aligned}
&(\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D(2)\mapsto d_1+d_2\in D})(X)) \\
&= (\mathrm{id}_{M^M} \otimes (W_{i_1^{D(2)}} \circ W_{(d_1,d_2)\in D(2)\mapsto d_1+d_2\in D}))(X) \\
&= X
\end{aligned}
$$

Thus we are done. □

The following corollary is essentially a variant of Proposition 4 in §§3.2 of [14].

Corollary 13.

$$
\begin{aligned}
&(\mathrm{id}_{M^M} \otimes W_{d\in D\mapsto(d,-d)\in D(2)})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), \\
&(\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))) \\
&= (\mathrm{id}_{M^M} \otimes W_{d\in D\mapsto(-d,d)\in D(2)})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), \\
&(\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))) \\
&= \mathbf{0}_{\mathrm{id}_M}
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&(\mathrm{id}_{M^M} \otimes W_{d\in D\mapsto(d,-d)\in D(2)})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), \\
&(\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))) \\
&= (\mathrm{id}_{M^M} \otimes W_{d\in D\mapsto(d,-d)\in D(2)})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D(2)\mapsto d_1+d_2\in D})(X)) \\
&\text{[by Proposition 12]} \\
&= (\mathrm{id}_{M^M} \otimes W_{d\in D\mapsto 0\in D})(X) \\
&= \mathbf{0}_{\mathrm{id}_M}
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&(\mathrm{id}_{M^M} \otimes W_{d\in D\mapsto(-d,d)\in D(2)})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), \\
&(\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))) \\
&= \mathbf{0}_{\mathrm{id}_M}
\end{aligned}
$$

□

The following proposition is essentially a variant of Proposition 6 in §§3.2 of [14].

Proposition 14. *For any $X, Y \in \aleph(M)$ we have*

$$
l_{(X,Y)}
$$

$$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(Y))$$
$$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(Y), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))$$

so that

$$X + Y$$
$$= (\mathrm{id}_{M^M} \otimes W_{+_D})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(Y)))$$
$$= (\mathrm{id}_{M^M} \otimes W_{+_D})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(Y), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X)))$$

Proof. With due regard to the limit diagram (4) of Weil algebras and the microlinearity of M^M, it suffices to see that

$$(\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(Y))) = X \quad (10)$$

$$(\mathrm{id}_{M^M} \otimes W_{i_2^{D(2)}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(Y))) = Y \quad (11)$$

$$(\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(Y), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))) = Y \quad (12)$$

$$(\mathrm{id}_{M^M} \otimes W_{i_2^{D(2)}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(Y), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(X))) = X \quad (13)$$

Here we deal only with (10), leaving similar treatments of (11)-(13) safely to the reader. Exploiting the commutativity of the diagram (9), we have

$$(\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}})((\circ_{M^M} \otimes \mathrm{id}_{W_{D(2)}})((\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X), (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(Y)))$$
$$= (\circ_{M^M} \otimes \mathrm{id}_{W_D})((\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}}) \circ (\mathrm{id}_{M^M} \otimes W_{p_1^{D(2)}})(X),$$
$$(\mathrm{id}_{M^M} \otimes W_{i_1^{D(2)}}) \circ (\mathrm{id}_{M^M} \otimes W_{p_2^{D(2)}})(Y)))$$
$$= (\circ_{M^M} \otimes \mathrm{id}_{W_D})((\mathrm{id}_{M^M} \otimes (W_{i_1^{D(2)}} \circ W_{p_1^{D(2)}}))(X),$$
$$(\mathrm{id}_{M^M} \otimes (W_{i_1^{D(2)}} \circ W_{p_2^{D(2)}})(Y))$$
$$= X$$

Thus we are done. □

Theorem 15. *For any $X, Y \in \aleph(M)$, there exists a unique $[X,Y] \in \aleph(M)$ such that*

$$\begin{aligned}&(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,d_2,-d_1,-d_2)\in D^4})(Y * X * Y * X)\\ &= (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto d_1 d_2 \in D})([X,Y])\end{aligned}$$

Proof. With due regard to the microlinearity of M^M and the limit diagram of Weil algebras

$$M^M \otimes W_D \quad \begin{array}{c} \mathrm{id}_{M^M} \otimes W_{i_1^{D^2}} \\ \longleftarrow \\ \mathrm{id}_{M^M} \otimes W_{i_2^{D^2}} \\ \longleftarrow \\ \mathrm{id}_{M^M} \otimes W_{d\in D \mapsto (0,0)\in D^2} \\ \longleftarrow \end{array} \quad M^M \otimes W_{D^2} \quad \begin{array}{c} \mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto d_1 d_2 \in D} \\ \longleftarrow \end{array} \quad M^M \otimes W_D$$

it suffices to see that

$$(\mathrm{id}_{M^M} \otimes W_{i_1^{D^2}})(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,d_2,-d_1,-d_2)\in D^4}(Y * X * Y * X)) = I_1 \quad (14)$$

$$(\mathrm{id}_{M^M} \otimes W_{i_2^{D^2}})(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,d_2,-d_1,-d_2)\in D^4}(Y * X * Y * X)) = I_1 \quad (15)$$

Here we deal only with (14), leaving a similar treatment of (15) safely to the reader. By dint of the bifunctionality of $\otimes$, it is easy to see that

$$\begin{array}{ccc} (M^M \otimes W_{D^4}) \times (M^M \otimes W_{D4}) & \xrightarrow{\circ_{M^M} \otimes \mathrm{id}_{W_{D^4}}} & M^M \otimes W_{D^4} \\ \downarrow & & \downarrow \\ (M^M \otimes W_D) \times (M^M \otimes W_D) & \underset{\circ_{M^M} \otimes \mathrm{id}_{W_D}}{\rightarrow} & M^M \otimes W_D \end{array}$$

with the left vertical arrow standing for $(\mathrm{id}_{M^M} \otimes W_{d\in D \mapsto (d,0,-d,0)\in D^4}) \times (\mathrm{id}_{M^M} \otimes W_{d\in D \mapsto (d,0,-d,0)\in D^4})$ and the right vertical arrow standing for $\mathrm{id}_{M^M} \otimes W_{d\in D \mapsto (d,0,-d,0)\in D^4}$, so that we have

$$\begin{aligned}&(\mathrm{id}_{M^M} \otimes W_{d\in D \mapsto (d,0,-d,0)\in D^4})(Y * X * Y * X))\\ &= (\circ_{M^M} \otimes \mathrm{id}_{W_D})((\mathrm{id}_{M^M} \otimes W_{d\in D \mapsto (d,0)\in D^2})(Y * X),\\ &(\mathrm{id}_{M^M} \otimes W_{d\in D \mapsto (-d,0)\in D^2})(Y * X))\end{aligned}$$

Therefore we have

$$(\mathrm{id}_{M^M} \otimes W_{i_1^{D^2}})(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1,d_2)\in D^4}(Y * X * Y * X))$$

$= (\mathrm{id}_{M^M} \otimes W_{d\in D\mapsto(d,0,-d,0)\in D^4})(Y * X * Y * X))$
$= (\circ_{M^M} \otimes \mathrm{id}_{W_D})((\mathrm{id}_{M^M} \otimes W_{d\in D\mapsto(d,0)\in D^2})(Y * X),$
$(\mathrm{id}_{M^M} \otimes W_{d\in D\mapsto(-d,0)\in D^2})(Y * X))$
$= (\circ_{M^M} \otimes \mathrm{id}_{W_D})(X, -X)$
$= \mathbf{0}_{\mathrm{id}_M}$

Thus we are done. □

Proposition 16. *For any $X, Y \in \aleph(M)$, we have*

$$[X, Y] = -[Y, X]$$

Proof. With due regard to Proposition 14, it suffices to show that

$$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_1d_2,d_1d_2)\in D^2})([X, Y] * [Y, X]) = I_2$$

This follows from

$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_1d_2,d_1d_2)\in D^2})([X, Y] * [Y, X])$
$= (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,d_1,-d_2,-d_1,d_1,d_2,-d_1,-d_2)\in D^8})$
$(Y * X * Y * X * X * Y * X * Y)$
$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,d_1,-d_2)\in D^3})(Y * X * Y),$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(-d_1,d_1)\in D^2})(X * X),$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,-d_1,-d_2)\in D^3})(Y * X * Y))$
[by the bifunctionality of $\otimes$]
$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,d_1,-d_2)\in D^3})(Y * X * Y), I_2,$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,-d_1,-d_2)\in D^3})(Y * X * Y))$
$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,d_1,-d_2)\in D^3})(Y * X * Y),$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,-d_1,-d_2)\in D^3})(Y * X * Y))$
$= (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,d_1,-d_2,d_2,-d_1,-d_2)\in D^6})(Y * X * Y * Y * X * Y)$
[by the bifunctionality of $\otimes$]
$= (\circ_{M^M} \otimes \mathrm{id}_{W_D})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,d_1)\in D^2})(X * Y),$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(-d_2,d_2)\in D^2})(Y * Y),$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(-d_1,-d_2)\in D^2})(Y * X))$
[by the bifunctionality of $\otimes$]
$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(d_2,d_1)\in D^2})(X * Y), I_2,$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2\mapsto(-d_1,-d_2)\in D^2})(Y * X))$

$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2})(X * Y),$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (-d_1,-d_2)\in D^2})(Y * X))$
$= (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,d_1,-d_1,-d_2)\in D^4})(Y * X * X * Y)$
[by the bifunctionality of $\otimes$]
$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto d_2\in D})(Y),$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,-d_1)\in D^2})(X * X),$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto -d_2\in D})(Y))$
[by the bifunctionality of $\otimes$]
$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto d_2\in D})(Y), I_2,$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto -d_2\in D})(Y))$
$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})((\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto d_2\in D})(Y),$
$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto -d_2\in D})(Y))$
$= (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,-d_2)\in D^2})(Y * Y)$
$= I_2$ □

The proof of the Jacobi identity is postponed to the subsequent two sections.

5. The General Jacobi Identity

The principal objective in this section is to give a proof of the general Jacobi identity. Our harder treatment of the general Jacobi identity is preceded by a simpler treatment of the primordial Jacobi identity, because the latter is easier to grasp intuitively.

Proposition 17. *The diagram*

$$\begin{array}{ccccc} & & \mathrm{id}_M \otimes W_\varphi & & \\ & M \otimes W_{D^3\{(1,3),(2,3)\}} & \rightarrow & M \otimes W_{D^2} & \\ \mathrm{id}_M \otimes W_\psi & \downarrow & & \downarrow & \mathrm{id}_M \otimes W_{i^{D^2}_{D(2)}} \\ & M \otimes W_{D^2} & \rightarrow & M \otimes W_{D(2)} & \\ & & \mathrm{id}_M \otimes W_{i^{D^2}_{D(2)}} & & \end{array}$$

is a pullback diagram, where the putative mapping $\varphi : D^2 \to D^3\{(1,3),(2,3)\}$ *is*

$$(d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0) \in D^3\{(1,3),(2,3)\}$$

while the putative mapping $\psi : D^2 \to D^3\{(1,3),(2,3)\}$ *is*

$$(d_1, d_2) \in D^2 \mapsto (d_1, d_2, d_1 d_2) \in D^3\{(1,3),(2,3)\}$$

Proof. This follows from the microlinearity of M and the pullback diagram of Weil algebras

$$\begin{array}{ccccc} & W_{D^3\{(1,3),(2,3)\}} & \overset{W_\varphi}{\rightarrow} & W_{D^2} & \\ W_\psi & \downarrow & & \downarrow & W_{i_{D(2)}^{D^2}} \\ & W_{D^2} & \underset{W_{i_{D(2)}^{D^2}}}{\rightarrow} & W_{D(2)} & \end{array}$$

□

Corollary 18. *For any* $\gamma_1, \gamma_2 \in M \otimes W_{D^2}$, *if* $\left(\mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}}\right)(\gamma_1) = \left(\mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}}\right)(\gamma_2)$, *then there exists unique* $\gamma \in M \otimes W_{D^3\{(1,3),(2,3)\}}$ *with* $(\mathrm{id}_M \otimes W_\varphi)(\gamma) = \gamma_1$ *and* $(\mathrm{id}_M \otimes W_\psi)(\gamma) = \gamma_2$.

Remark 19. Thus γ encodes γ_1 and γ_2, which are in turn recovered from γ via $\mathrm{id}_M \otimes W_\varphi$ and $\mathrm{id}_M \otimes W_\psi$ respectively.

Notation 20. *We will write* $g_{(\gamma_1,\gamma_2)}$ *for* γ *in the above corollary.*

Definition 21. The *strong difference* $\gamma_2 \overset{\cdot}{-} \gamma_1 \in M \otimes W_D$ is defined to be

$$(\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,d)\in D^3\{(1,3),(2,3)\}})(g_{(\gamma_1,\gamma_2)})$$

The following is the prototype for the general Jacobi identity.

Theorem 22. *(The Primordial Jacobi Identity) Let* $\gamma_1, \gamma_2, \gamma_3 \in M \otimes W_{D^2}$. *As long as the following three expressions are well defined (i.e.,*

$$\left(\mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}}\right)(\gamma_1) = \left(\mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}}\right)(\gamma_2) = \left(\mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}}\right)(\gamma_3)),$$

they sum up only to vanish:

$$\gamma_2 \overset{\cdot}{-} \gamma_1$$
$$\gamma_3 \overset{\cdot}{-} \gamma_2$$
$$\gamma_1 \overset{\cdot}{-} \gamma_3$$

Its proof is based completely upon the following theorem.

Theorem 23. *The diagram*

$$\begin{array}{ccccccc}
 & \mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}} & & M \otimes W_{D^2} & & \mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}} & \\
 & & \swarrow & \uparrow & \searrow & & \\
 & M \otimes W_{D(2)} & & M \otimes W_E & & M \otimes W_{D(2)} & \\
\mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}} & \uparrow & \swarrow & & \searrow & \uparrow & \mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}} \\
 & M \otimes W_{D^2} & & & & M \otimes W_{D^2} & \\
 & & \searrow & & \swarrow & & \\
 & \mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}} & & M \otimes W_{D(2)} & & \mathrm{id}_M \otimes W_{i_{D(2)}^{D^2}} &
\end{array}$$

is a limit diagram, where the putative object E is

$$D^4\{(1,3),(2,3),(1,4),(2,4),(3,4)\}$$

and the putative mapping $i_{D(2)}^{D^2} : D(2) \to D^2$ is $(d_1, d_2) \in D(2) \mapsto (d_1, d_2) \in D^2$, while the three unnamed arrows $M \otimes W_E \to M \otimes W_{D^2}$ are $\mathrm{id}_M \otimes W_{l_i}$ $(i = 1, 2, 3)$ with the putative mappings $l_i : D^2 \to E$ $(i = 1, 2, 3)$ being

$$l_1 : (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0, 0) \in E$$
$$l_2 : (d_1, d_2) \in D^2 \mapsto (d_1, d_2, d_1 d_2, 0) \in E$$
$$l_3 : (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0, d_1 d_2) \in E$$

This theorem follows directly from the following lemma.

Lemma 24. *The following diagram is a limit diagram of Weil algebras:*

$$\begin{array}{ccccccc}
 & W_{i_{D(2)}^{D^2}} & & W_{D^2} & & W_{i_{D(2)}^{D^2}} & \\
 & & \swarrow & \uparrow & \searrow & & \\
 & W_{D(2)} & & W_E & & W_{D(2)} & \\
W_{i_{D(2)}^{D^2}} & \uparrow & \swarrow & & \searrow & \uparrow & W_{i_{D(2)}^{D^2}} \\
 & W_{D^2} & & & & W_{D^2} & \\
 & & \searrow & & \swarrow & & \\
 & W_{i_{D(2)}^{D^2}} & & W_{D(2)} & & W_{i_{D(2)}^{D^2}} &
\end{array}$$

Proof. Let $\gamma_1, \gamma_2, \gamma_3 \in W_{D^2}$ and $\gamma \in W_E$ so that they are the polynomials with real coefficients in the following form:

$$\gamma_1(X_1, X_2) = a + a_1 X_1 + a_2 X_2 + a_{12} X_1 X_2$$
$$\gamma_2(X_1, X_2) = b + b_1 X_1 + b_2 X_2 + b_{12} X_1 X_2$$
$$\gamma_3(X_1, X_2) = c + c_1 X_1 + c_2 X_2 + c_{12} X_1 X_2$$
$$\gamma(X_1, X_2, X_3, X_4) = e + e_1 X_1 + e_2 X_2 + e_{12} X_1 X_2 + e_3 X_3 + e_4 X_4$$

The condition that $W_{i_{D(2)}^{D^2}}(\gamma_1) = W_{i_{D(2)}^{D^2}}(\gamma_2) = W_{i_{D(2)}^{D^2}}(\gamma_3)$ is equivalent to the following three conditions as a whole:

$$a = b = c$$
$$a_1 = b_1 = c_1$$
$$a_2 = b_2 = c_2$$

Therefore, in order that $W_{l_1}(\gamma) = \gamma_1$, $W_{l_2}(\gamma) = \gamma_2$ and $W_{l_3}(\gamma) = \gamma_3$ in this case, it is necessary and sufficient that the polynomial γ should be of the following form:

$$\gamma(X_1, X_2, X_3, X_4) = a + a_1 X_1 + a_2 X_2 + a_{12} X_1 X_2 + (b_{12} - a_{12}) X_3 + (c_{12} - a_{12}) X_4$$

This completes the proof. □

Corollary 25. *Given* $\gamma_1, \gamma_2, \gamma_3 \in M \otimes W_{D^2}$ *with* $W_{i_{D(2)}^{D^2}}(\gamma_1) = W_{i_{D(2)}^{D^2}}(\gamma_2) = W_{i_{D(2)}^{D^2}}(\gamma_3)$, *there exists a unique* $\gamma \in M \otimes W_E$, *usally denoted by* $h_{(\gamma_1,\gamma_2,\gamma_3)}$, *such that* $\gamma_i = (\mathrm{id}_M \otimes W_{l_i})(\gamma)$ $(i = 1, 2, 3)$.

Remark 26. Thus $h_{(\gamma_1,\gamma_2,\gamma_3)}$ encodes γ_1, γ_2 and γ_3, which are in turn recovered from $h_{(\gamma_1,\gamma_2,\gamma_3)}$ via $\mathrm{id}_M \otimes W_{l_i}$ $(i = 1, 2, 3)$.

Proof. (of the primordial Jacobi identity). Let t_i $(i = 1, 2, 3)$ be the three expression in Theorem 22 in order. It is easy to see that

$$g_{(\gamma_1,\gamma_2)} = (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\}\mapsto(d_1,d_2,d_3,0)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)})$$
$$g_{(\gamma_2,\gamma_3)} = (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\}\mapsto(d_1,d_2,d_1d_2-d_3,d_3)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)})$$
$$g_{(\gamma_3,\gamma_1)} = (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\}\mapsto(d_1,d_2,0,d_1d_2-d_3)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)})$$

Therefore we have

$$\begin{aligned} &t_1 \\ &= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,d)\in D^3\{(1,3),(2,3)\}})(g_{(\gamma_1,\gamma_2)}) \\ &= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,d)\in D^3\{(1,3),(2,3)\}}) \circ (\mathrm{id}_M \\ &\otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\}\mapsto(d_1,d_2,d_3,0)\in E}) \\ &(h_{(\gamma_1,\gamma_2,\gamma_3)}) \\ &= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,d,0)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)}) \end{aligned}$$

$$\begin{aligned} &t_2 \\ &= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,d)\in D^3\{(1,3),(2,3)\}})(g_{(\gamma_2,\gamma_3)}) \\ &= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,d)\in D^3\{(1,3),(2,3)\}}) \circ (\mathrm{id}_M \\ &\otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\}\mapsto(d_1,d_2,d_1d_2-d_3,d_3)\in E}) \\ &(h_{(\gamma_1,\gamma_2,\gamma_3)}) \end{aligned}$$

$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,-d,d)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)})$$

$$t_3$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,d)\in D^3\{(1,3),(2,3)\}})(g_{(\gamma_2,\gamma_3)})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,d)\in D^3\{(1,3),(2,3)\}}) \circ (\mathrm{id}_M$$
$$\otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\}\mapsto(d_1,d_2,0,d_1d_2-d_3)\in E})$$
$$(h_{(\gamma_1,\gamma_2,\gamma_3)})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,-d)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)})$$

Thus we have

$$l_{(t_1,t_2,t_3)} = (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D(3)\mapsto(0,0,d_1-d_2,d_2-d_3)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)})$$

This means that

$$t_1 + t_2 + t_3$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(d,d,d)\in D(3)})(l_{(t_1,t_2,t_3)})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(d,d,d)\in D(3)}) \circ (\mathrm{id}_M$$
$$\otimes W_{(d_1,d_2,d_3)\in D(3)\mapsto(0,0,d_1-d_2,d_2-d_3)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,d-d,d-d)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0)\in E})(h_{(\gamma_1,\gamma_2,\gamma_3)})$$

Thus the proof of the primordial Jacobi identity is complete. □

Proposition 27. *The diagram*

$$\begin{array}{ccccc} & M\otimes W_{D^4\{(2,4),(3,4)\}} & \overset{\mathrm{id}_M\otimes W_{\varphi_1^3}}{\rightarrow} & M\otimes W_{D^3} & \\ \mathrm{id}_M\otimes W_{\psi_1^3} & \downarrow & & \downarrow & \mathrm{id}_M\otimes W_{i^{D^3}_{D^3\{(2,3)\}}} \\ & M\otimes W_{D^3} & \underset{\mathrm{id}_M\otimes W_{i^{D^3}_{D^3\{(2,3)\}}}}{\rightarrow} & M\otimes W_{D^3\{(2,3)\}} & \end{array}$$

is a pullback diagram, where the putative mapping $\varphi_1^3 : D^3 \to D^4\{(2,4),(3,4)\}$ *is*

$$(d_1,d_2,d_3) \in D^3 \mapsto (d_1,d_2,d_3,0) \in D^4\{(2,4),(3,4)\}$$

while the putative mapping $\psi_1^3 : D^3 \to D^4\{(2,4),(3,4)\}$ *is*

$$(d_1,d_2,d_3) \in D^3 \mapsto (d_1,d_2,d_3,d_2d_3) \in D^4\{(2,4),(3,4)\}$$

Proof. This follows from the microlinearity of M and the pullback diagram

of Weil algebras

$$\begin{array}{ccccc} & W_{D^4\{(2,4),(3,4)\}} & \overset{W_{\varphi_1^3}}{\rightarrow} & W_{D^3} & \\ W_{\psi_1^3} & \downarrow & & \downarrow & W_{i_{D^3\{(2,3)\}}^{D^3}} \\ & W_{D^3} & \underset{W_{i_{D^3\{(2,3)\}}^{D^3}}}{\rightarrow} & W_{D^3\{(2,3)\}} & \end{array}$$

□

Corollary 28. *For any $\gamma_1, \gamma_2 \in M \otimes W_{D^3}$, if $\left(\mathrm{id}_M \otimes W_{i_{D^3\{(2,3)\}}^{D^3}}\right)(\gamma_1) = \left(\mathrm{id}_M \otimes W_{i_{D^3\{(2,3)\}}^{D^3}}\right)(\gamma_2)$, then there exists unique $\gamma \in M \otimes W_{D^4\{(2,4),(3,4)\}}$ with $(\mathrm{id}_M \otimes W_{\varphi_1^3})(\gamma) = \gamma_1$ and $(\mathrm{id}_M \otimes W_{\psi_1^3})(\gamma) = \gamma_2$.*

Remark 29. Thus γ encodes γ_1 and γ_2, which are in turn recovered from γ via $\mathrm{id}_M \otimes W_{\varphi_1^3}$ and $\mathrm{id}_M \otimes W_{\psi_1^3}$ respectively.

Notation 30. *We will write $g^1_{(\gamma_1,\gamma_2)}$ for γ in the above corollary.*

Definition 31. The *(first) strong difference* $\gamma_2 \underset{1}{\dot{-}} \gamma_1 \in M \otimes W_{D^2}$ is defined to be

$$(\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,0,0,d_2)\in D^4\{(2,4),(3,4)\}})(g^1_{(\gamma_1,\gamma_2)})$$

Proposition 32. *The diagram*

$$\begin{array}{ccccc} & M \otimes W_{D^4\{(1,4),(3,4)\}} & \overset{\mathrm{id}_M \otimes W_{\varphi_2^3}}{\rightarrow} & M \otimes W_{D^3} & \\ \mathrm{id}_M \otimes W_{\psi_2^3} & \downarrow & & \downarrow & \mathrm{id}_M \otimes W_{i_{D^3\{(1,3)\}}^{D^3}} \\ & M \otimes W_{D^3} & \underset{\mathrm{id}_M \otimes W_{i_{D^3\{(1,3)\}}^{D^3}}}{\rightarrow} & M \otimes W_{D^3\{(1,3)\}} & \end{array}$$

is a pullback diagram, where the putative mapping $\varphi_2^3 : D^3 \to D^4\{(1,4),(3,4)\}$ is

$$(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0) \in D^4\{(1,4),(3,4)\}$$

while the putative mapping $\psi_2^3 : D^3 \to D^4\{(1,4),(3,4)\}$ is

$$(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, d_1 d_3) \in D^4\{(1,4),(3,4)\}$$

Proof. This follows from the microlinearity of M and the pullback diagram

of Weil algebras

$$\begin{array}{ccccc} & W_{D^4\{(1,4),(3,4)\}} & \overset{W_{\varphi_2^3}}{\rightarrow} & W_{D^3} & \\ W_{\psi_2^3} & \downarrow & & \downarrow & W_{i^{D^3}_{D^3\{(1,3)\}}} \\ & W_{D^3} & \underset{W_{i^{D^3}_{D^3\{(1,3)\}}}}{\rightarrow} & W_{D^3\{(1,3)\}} & \end{array}$$

□

Corollary 33. *For any* $\gamma_1, \gamma_2 \in M \otimes W_{D^3}$, *if* $\left(\mathrm{id}_M \otimes W_{i^{D^3}_{D^3\{(1,3)\}}}\right)(\gamma_1) = \left(\mathrm{id}_M \otimes W_{i^{D^3}_{D^3\{(1,3)\}}}\right)(\gamma_2)$, *then there exists unique* $\gamma \in M \otimes W_{D^4\{(1,4),(3,4)\}}$ *with* $(\mathrm{id}_M \otimes W_{\varphi_2^3})(\gamma) = \gamma_1$ *and* $(\mathrm{id}_M \otimes W_{\psi_2^3})(\gamma) = \gamma_2$.

Remark 34. Thus γ encodes γ_1 and γ_2, which are in turn recovered from γ via $\mathrm{id}_M \otimes W_{\varphi_2^3}$ and $\mathrm{id}_M \otimes W_{\psi_2^3}$.

Notation 35. *We will write* $g^2_{(\gamma_1,\gamma_2)}$ *for* γ *in the above corollary.*

Definition 36. The *(second) strong difference* $\gamma_2 \overset{\cdot}{\underset{2}{-}} \gamma_1 \in M \otimes W_{D^2}$ is defined to be

$$(\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,d_1,0,d_2)\in D^4\{(1,4),(3,4)\}})(g^2_{(\gamma_1,\gamma_2)})$$

Proposition 37. *The diagram*

$$\begin{array}{ccccc} & M \otimes W_{D^4\{(1,4),(2,4)\}} & \overset{\mathrm{id}_M \otimes W_{\varphi_3^3}}{\rightarrow} & M \otimes W_{D^3} & \\ \mathrm{id}_M \otimes W_{\psi_3^3} & \downarrow & & \downarrow & \mathrm{id}_M \otimes W_{i^{D^3}_{D^3\{(1,2)\}}} \\ & M \otimes W_{D^3} & \underset{\mathrm{id}_M \otimes W_{i^{D^3}_{D^3\{(1,2)\}}}}{\rightarrow} & M \otimes W_{D^3\{(1,2)\}} & \end{array}$$

is a pullback diagram, where the putative mapping $\varphi_3^3 : D^3 \to D^4\{(1,4),(2,4)\}$ *is*

$$(d_1,d_2,d_3) \in D^3 \mapsto (d_1,d_2,d_3,0) \in D^4\{(1,4),(2,4)\}$$

while the putative mapping $\psi_3^3 : D^3 \to D^4\{(1,4),(2,4)\}$ *is*

$$(d_1,d_2,d_3) \in D^3 \mapsto (d_1,d_2,d_3,d_1 d_2) \in D^4\{(1,4),(2,4)\}$$

Proof. This follows from the microlinearity of M and the pullback diagram

of Weil algebras

$$\begin{array}{ccccc} & W_{D^4\{(1,4),(2,4)\}} & \xrightarrow{W_{\varphi_3^3}} & W_{D^3} & \\ W_{\psi_3^3} & \downarrow & & \downarrow & W_{i_{D^3\{(1,2)\}}^{D^3}} \\ & W_{D^3} & \xrightarrow[W_{i_{D^3\{(1,2)\}}^{D^3}}]{} & W_{D^3\{(1,2)\}} & \end{array}$$

□

Corollary 38. *For any* $\gamma_1, \gamma_2 \in M \otimes W_{D^3}$, *if* $\left(\mathrm{id}_M \otimes W_{i_{D^3\{(1,2)\}}^{D^3}}\right)(\gamma_1) = \left(\mathrm{id}_M \otimes W_{i_{D^3\{(1,2)\}}^{D^3}}\right)(\gamma_2)$, *then there exists unique* $\gamma \in M \otimes W_{D^4\{(1,4),(2,4)\}}$ *with* $(\mathrm{id}_M \otimes W_{\varphi_3^3})(\gamma) = \gamma_1$ *and* $(\mathrm{id}_M \otimes W_{\psi_3^3})(\gamma) = \gamma_2$.

Remark 39. Thus γ encodes γ_1 and γ_2, which are in turn recovered from γ via $\mathrm{id}_M \otimes W_{\varphi_3^3}$ and $\mathrm{id}_M \otimes W_{\psi_3^3}$.

Notation 40. *We will write* $g^3_{(\gamma_1,\gamma_2)}$ *for* γ *in the above corollary.*

Definition 41. The *(third) strong difference* $\gamma_2 \underset{3}{\dot{-}} \gamma_1 \in M \otimes W_{D^2}$ is defined to be

$$(\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,0,d_1,d_2)\in D^4\{(1,4),(3,4)\}})(g^3_{(\gamma_1,\gamma_2)})$$

The general Jacobi identity discovered by Nishimura [18] goes as follows:

Theorem 42. *(The General Jacobi Identity) Let* $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in M \otimes W_{D^3}$. *As long as the following three expressions are well defined, they sum up only to vanish:*

$$(\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321})$$
$$(\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}) \dot{-} (\gamma_{312} \underset{2}{\dot{-}} \gamma_{132})$$
$$(\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}) \dot{-} (\gamma_{123} \underset{3}{\dot{-}} \gamma_{213})$$

Now we set out on a long journey to establish the above theorem. Let us begin with

Proposition 43. *The diagram*

$$\begin{array}{ccccc} & M\otimes W_{E[1]} & \overset{\mathrm{id}_M\otimes W_{\eta_1^1}}{\rightarrow} & M\otimes W_{D^4\{(2,4),(3,4)\}} & \\ \mathrm{id}_M\otimes W_{\eta_2^1} & \downarrow & & \downarrow & \mathrm{id}_M\otimes W_{i_{14}^1} \\ & M\otimes W_{D^4\{(2,4),(3,4)\}} & \underset{\mathrm{id}_M\otimes W_{i_{14}^1}}{\rightarrow} & M\otimes W_{D(2)} & \end{array}$$

is a pullback, where the putative object $E[1]$ *is*

$D^7\{(2,6),(3,6),(4,6),(5,6),(1,7),(2,7),(3,7),(4,7),(5,7),(6,7),(2,4),$
$(2,5),(3,4),(3,5)\}$

the putative mapping $i_{14}^1 : D(2) \to D^4\{(2,4),(3,4)\}$ *is*

$$(d_1,d_2)\in D(2)\mapsto (d_1,0,0,d_2)\in D^4\{(2,4),(3,4)\}$$

the putative mapping $\eta_1^1 : D^4\{(2,4),(3,4)\}\to E[1]$ *is*

$$(d_1,d_2,d_3,d_4)\in D^4\{(2,4),(3,4)\}\mapsto$$
$$(d_1,d_2,d_3,0,0,d_4,0)\in E[1]$$

and the putative mapping $\eta_2^1 : D^4\{(2,4),(3,4)\}\to E[1]$ *is*

$$(d_1,d_2,d_3,d_4)\in D^4\{(2,4),(3,4)\}\mapsto$$
$$(d_1,0,0,d_2,d_3,d_4,d_1d_4)\in E[1]$$

Proof. This follows from the microlinearity of M and the pullback diagram of Weil algebras

$$\begin{array}{ccccc} & W_{E[1]} & \overset{W_{\eta_1^1}}{\rightarrow} & W_{D^4\{(2,4),(3,4)\}} & \\ W_{\eta_2^1} & \downarrow & & \downarrow & W_{i_{14}^1} \\ & W_{D^4\{(2,4),(3,4)\}} & \underset{W_{i_{14}^1}}{\rightarrow} & W_{D(2)} & \end{array}$$

□

Notation 44. *We will write* ι_1^1, ι_2^1, ι_3^1 *and* ι_4^1 *for the putative mappings* $\eta_1^1\circ\varphi_1^3$, $\eta_1^1\circ\psi_1^3$, $\eta_2^1\circ\varphi_1^3$ *and* $\eta_2^1\circ\psi_1^3$ *respectively. That is to say, we have*

$$\iota_1^1 : (d_1,d_2,d_3)\in D^3\mapsto (d_1,d_2,d_3,0,0,0,0)\in E[1]$$
$$\iota_2^1 : (d_1,d_2,d_3)\in D^3\mapsto (d_1,d_2,d_3,0,0,d_2d_3,0)\in E[1]$$
$$\iota_3^1 : (d_1,d_2,d_3)\in D^3\mapsto (d_1,0,0,d_2,d_3,0,0)\in E[1]$$
$$\iota_4^1 : (d_1,d_2,d_3)\in D^3\mapsto (d_1,0,0,d_2,d_3,d_2d_3,d_1d_2d_3)\in E[1]$$

Corollary 45. *For any* $\gamma_1,\gamma_2,\gamma_3,\gamma_4\in M\otimes W_{D^3}$, *if the expression*

$$(\gamma_4 \underset{1}{\dot{-}} \gamma_3) \dot{-} (\gamma_2 \underset{1}{\dot{-}} \gamma_1)$$

is well defined, then there exists unique $\gamma \in M \otimes W_{E[1]}$ *such that* $(\mathrm{id}_M \otimes W_{\iota_i^1})(\gamma) = \gamma_i$ $(i = 1, 2, 3, 4)$.

Remark 46. This means that γ encodes γ_1, γ_2, γ_3 and γ_4, which are in turn recovered from γ via $\mathrm{id}_M \otimes W_{\iota_i^1}$'s.

Notation 47. *We will write* $h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}$ *for* γ *in the above corollary.*

Remark 48. We note that

$$g^1_{(\gamma_1,\gamma_2)} = (\mathrm{id}_M \otimes W_{\eta_1^1})(h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})$$
$$g^1_{(\gamma_3,\gamma_4)} = (\mathrm{id}_M \otimes W_{\eta_2^1})(h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})$$

Therefore we have

$$\begin{aligned}\gamma_2 \underset{1}{\dot{-}} \gamma_1 &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,0,0,d_2)\in D^4\{(2,4),(3,4)\}})(g^1_{(\gamma_1,\gamma_2)}) \\ &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,0,0,d_2)\in D^4\{(2,4),(3,4)\}}) \circ (\mathrm{id}_M \otimes W_{\eta_1^1})(h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}) \\ &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,0,0,0,0,d_2,0)\in E[1]})(h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}),\end{aligned}$$

$$\begin{aligned}\gamma_4 \underset{1}{\dot{-}} \gamma_3 &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,0,0,d_2)\in D^4\{(2,4),(3,4)\}})(g^1_{(\gamma_3,\gamma_4)}) \\ &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,0,0,d_2)\in D^4\{(2,4),(3,4)\}}) \circ (\mathrm{id}_M \otimes W_{\eta_2^1})(h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}) \\ &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_1,0,0,0,0,d_2,d_1d_2)\in E[1]})(h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}).\end{aligned}$$

Thus we have

$$g_{(\gamma_2 \underset{1}{\dot{-}} \gamma_1, \gamma_4 \underset{1}{\dot{-}} \gamma_3)} = (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\} \mapsto (d_1,0,0,0,0,d_2,d_3)\in E[1]})(h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})$$

Finally we have

$$\begin{aligned}&(\gamma_4 \underset{1}{\dot{-}} \gamma_3) \dot{-} (\gamma_2 \underset{1}{\dot{-}} \gamma_1) \\ &= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,d)\in D^3\{(1,3),(2,3)\}})(g_{(\gamma_2 \underset{1}{\dot{-}} \gamma_1, \gamma_4 \underset{1}{\dot{-}} \gamma_3)}) \\ &= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,d)\in D^3\{(1,3),(2,3)\}}) \circ (\mathrm{id}_M \\ &\otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\} \mapsto (d_1,0,0,0,0,d_2,d_3)\in E[1]}) \\ &(h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}) \\ &= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,0,0,0,0,d)\in E[1]})(h^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})\end{aligned}$$

Proposition 49. *The diagram*

$$\begin{array}{ccccc} & M\otimes W_{E[2]} & \overset{\mathrm{id}_M\otimes W_{\eta_1^2}}{\rightarrow} & M\otimes W_{D^4\{(1,4),(3,4)\}} & \\ \mathrm{id}_M\otimes W_{\eta_2^2} & \downarrow & & \downarrow & \mathrm{id}_M\otimes W_{i_{24}^2} \\ & M\otimes W_{D^4\{(1,4),(3,4)\}} & \underset{\mathrm{id}_M\otimes W_{i_{24}^2}}{\rightarrow} & M\otimes W_{D(2)} & \end{array}$$

is a pullback, where the putative object $E[2]$ *is*

$D^7\{(1,6),(3,6),(4,6),(5,6),(1,7),(2,7),(3,7),(4,7),(5,7),(6,7),(1,4),$
$(1,5),(3,4),(3,5)\}$

the putative mapping $i_{24}^2 : D(2) \rightarrow D^4\{(1,4),(3,4)\}$ *is*

$$(d_1,d_2)\in D(2)\mapsto (0,d_1,0,d_2)\in D^4\{(1,4),(3,4)\}$$

the putative mapping $\eta_1^2 : D^4\{(1,4),(3,4)\}\} \rightarrow E[2]$ *is*

$$(d_1,d_2,d_3,d_4)\in D^4\{(1,4),(3,4)\}\mapsto$$
$$(d_1,d_2,d_3,0,0,d_4,0)\in E[2]$$

and the putative mapping $\eta_2^2 : D^4\{(1,4),(3,4)\} \rightarrow E[2]$ *is*

$$(d_1,d_2,d_3,d_4)\in D^4\{(1,4),(3,4)\}\mapsto$$
$$(0,d_2,0,d_1,d_3,d_4,d_2d_4)\in E[2]$$

Proof. This follows from the microlinearity of M and the pullback diagram of Weil algebras

$$\begin{array}{ccccc} & W_{E[2]} & \overset{W_{\eta_1^2}}{\rightarrow} & W_{D^4\{(1,4),(3,4)\}} & \\ W_{\eta_2^2} & \downarrow & & \downarrow & W_{i_{24}^2} \\ & W_{D^4\{(1,4),(3,4)\}} & \underset{W_{i_{24}^2}}{\rightarrow} & W_{D(2)} & \end{array}$$

□

Notation 50. *We will write* ι_1^2, ι_2^2, ι_3^2 *and* ι_4^2 *for the putative mappings* $\eta_1^2\circ\varphi_2^3$, $\eta_1^2\circ\psi_2^3$, $\eta_2^2\circ\varphi_2^3$ *and* $\eta_2^2\circ\psi_2^3$ *respectively. That is to say, we have*

$$\iota_1^2 : (d_1,d_2,d_3)\in D^3\mapsto (d_1,d_2,d_3,0,0,0,0)\in E[2]$$
$$\iota_2^2 : (d_1,d_2,d_3)\in D^3\mapsto (d_1,d_2,d_3,0,0,d_2d_3,0)\in E[2]$$
$$\iota_3^2 : (d_1,d_2,d_3)\in D^3\mapsto (0,d_2,0,d_3,d_1,0,0)\in E[2]$$
$$\iota_4^2 : (d_1,d_2,d_3)\in D^3\mapsto (0,d_2,0,d_3,d_1,d_1d_3,d_1d_2d_3)\in E[2]$$

Corollary 51. *For any* $\gamma_1,\gamma_2,\gamma_3,\gamma_4\in M\otimes W_{D^3}$, *if the expression*

$$(\gamma_4 \underset{2}{\dot{-}} \gamma_3) \dot{-} (\gamma_2 \underset{2}{\dot{-}} \gamma_1)$$

is well defined, then there exists unique $\gamma \in M \otimes W_{E[2]}$ *such that* $(\mathrm{id}_M \otimes W_{\iota_i^2})(\gamma) = \gamma_i$ $(i = 1, 2, 3, 4)$.

Remark 52. This means that γ encodes γ_1, γ_2, γ_3 and γ_4, which are in turn recovered from γ via $\mathrm{id}_M \otimes W_{\iota_i^2}$'s.

Notation 53. *We will write* $h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}$ *for* γ *in the above corollary.*

Remark 54. We note that

$$g^2_{(\gamma_1,\gamma_2)} = (\mathrm{id}_M \otimes W_{\eta_1^2})(h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})$$
$$g^2_{(\gamma_3,\gamma_4)} = (\mathrm{id}_M \otimes W_{\eta_2^2})(h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})$$

Therefore we have

$$\begin{aligned}\gamma_2 \underset{2}{\dot{-}} \gamma_1 &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,d_1,0,d_2)\in D^4\{(1,4),(3,4)\}})(g^2_{(\gamma_1,\gamma_2)}) \\ &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,d_1,0,d_2)\in D^4\{(1,4),(3,4)\}}) \circ (\mathrm{id}_M \otimes W_{\eta_1^2})(h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}) \\ &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,d_1,0,0,0,d_2,0)\in E[2]})(h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})\end{aligned}$$

$$\begin{aligned}\gamma_4 \underset{2}{\dot{-}} \gamma_3 &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,d_1,0,d_2)\in D^4\{(1,4),(3,4)\}})(g^2_{(\gamma_3,\gamma_4)}) \\ &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,d_1,0,d_2)\in D^4\{(1,4),(3,4)\}}) \circ (\mathrm{id}_M \otimes W_{\eta_2^2})(h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}) \\ &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,d_1,0,0,0,d_2,d_1d_2)\in E[2]})(h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})\end{aligned}$$

Thus we have

$$g_{(\gamma_2 \underset{2}{\dot{-}} \gamma_1, \gamma_4 \underset{2}{\dot{-}} \gamma_3)} = (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\} \mapsto (0,d_1,0,0,0,d_2,d_3)\in E[2]})(h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})$$

Finally we have

$$\begin{aligned}&(\gamma_4 \underset{2}{\dot{-}} \gamma_3) \dot{-} (\gamma_2 \underset{2}{\dot{-}} \gamma_1) \\ &= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,d)\in D^3\{(1,3),(2,3)\}})(g_{(\gamma_2 \underset{2}{\dot{-}} \gamma_1, \gamma_4 \underset{2}{\dot{-}} \gamma_3)}) \\ &= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,d)\in D^3\{(1,3),(2,3)\}}) \circ (\mathrm{id}_M \\ &\otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\} \mapsto (0,d_1,0,0,0,d_2,d_3)\in E[2]}) \\ &(h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}) \\ &= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,0,0,0,0,d)\in E[2]})(h^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})\end{aligned}$$

Proposition 55. *The diagram*

$$\begin{array}{ccccc} & & \mathrm{id}_M \otimes W_{\eta_1^3} & & \\ & M \otimes W_{E[3]} & \rightarrow & M \otimes W_{D^4\{(1,4),(2,4)\}} & \\ \mathrm{id}_M \otimes W_{\eta_2^3} & \downarrow & & \downarrow & \mathrm{id}_M \otimes W_{i_{34}^3} \\ & M \otimes W_{D^4\{(1,4),(2,4)\}} & \rightarrow & M \otimes W_{D(2)} & \\ & & \mathrm{id}_M \otimes W_{i_{34}^3} & & \end{array}$$

is a pullback, where the putative object $E[3]$ *is*

$D^7\{(1,6),(2,6),(4,6),(5,6),(1,7),(2,7),(3,7),(4,7),(5,7),(6,7),(1,4),$
$(1,5),(2,4),(2,5)\}\}$

the putative mapping $i_{34}^3 : D(2) \rightarrow D^4\{(1,4),(2,4)\}$ *is*

$$(d_1, d_2) \in D(2) \mapsto (0, 0, d_1, d_2) \in D^4\{(1,4),(2,4)\}$$

the putative mapping $\eta_1^3 : D^4\{(1,4),(2,4)\}\} \rightarrow E[3]$ *is*

$$(d_1, d_2, d_3, d_4) \in D^4\{(1,4),(2,4)\} \mapsto$$
$$(d_1, d_2, d_3, 0, 0, d_4, 0) \in E[3]$$

and the putative mapping $\eta_2^3 : D^4\{(1,4),(3,4)\} \rightarrow E[3]$ *is*

$$(d_1, d_2, d_3, d_4) \in D^4\{(1,4),(2,4)\} \mapsto$$
$$(0, 0, d_3, d_1, d_2, d_4, d_3 d_4) \in E[3]$$

Proof. This follows from the microlinearity of M and the pullback diagram of Weil algebras

$$\begin{array}{ccccc} & & W_{\eta_1^3} & & \\ & W_{E[3]} & \rightarrow & W_{D^4\{(1,4),(2,4)\}} & \\ W_{\eta_2^3} & \downarrow & & \downarrow & W_{i_{34}^3} \\ & W_{D^4\{(1,4),(2,4)\}} & \rightarrow & W_{D(2)} & \\ & & W_{i_{34}^3} & & \end{array}$$

□

Notation 56. *We will write* ι_1^3, ι_2^3, ι_3^3 *and* ι_4^3 *for the putative mappings* $\eta_1^3 \circ \varphi_3^3$, $\eta_1^3 \circ \psi_3^3$, $\eta_2^3 \circ \varphi_3^3$ *and* $\eta_2^3 \circ \psi_3^3$ *respectively. That is to say, we have*

$$\iota_1^3 : (d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0, 0, 0, 0) \in E[3]$$
$$\iota_2^3 : (d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0, 0, d_1 d_2, 0) \in E[3]$$
$$\iota_3^3 : (d_1, d_2, d_3) \in D^3 \mapsto (0, 0, d_3, d_1, d_2, 0, 0) \in E[3]$$
$$\iota_4^3 : (d_1, d_2, d_3) \in D^3 \mapsto (0, 0, d_3, d_1, d_2, d_1 d_2, d_1 d_2 d_3) \in E[3]$$

Corollary 57. *For any* $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in M \otimes W_{D^3}$*, if the expression*

$$(\gamma_4 \underset{3}{\dot{-}} \gamma_3) \dot{-} (\gamma_2 \underset{3}{\dot{-}} \gamma_1)$$

is well defined, then there exists unique $\gamma \in M \otimes W_{E[3]}$ *such that* $(\mathrm{id}_M \otimes W_{\iota_i^3})(\gamma) = \gamma_i$ *(*$i = 1, 2, 3, 4$*).*

Remark 58. This means that γ encodes γ_1, γ_2, γ_3 and γ_4, which are in turn recovered from γ via $\mathrm{id}_M \otimes W_{\iota_i^3}$'s.

Notation 59. *We will write* $h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}$ *for* γ *in the above corollary.*

Remark 60. We note that

$$g^3_{(\gamma_1,\gamma_2)} = (\mathrm{id}_M \otimes W_{\eta_1^3})(h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})$$
$$g^3_{(\gamma_3,\gamma_4)} = (\mathrm{id}_M \otimes W_{\eta_2^3})(h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})$$

Therefore we have

$$\begin{aligned}
\gamma_2 \underset{3}{\dot{-}} \gamma_1 &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,0,d_1,d_2)\in D^4\{(1,4),(2,4)\}})(g^3_{(\gamma_1,\gamma_2)}) \\
&= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,0,d_1,d_2)\in D^4\{(1,4),(2,4)\}}) \circ (\mathrm{id}_M \otimes W_{\eta_1^3})(h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}) \\
&= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,0,d_1,0,0,d_2,0)\in E[3]})(h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}),
\end{aligned}$$

$$\begin{aligned}
\gamma_4 \underset{3}{\dot{-}} \gamma_3 &= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,0,d_1,d_2)\in D^4\{(1,4),(2,4)\}})(g^3_{(\gamma_3,\gamma_4)}) \\
&= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,0,d_1,d_2)\in D^4\{(1,4),(2,4)\}}) \circ (\mathrm{id}_M \otimes W_{\eta_2^3})(h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}) \\
&= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (0,0,d_1,0,0,d_2,d_1 d_2)\in E[3]})(h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}).
\end{aligned}$$

Thus we have

$$g_{(\gamma_2 \underset{3}{\dot{-}} \gamma_1, \gamma_4 \underset{3}{\dot{-}} \gamma_3)} = (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\} \mapsto (0,0,d_1,0,0,d_2,d_3)\in E[3]})(h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})$$

Finally we have

$$\begin{aligned}
&(\gamma_4 \underset{3}{\dot{-}} \gamma_3) \dot{-} (\gamma_2 \underset{3}{\dot{-}} \gamma_1) \\
&= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,d)\in D^3\{(1,3),(2,3)\}})(g_{(\gamma_2 \underset{3}{\dot{-}} \gamma_1, \gamma_4 \underset{3}{\dot{-}} \gamma_3)}) \\
&= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,d)\in D^3\{(1,3),(2,3)\}}) \\
&\circ (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\} \mapsto (0,0,d_1,0,0,d_2,d_3)\in E[3]}) \\
&(h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}) \\
&= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,0,0,0,0,d)\in E[3]})(h^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)})
\end{aligned}$$

Now we come to the crucial step in the proof of the general Jacobi identity.

Theorem 61. *The diagram*

$$
\begin{array}{ccccccc}
 & \mathrm{id}_M \otimes W_{h^1_{12}} & & M \otimes W_{E[1]} & & \mathrm{id}_M \otimes W_{h^1_{31}} & \\
 & & \swarrow & \uparrow & \searrow & & \\
 & M \otimes W_{D^3 \oplus D^3} & & M \otimes W_G & & M \otimes W_{D^3 \oplus D^3} & \\
\mathrm{id}_M \otimes W_{h^2_{12}} & \uparrow & \swarrow & & \searrow & \uparrow & \mathrm{id}_M \otimes W_{h^3_{31}} \\
 & M \otimes W_{E[2]} & & & & M \otimes W_{E[3]} & \\
 & & \searrow & & \swarrow & & \\
 & \mathrm{id}_M \otimes W_{h^2_{23}} & & M \otimes W_{D^3 \oplus D^3} & & \mathrm{id}_M \otimes W_{h^3_{23}} &
\end{array}
$$

is a limit diagram with the three unnamed arrows being

$$\mathrm{id}_M \otimes W_{k_1} : M \otimes W_G \to M \otimes W_{E[1]}$$
$$\mathrm{id}_M \otimes W_{k_2} : M \otimes W_G \to M \otimes W_{E[2]}$$
$$\mathrm{id}_M \otimes W_{k_3} : M \otimes W_G \to M \otimes W_{E[3]}$$

where the putative object G is

$D^8\{(2,4),(3,4),(1,5),(3,5),(1,6),(2,6),(4,5),(4,6),(5,6),(1,7),(2,7),(3,7),(4,7),(5,7),(6,7),(1,8),(2,8),(3,8),(4,8),(5,8),(6,8),(7,8)\}$,

the putative mapping $k_1 : E[1] \to G$ is

$$(d_1,d_2,d_3,d_4,d_5,d_6,d_7) \in E[1] \mapsto$$
$$(d_1, d_2+d_4, d_3+d_5, d_6-d_2d_3-d_4d_5, -d_1d_5, d_1d_4, d_7+d_1d_2d_3, d_1d_2d_3) \in G,$$

the putative mapping $k_2 : E[2] \to G$ is

$$(d_1,d_2,d_3,d_4,d_5,d_6,d_7) \in E[2] \mapsto$$
$$(d_1+d_5, d_2, d_3+d_4, -d_2d_3, d_6-d_1d_3-d_4d_5, d_1d_2, d_2d_4d_5, d_7) \in G,$$

the putative mapping $k_3 : E[3] \to G$ is

$$(d_1,d_2,d_3,d_4,d_5,d_6,d_7) \in E[3] \mapsto$$
$$(d_1+d_4, d_2+d_5, d_3, -d_4d_5, -d_1d_3, d_6, -d_7, -d_7+d_3d_4d_5) \in G,$$

the putative mapping h^1_{12} is $\iota^1_2 \oplus \iota^1_3$, the putative mapping h^2_{12} is $\iota^2_4 \oplus \iota^2_1$, the putative mapping h^2_{23} is $\iota^2_2 \oplus \iota^2_3$, the putative mapping h^3_{23} is $\iota^3_4 \oplus \iota^3_1$, the putative mapping h^3_{31} is $\iota^3_2 \oplus \iota^3_3$, and the putative mapping h^1_{31} is $\iota^1_4 \oplus \iota^2_1$.

The proof of the above theorem follows directly from the following lemma.

Lemma 62. *The following diagram is a limit diagram of Weil algebras:*

$$\begin{array}{ccccccc}
 & W_{h_{12}^{1}} & & W_{E[1]} & & W_{h_{31}^{1}} & \\
 & & \swarrow & \uparrow & \searrow & & \\
 & W_{D^{3}\oplus D^{3}} & & W_{G} & & W_{D^{3}\oplus D^{3}} & \\
W_{h_{12}^{2}} & \uparrow & \swarrow & & \searrow & \uparrow & W_{h_{31}^{3}} \\
 & W_{E[2]} & & & & W_{E[3]} & \\
 & & \searrow & & \swarrow & & \\
 & W_{h_{23}^{2}} & & W_{D^{3}\oplus D^{3}} & & W_{h_{23}^{3}} &
\end{array}$$

Proof. Let $\gamma_1 \in W_{E[1]}$, $\gamma_2 \in W_{E[2]}$, $\gamma_3 \in W_{E[3]}$ and $\gamma \in W_G$ so that they are polynomials with real coefficients in the following forms:

$$\begin{aligned}
&\gamma_1(X_1,X_2,X_3,X_4,X_5,X_6,X_7)\\
&= a^1 + a_1^1X_1 + a_2^1X_2 + a_3^1X_3 + a_4^1X_4 + a_5^1X_5 + a_6^1X_6 + a_7^1X_7 + a_{12}^1X_1X_2\\
&\quad + a_{13}^1X_1X_3 + a_{14}^1X_1X_4 + a_{15}^1X_1X_5 + a_{16}^1X_1X_6 + a_{23}^1X_2X_3 + a_{45}^1X_4X_5\\
&\quad + a_{123}^1X_1X_2X_3 + a_{145}^1X_1X_4X_5
\end{aligned}$$

$$\begin{aligned}
&\gamma_2(X_1,X_2,X_3,X_4,X_5,X_6,X_7)\\
&= a^2 + a_1^2X_1 + a_2^2X_2 + a_3^2X_3 + a_4^2X_4 + a_5^2X_5 + a_6^2X_6 + a_7^2X_7 + a_{12}^2X_1X_2\\
&\quad + a_{13}^2X_1X_3 + a_{23}^2X_2X_3 + a_{24}^2X_2X_4 + a_{25}^2X_2X_5 + a_{26}^2X_2X_6 + a_{45}^2X_4X_5\\
&\quad + a_{123}^2X_1X_2X_3 + a_{245}^2X_2X_4X_5
\end{aligned}$$

$$\begin{aligned}
&\gamma_3(X_1,X_2,X_3,X_4,X_5,X_6,X_7)\\
&= a^3 + a_1^3X_1 + a_2^3X_2 + a_3^3X_3 + a_4^3X_4 + a_5^3X_5 + a_6^3X_6 + a_7^3X_7 + a_{12}^3X_1X_2\\
&\quad + a_{13}^3X_1X_3 + a_{23}^3X_2X_3 + a_{34}^3X_3X_4 + a_{35}^3X_3X_5 + a_{36}^3X_3X_6 + a_{45}^3X_4X_5\\
&\quad + a_{123}^3X_1X_2X_3 + a_{345}^3X_3X_4X_5
\end{aligned}$$

$$\begin{aligned}
&\gamma(X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8)\\
&= b + b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4 + b_5X_5 + b_6X_6 + b_7X_7 + b_8X_8 + b_{12}X_1X_2\\
&\quad + b_{13}X_1X_3 + b_{14}X_1X_4 + b_{23}X_2X_3 + b_{25}X_2X_5 + b_{36}X_3X_6
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&W_{h_{12}^1}(\gamma_1)(X_1,X_2,X_3,X_4,X_5,X_6)\\
&= a^1 + a_1^1X_1 + a_2^1X_2 + a_3^1X_3 + a_6^1X_2X_3 + a_{12}^1X_1X_2 + a_{13}^1X_1X_3 + a_{16}^1X_1X_2X_3 + a_{23}^1X_2X_3\\
&\quad + a_{123}^1X_1X_2X_3 + a_1^1X_4 + a_4^1X_5 + a_5^1X_6 + a_{14}^1X_4X_5 + a_{15}^1X_4X_6 + a_{45}^1X_5X_6\\
&\quad + a_{145}^1X_4X_5X_6
\end{aligned}$$

$$= a^1 + a^1_1 X_1 + a^1_2 X_2 + a^1_3 X_3 + a^1_{12} X_1 X_2 + a^1_{13} X_1 X_3 + (a^1_6 + a^1_{23}) X_2 X_3$$
$$+ (a^1_{16} + a^1_{123}) X_1 X_2 X_3 + a^1_1 X_4 + a^1_4 X_5 + a^1_5 X_6 + a^1_{14} X_4 X_5 + a^1_{15} X_4 X_6 + a^1_{45} X_5 X_6$$
$$+ a^1_{145} X_4 X_5 X_6$$

$$W_{h^2_{12}}(\gamma_2)(X_1, X_2, X_3, X_4, X_5, X_6)$$
$$= a^2 + a^2_2 X_2 + a^2_4 X_3 + a^2_5 X_1 + a^2_6 X_1 X_3 + a^2_7 X_1 X_2 X_3 + a^2_{24} X_2 X_3 + a^2_{25} X_1 X_2$$
$$+ a^2_{26} X_1 X_2 X_3 + a^2_{45} X_1 X_3 + a^2_{245} X_1 X_2 X_3 + a^2_1 X_4 + a^2_2 X_5 + a^2_3 X_6 + a^2_{12} X_4 X_5$$
$$+ a^2_{13} X_4 X_6 + a^2_{23} X_5 X_6 + a^2_{123} X_4 X_5 X_6$$
$$= a^2 + a^2_5 X_1 + a^2_2 X_2 + a^2_4 X_3 + a^2_{25} X_1 X_2 + (a^2_6 + a^2_{45}) X_1 X_3 + a^2_{24} X_2 X_3 +$$
$$(a^2_7 + a^2_{26} + a^2_{245}) X_1 X_2 X_3 + a^2_1 X_4 + a^2_2 X_5 + a^2_3 X_6 + a^2_{12} X_4 X_5 + a^2_{13} X_4 X_6 + a^2_{23} X_5 X_6$$
$$+ a^2_{123} X_4 X_5 X_6$$

Therefore the condition that $W_{h^1_{12}}(\gamma_1) = W_{h^2_{12}}(\gamma_2)$ is equivalent to the following conditions as a whole:

$$a^1 = a^2 \tag{16}$$
$$a^1_1 = a^2_5,\ a^1_2 = a^2_2,\ a^1_3 = a^2_4,\ a^1_1 = a^2_1,\ a^1_4 = a^2_2,\ a^1_5 = a^2_3 \tag{17}$$
$$a^1_{12} = a^2_{25},\ a^1_{13} = a^2_6 + a^2_{45},\ a^1_6 + a^1_{23} = a^2_{24},\ a^1_{14} = a^2_{12},\ a^1_{15} = a^2_{13},$$
$$a^1_{45} = a^2_{23} \tag{18}$$
$$a^1_{16} + a^1_{123} = a^2_7 + a^2_{26} + a^2_{245},\ a^1_{145} = a^2_{123} \tag{19}$$

By the same token, the condition that $W_{h^2_{23}}(\gamma_2) = W_{h^3_{23}}(\gamma_3)$ is equivalent to the following conditions as a whole:

$$a^2 = a^3 \tag{20}$$
$$a^2_2 = a^3_5,\ a^2_3 = a^3_3,\ a^2_1 = a^3_4,\ a^2_2 = a^3_2,\ a^2_4 = a^3_3,\ a^2_5 = a^3_1 \tag{21}$$
$$a^2_{23} = a^3_{35},\ a^2_{12} = a^3_6 + a^3_{45},\ a^2_6 + a^2_{13} = a^3_{34},\ a^2_{24} = a^3_{23},\ a^2_{25} = a^3_{12},$$
$$a^2_{45} = a^3_{13} \tag{22}$$
$$a^2_{26} + a^2_{123} = a^3_7 + a^3_{36} + a^3_{345},\ a^2_{245} = a^3_{123} \tag{23}$$

By the same token again, the condition that $W_{h^3_{31}}(\gamma_3) = W_{h^1_{31}}(\gamma_1)$ is equivalent to the following conditions as a whole:

$$a^3 = a^1 \tag{24}$$
$$a^3_3 = a^1_5,\ a^3_1 = a^1_1,\ a^3_2 = a^1_4,\ a^3_3 = a^1_3,\ a^3_4 = a^1_1,\ a^3_5 = a^1_2 \tag{25}$$
$$a^3_{13} = a^1_{15},\ a^3_{23} = a^1_6 + a^1_{45},\ a^3_6 + a^3_{12} = a^1_{14},\ a^3_{34} = a^1_{13},\ a^3_{35} = a^1_{23},$$
$$a^3_{45} = a^1_{12} \tag{26}$$
$$a^3_{36} + a^3_{123} = a^1_7 + a^1_{16} + a^1_{145},\ a^3_{345} = a^1_{123} \tag{27}$$

The three conditions (16), (20) and (24) can be combined into

$$a^1 = a^2 = a^3 \tag{28}$$

The three conditions (17), (21) and (25) are to be superseded by the following three conditions as a whole:

$$a_1^1 = a_1^2 = a_1^3 = a_5^2 = a_4^3 \tag{29}$$

$$a_2^1 = a_2^2 = a_2^3 = a_4^1 = a_5^3 \tag{30}$$

$$a_3^1 = a_3^2 = a_3^3 = a_5^1 = a_4^2 \tag{31}$$

The three conditions (18), (22) and (26) are equivalent to the following six conditions as a whole:

$$a_{12}^1 = a_{12}^2 = a_{12}^3 \tag{32}$$

$$a_{13}^1 = a_{13}^2 = a_{13}^3 \tag{33}$$

$$a_{23}^1 = a_{23}^2 = a_{23}^3 \tag{34}$$

$$a_{14}^1 = a_{12}^1 + a_6^3,\ a_{15}^1 = a_{13}^1 - a_6^2,\ a_{45}^1 = a_{23}^1 \tag{35}$$

$$a_{24}^2 = a_{23}^2 + a_6^1,\ a_{25}^2 = a_{12}^2 - a_6^3,\ a_{45}^2 = a_{13}^2 \tag{36}$$

$$a_{34}^3 = a_{13}^3 + a_6^2,\ a_{35}^3 = a_{23}^3 - a_6^1,\ a_{45}^3 = a_{12}^3 \tag{37}$$

The conditions (19), (23) and (27) imply that

$$\begin{aligned} & a_7^1 + a_7^2 + a_7^3 \\ &= (a_{36}^3 + a_{123}^3 - a_{16}^1 - a_{145}^1) + (a_{16}^1 + a_{123}^1 - a_{26}^2 - a_{245}^2) + (a_{26}^2 + a_{123}^2 - a_{36}^3 - a_{345}^3) \\ &= (a_{36}^3 + a_{123}^3 - a_{16}^1 - a_{123}^2) + (a_{16}^1 + a_{123}^1 - a_{26}^2 - a_{123}^3) + (a_{26}^2 + a_{123}^2 - a_{36}^3 - a_{123}^1) \\ &= 0 \end{aligned} \tag{38}$$

Therefore the three conditions (19), (23) and (27) are to be replaced by the following five conditions as a whole:

$$a_{145}^1 - a_{123}^1 = a_7^3 + a_{36}^3 - a_{26}^2 \tag{39}$$

$$a_{245}^2 - a_{123}^2 = a_7^1 + a_{16}^1 - a_{36}^3 \tag{40}$$

$$a_{345}^3 - a_{123}^3 = a_7^2 + a_{26}^2 - a_{16}^1 \tag{41}$$

$$a_{145}^1 = a_{123}^2,\ a_{245}^2 = a_{123}^3 \tag{42}$$

$$a_7^1 + a_7^2 + a_7^3 = 0 \tag{43}$$

Indeed, the condition that $a_{345}^3 = a_{123}^1$ is derivable from the above five conditions, as is to be demonstrated in the following:

$$\begin{aligned} & a_{345}^3 \\ &= a_{123}^3 + a_7^2 + a_{26}^2 - a_{16}^1 \quad [(41)] \end{aligned}$$

$$= a^2_{245} + a^2_7 + a^2_{26} - a^1_{16} \quad [(42)]$$
$$= a^2_{123} + a^1_7 - a^3_{36} + a^2_7 + a^2_{26} \quad [(40)]$$
$$= a^1_{145} + a^1_7 - a^3_{36} + a^2_7 + a^2_{26} \quad [(42)]$$
$$= a^1_{123} + a^1_7 + a^2_7 + a^3_7 \quad [(39)]$$
$$= a^1_{123} \quad [(43)]$$

Now it is not difficult to see that $W_{h^1_{12}}(\gamma_1) = W_{h^2_{12}}(\gamma_2)$, $W_{h^2_{23}}(\gamma_2) = W_{h^3_{23}}(\gamma_3)$ and $W_{h^3_{31}}(\gamma_3) = W_{h^1_{31}}(\gamma_1)$ exactly when there exists $\gamma \in W_G$ with $\gamma_i = W_{k_i}(\gamma)$ $(i = 1, 2, 3)$, in which γ should uniquely be of the following form:

$$\gamma(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$
$$= a^1 + a^1_1X_1 + a^1_2X_2 + a^1_3X_3 + a^1_6X_4 + a^2_6X_5 + a^3_6X_6 + a^1_7X_7 + a^2_7X_8 + a^1_{12}X_1X_2$$
$$+ a^1_{13}X_1X_3 + a^1_{16}X_1X_4 + (a^2_{23} + a^1_6)X_2X_3 + a^2_{26}X_2X_5 + a^3_{36}X_3X_6$$

This completes the proof of the theorem. □

Corollary 63. *For any γ_{123}, γ_{132}, γ_{213}, γ_{231}, γ_{312}, $\gamma_{321} \in M \otimes W_{D^3}$, if all expressions (2.9)-(2.11) are well defined, then there exists unique $\gamma \in M \otimes W_G$ such that*

$$(\mathrm{id}_M \otimes W_{k_1})(\gamma) = h^1_{(\gamma_{321},\gamma_{231},\gamma_{132},\gamma_{123})}$$
$$(\mathrm{id}_M \otimes W_{k_2})(\gamma) = h^2_{(\gamma_{132},\gamma_{312},\gamma_{213},\gamma_{231})}$$
$$(\mathrm{id}_M \otimes W_{k_3})(\gamma) = h^3_{(\gamma_{213},\gamma_{123},\gamma_{321},\gamma_{312})}$$

Remark 64. This means that γ encodes γ_{123}, γ_{132}, γ_{213}, γ_{231}, γ_{312} and γ_{321}. We can decode γ, by way of example, into γ_{123} via $\mathrm{id}_M \otimes W_{k_1 \circ \iota^1_4}$ or $\mathrm{id}_M \otimes W_{k_3 \circ \iota^3_2}$.

Proof. (of the Corollary) Since

$$(\mathrm{id}_M \otimes W_{h^1_{12}})(h^1_{(\gamma_{321},\gamma_{231},\gamma_{132},\gamma_{123})}) = l_{(\gamma_{231},\gamma_{132})} = (\mathrm{id}_M \otimes W_{h^2_{12}})(h^2_{(\gamma_{132},\gamma_{312},\gamma_{213},\gamma_{231})}),$$
$$(\mathrm{id}_M \otimes W_{h^2_{23}})(h^2_{(\gamma_{132},\gamma_{312},\gamma_{213},\gamma_{231})}) = l_{(\gamma_{312},\gamma_{213})} = (\mathrm{id}_M \otimes W_{h^3_{23}})(h^3_{(\gamma_{213},\gamma_{123},\gamma_{321},\gamma_{312})})$$

and

$$(\mathrm{id}_M \otimes W_{h^3_{31}})(h^3_{(\gamma_{213},\gamma_{123},\gamma_{321},\gamma_{312})}) = l_{(\gamma_{123},\gamma_{321})} = (\mathrm{id}_M \otimes W_{h^1_{31}})(h^1_{(\gamma_{321},\gamma_{231},\gamma_{132},\gamma_{123})}),$$

the desired conclusion follows directly from the above theorem. □

Notation 65. *We will write $m_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})}$ or m for short for the above γ.*

Once the above theorem is established, we can easily establish the general Jacobi identity as follows:

Proof. (of the general Jacobi identity) Indeed we note that for any $d \in D$, we have

$$(\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,d)\in E[1]})(h^1_{(\gamma_{321},\gamma_{231},\gamma_{132},\gamma_{123})})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,d)\in E[1]}) \circ (\mathrm{id}_M \otimes W_{k_1})(m)$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,d,0)\in G})(m)$$
$$(\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}) \dot{-} (\gamma_{312} \underset{2}{\dot{-}} \gamma_{132})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,d)\in E[2]})(h^2_{(\gamma_{132},\gamma_{312},\gamma_{213},\gamma_{231})})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,d)\in E[2]}) \circ (\mathrm{id}_M \otimes W_{k_2})(m)$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,0,d)\in G})(m)$$
$$(\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}) \dot{-} (\gamma_{123} \underset{3}{\dot{-}} \gamma_{213})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,d)\in E[3]})(h^3_{(\gamma_{213},\gamma_{123},\gamma_{321},\gamma_{312})})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,d)\in E[3]}) \circ (\mathrm{id}_M \otimes W_{k_3})(m)$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,-d,-d)\in G})(m)$$

Therefore, letting t_1, t_2 and t_3 denote the three expressions in Theorem 42 in order, we have

$$l_{(t_1,t_2,t_3)} = (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D(3)\mapsto(0,0,0,0,0,0,d_1-d_3,d_2-d_3)\in G})(m)$$

This means that

$$t_1 + t_2 + t_3$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(d,d,d)\in D(3)})(l_{(t_1,t_2,t_3)})$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(d,d,d)\in D(3)})$$
$$\circ (\mathrm{id}_M \otimes W_{(d_1,d_2,d_3)\in D(3)\mapsto(0,0,0,0,0,0,d_1-d_3,d_2-d_3)\in G})(m)$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,d-d,d-d)\in G})(m)$$
$$= (\mathrm{id}_M \otimes W_{d\in D\mapsto(0,0,0,0,0,0,0,0)\in G})(m)$$

This completes the proof of the general Jacobi identity. □

6. From the General Jacobi Identity to the Jacobi Identity of Lie Brackets

Theorem 66. *For any $X, Y \in \aleph(M)$, we have*

$$[X,Y] = Y * X \dot{-} (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2})(X * Y)$$

Proof. Let the putative mapping $i : D^3\{(1,3),(2,3)\} \to D^3$ be the canonical embedding. Since

$$(\mathrm{id}_M \otimes W_\varphi)(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\} \mapsto (d_2,d_3,d_1)\in D^3})(X * [X,Y] * Y)$$
$$= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,0,d_1)\in D^3})(X * [X,Y] * Y)$$
$$= (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2})(X * Y)$$

and

$$(\mathrm{id}_M \otimes W_\psi)(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\} \mapsto (d_2,d_3,d_1)\in D^3})(X * [X,Y] * Y)$$
$$= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,d_1 d_2,d_1)\in D^3})(X * [X,Y] * Y)$$
$$= (\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,-d_2,d_1,d_2,-d_1,d_1)\in D^6})(X * X * Y * X * Y * Y)$$
$$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})((\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,-d_2)\in D^2})(Y * Y), Y * X,$$
$$(\mathrm{id}_M \otimes W_{(d_1,d_2)\in D^2 \mapsto (-d_1,d_1)\in D^2})(X * X))$$
[by the bifunctionality of $\otimes$]
$$= (\circ_{M^M} \otimes \mathrm{id}_{W_{D^2}})(I_2, Y * X, I_2)$$
$$= Y * X$$

we have

$$Y * X \dot{-} (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2})(X * Y)$$
$$= (\mathrm{id}_M \otimes W_{d\in D \mapsto (0,0,d)\in D^3\{(1,3),(2,3)\}})$$
$$(\mathrm{id}_{M^M} \otimes W_{(d_1,d_2,d_3)\in D^3\{(1,3),(2,3)\} \mapsto (d_2,d_3,d_1)\in D^3})$$
$$(X * [X,Y] * Y)$$
$$= (\mathrm{id}_{M^M} \otimes W_{d\in D \mapsto (0,d,0)\in D^3})(X * [X,Y] * Y)$$
$$= [X,Y]$$

Thus we are done. □

The following proposition should be obvious.

Proposition 67. *For any $X \in \aleph(M)$ and any $\gamma_1, \gamma_2 \in \aleph^2(M)$ with $(\mathrm{id}_{M^M} \otimes W_{i^{D^2}_{D(2)}})(\gamma_1) = (\mathrm{id}_{M^M} \otimes W_{i^{D^2}_{D(2)}})(\gamma_2)$, we have the following:*

$$\gamma_1 * X|_{D^3\{(2,3)\}} = \gamma_2 * X|_{D^3\{(2,3)\}}$$

$$X * \gamma_1|_{D^3\{(1,2)\}} = X * \gamma_2|_{D^3\{(1,2)\}}$$

$$\gamma_1 * X \underset{1}{\dot{-}} \gamma_2 * X = (\gamma_1 \dot{-} \gamma_2) * X$$

$$X * \gamma_1 \underset{3}{\dot{-}} X * \gamma_2 = (\mathrm{id}_{M^M} \otimes W_{\tau_{(12)}})(X * (\gamma_1 \dot{-} \gamma_2))$$

Theorem 68. *For any $X, Y, Z \in \aleph(M)$, we have*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Proof. we define $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in \aleph^3(M)$ as follows:

$$\gamma_{123} = Z * Y * X$$

$$\gamma_{132} = (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2,d_3)\in D^3 \mapsto (d_1,d_3,d_2)\in D^3})(Y * Z * X)$$

$$\gamma_{213} = (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2,d_3)\in D^3 \mapsto (d_2,d_1,d_3)\in D^3})(Z * X * Y)$$

$$\gamma_{231} = (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2,d_3)\in D^3 \mapsto (d_2,d_3,d_1)\in D^3})(X * Z * Y)$$

$$\gamma_{312} = (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2,d_3)\in D^3 \mapsto (d_3,d_1,d_2)\in D^3})(Y * X * Z)$$

$$\gamma_{321} = (\mathrm{id}_{M^M} \otimes W_{(d_1,d_2,d_3)\in D^3 \mapsto (d_3,d_2,d_1)\in D^3})(X * Y * Z)$$

Then the right-hand sides of the following three identities are meaningful, and all the three identities hold:

$$[X, [Y, Z]] = (\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321})$$

$$[Y, [Z, X]] = (\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}) \dot{-} (\gamma_{312} \underset{2}{\dot{-}} \gamma_{132})$$

$$[Z, [X, Y]] = (\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}) \dot{-} (\gamma_{123} \underset{3}{\dot{-}} \gamma_{213})$$

Therefore the desired result follows from the general Jacobi identity. □

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