International Journal of Pure and Applied Mathematics
Volume 64 No. 1 2010, 103-120

THEORY OF GENERALIZED DIFFERENCE OPERATOR OF n-TH KIND AND ITS APPLICATIONS IN NUMBER THEORY (PART - I)

R. Pugalarasu[1], M. Maria Susai Manuel[2 §], V. Chandrasekar[3], G. Britto Antony Xavier[4]

[1,3,4]Department of Mathematics
Sacred Heart College
Tirupattur, 635 601, Tamil Nadu, INDIA

[2]Department of Science and Humanities
R.M.D. Engineering College
Kavaraipettai, 601 206, Tamil Nadu, INDIA
e-mail: manuelmsm_03@yahoo.co.in

Abstract: In this paper, we define the generalized difference operator of n-th kind denoted as $\Delta_{\ell_1,\ell_2,\cdots\ell_n}$ for the real or complex valued function $u(k)$, $k \in [0,\infty)$ as

$$\begin{aligned}\Delta_{\ell_1,\ell_2,\cdots\ell_n}u(k) &= u(k+\ell_1+\ell_2+\cdots+\ell_n) - \big\{u(k+\ell_1+\ell_2+\cdots+\ell_{n-1}) \\ &+ u(k+\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n) + u(k+\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n)+\cdots \\ &+ u(k+\ell_1+\ell_3+\cdots+\ell_n) + u(k+\ell_2+\ell_3+\cdots+\ell_n)\big\} + \big\{u(k+\ell_1+\cdots+\ell_{n-2}) \\ &+ u(k+\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}) + \cdots + u(k+\ell_1+\ell_3+\cdots+\ell_{n-1}) \\ &+ u(k+\ell_2+\ell_3+\cdots+\ell_{n-1}) + u(k+\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n) + u(k+\ell_1+\ell_2 \\ &+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n) + \cdots + u(k+\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n) + \cdots + u(k+\ell_1 \\ &+ \ell_4+\cdots+\ell_n) + u(k+\ell_2+\ell_4+\cdots+\ell_n) + u(k+\ell_3+\ell_4+\cdots+\ell_n)\big\} \\ &+\cdots+(-1)^{n-1}\big\{u(k+\ell_1)+\cdots+u(k+\ell_n)\big\} + (-1)^n u(k). \quad (1)\end{aligned}$$

and obtain its relation with Δ_ℓ and E, the generalized difference and shift operators respectively. Also we present the discrete version of Leibnitz Theorem, binomial theorem and Newton's formula according to $\Delta_{\ell_1,\ell_2,\cdots\ell_n}$. By defining the inverse, $\Delta^{-1}_{\ell_1,\ell_2,\cdots\ell_n}$ and using S^n_t's, the Stirling numbers of the second kind,

Received: July 18, 2010

§Correspondence author

we establish a formula for the sum of $(n-1)$ times partial sums of the m-th powers of an arithmetic progression in number theory.

AMS Subject Classification: 39A12
Key Words: generalized difference operator, generalized polynomial factorial, partial sums

1. Introduction

The theory of the generalized difference operator Δ_ℓ defined as $\Delta_\ell u(k) = u(k+\ell) - u(k), \ell \in \mathbb{N}$ is developed in [4]. In [4], the authors obtained the generalized version of Leibnitz, binomial, Montmorte's theorems, Newton's formula, formulae for sum of the n-th powers of an arithmetic progression, the sum of the products of n consecutive terms of an arithmetic progression and the sum of an arithmetico-geometric progression with respect to Δ_ℓ. Qualitative properties of certain class of generalized difference equations (see [5]), theory of $\Delta_{\pm\ell}$, generalized Bernoulli polynomials $B_n(k,-\ell)$ for $-\ell$ with applications [7], rotatory behaviors of certain class of generalized difference equations (see [6, 8]) are some of the applications of equations involving Δ_ℓ. In this paper, we extend Δ_ℓ to $\Delta_{\ell_1,\ell_2,\cdots,\ell_n}$ the generalized difference operator of the n-th kind and obtain a formula for the sum $P^{n-1}S$, where S is the sum of all the m-th powers of the A.P. $a^m, (a+d)^m, (a+2d)^m, \cdots, (a+(k-1)d)^m$

$$S = a^m + (a+d)^m + (a+2d)^m + \cdots + (a+(k-1)d)^m,$$

PS is the sum of all the partial sums of terms of S

$$PS = a^m + \{a^m + (a+d)^m\} + \{a^m + (a+d)^m + (a+2d)^m\} + \cdots \\ + \{a^m + (a+d)^m + (a+2d)^m +, \cdots, +(a+(k-1)d)^m\},$$

P^2S is the sum of all the partial sums of terms of PS,

$$P^2S = a^m + \{a^m + (a^m + (a+d)^m)\} + \{a^m + (a^m + (a+d)^m) \\ + (a^m + (a+d)^m + (a+2d)^m)\} + \cdots + \{a^m + (a^m + (a+d)^m) \\ + (a^m + (a+d)^m + (a+2d)^m + \ldots + (a+(k-1)d)^m)\}.$$

Similarly, P^nS is the sum of all the partial sums of terms of $P^{n-1}S$. When $n = 1, 2, 3, \cdots$, we get values of the sums $S, PS, P^2S, \cdots$, respectively.

Throughout this paper, we use the following assumptions:

(i) $[x]$ is integer part of x,

(ii) $c, c_0, c_1, \cdots, c_n$ are constants,

(iii) $rC_i = \frac{r!}{(r-i)!i!}$ where $0! = 1, r! = 1.2.\cdots r$ and,

(iv) $\mathbb{N}(a) = \{a, a+1, a+2, \ldots, \}$.

2. Preliminaries

Definition 2.1. For the real or complex valued function $u(k), k \in [0, \infty)$ is a complex valued function, we define the generalized difference operator of the n-th kind as given in (1).

The following is the relations among $\Delta_{\ell_1,\ell_2,\cdots\ell_n}$, the shift operator E and lower orders of the generalized difference operator of n-th kind.

Lemma 2.2. *If E is the usual shift operator and $\ell_j, j = 1, 2, \ldots, n$ are positive reals, then*

$$(i) \quad \Delta_{\ell_1,\ell_2,\cdots\ell_n} = E^{\ell_1+\ell_2+\cdots+\ell_n} - \big\{E^{\ell_1+\ell_2+\cdots+\ell_{n-1}} + E^{\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n} + \cdots + E^{\ell_2+\ell_3+\cdots+\ell_n}\big\} + \big\{E^{\ell_1+\ell_2+\cdots+\ell_{n-2}} + E^{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}} + \cdots + E^{\ell_2+\ell_3+\cdots+\ell_{n-1}} + E^{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n} + E^{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n} + \cdots + E^{\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n} + E^{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n} + E^{\ell_1+\ell_2+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n} + \cdots + E^{\ell_2+\ell_3+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n} + \cdots + E^{\ell_1+\ell_4+\cdots+\ell_n} + E^{\ell_2+\ell_4+\cdots+\ell_n} + E^{\ell_3+\ell_4+\cdots+\ell_n}\big\} + \cdots + (-1)^{n-1}\big\{E^{\ell_1} + E^{\ell_2} + \cdots + E^{\ell_n}\big\} + (-1)^n. \quad (2)$$

$$(ii) \quad \Delta_{\ell_1,\ell_2,\cdots\ell_n} = \Delta_{\ell_1+\ell_2+\cdots+\ell_n} - \big\{\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-1}} + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n} + \cdots + \Delta_{\ell_2+\ell_3+\cdots+\ell_n}\big\} + \big\{\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}} + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}} + \cdots + \Delta_{\ell_2+\ell_3+\cdots+\ell_{n-1}} + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n} + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n} + \cdots + \Delta_{\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n} + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n} + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n} + \cdots + \Delta_{\ell_2+\ell_3+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n}\big\} + \cdots + (-1)^{n-1}\big\{\Delta_{\ell_1} + \Delta_{\ell_2} + \cdots + \Delta_{\ell_n}\big\}. \quad (3)$$

$$(iii) \quad \Delta_{\ell_1,\ell_2,\cdots\ell_n} = \Delta_{\ell_1}\Delta_{\ell_2}\cdots\Delta_{\ell_n}. \quad (4)$$

Proof. (i) follows from equation (1) and $E^{\ell}u(k) = u(k+\ell)$.

(ii) follows from (1), $E^{\ell} = (1 + \Delta_{\ell})$ and $\sum\limits_{i=0}^{n} nC_i(-1)^i = 0$.

(iii) follows from (1) and $\Delta_{\ell_1,\ell_2,\cdots\ell_n}u(k) = \Delta_{\ell_1}(\Delta_{\ell_2}(\cdots(\Delta_{\ell_n}u(k))\cdots))$. □

Lemma 2.3. *If ℓ_j's, $j = 1, 2, \ldots, n$ are positive integers, then*

$$\Delta_{\ell_1,\ell_2,\cdots\ell_n} = \prod_{j=1}^{n} \Big(\sum_{i_j=1}^{\ell_j} \ell_j C_{i_j} \Delta^{i_j} \Big). \tag{5}$$

The following lemma is an immediate consequence of Definition 2.1.

Lemma 2.4. *Let $u(k)$ and $v(k), k \in [0, \infty)$ be real or complex valued functions. Then:*

(i) $\Delta_{\ell_1,\ell_2,\cdots\ell_n}\{c_1 u(k) + c_2 v(k)\} = c_1 \Delta_{\ell_1,\ell_2,\cdots\ell_n} u(k) + c_2 \Delta_{\ell_1,\ell_2,\cdots\ell_n} v(k)$.

(ii) $\Delta_{\ell_1,\ell_2,\cdots\ell_n}\{u(k)v(k)\} = u(k)\Delta_{\ell_1,\ell_2,\cdots,\ell_n}\big(v(k)\big) + \Big\{v(k+\ell_1+\ell_2+\cdots+\ell_n)\Delta_{\ell_1+\ell_2+\cdots+\ell_n} - \big(v(k+\ell_1+\ell_2+\cdots+\ell_{n-1})\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-1}} + v(k+\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n)\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n} + \cdots + v(k+\ell_2+\ell_3+\cdots+\ell_n)\Delta_{\ell_2+\ell_3+\cdots+\ell_n}\big) + \big(v(k+\ell_1+\ell_2+\cdots+\ell_{n-2})\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}} + v(k+\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1})\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}} + \cdots + v(k+\ell_2+\ell_3+\cdots+\ell_{n-1})\Delta_{\ell_2+\ell_3+\cdots+\ell_{n-1}} + v(k+\ell_1+\cdots+\ell_{n-3}+\ell_n)\Delta_{\ell_1+\cdots+\ell_{n-3}+\ell_n} + v(k+\ell_1+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n)\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n} + \cdots + v(k+\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n)\Delta_{\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n} + v(k+\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n)\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n} + v(k+\ell_1+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n)\Delta_{\ell_1+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n} + \cdots + v(k+\ell_2+\ell_3+\cdots\ell_{n-3}+\ell_{n-1}+\ell_n)\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n}\big) + \cdots + (-1)^n\big(v(k+\ell_1)\Delta_{\ell_1} + v(k+\ell_2)\Delta_{\ell_2} + \cdots + v(k+\ell_n)\Delta_{\ell_n}\big)\Big\}u(k)$.

Proof. The proof follows from (1) and $\Delta_\ell u(k) = u(k+\ell) - u(k)$. □

Definition 2.5. The second order of the generalized difference operator of the n-th kind is $\Delta^2_{\ell_1,\ell_2,\cdots\ell_n} = \Delta_{\ell_1,\ell_2,\cdots\ell_n}(\Delta_{\ell_1,\ell_2,\cdots\ell_n})$ and in general, the m-th order of the generalized difference operator of the n-th kind is defined as $\Delta^m_{\ell_1,\ell_2,\cdots\ell_n} = \Delta_{\ell_1,\ell_2,\cdots\ell_n}(\Delta^{m-1}_{\ell_1,\ell_2,\cdots\ell_n})$.

Lemmas 2.6 to 2.12 given below are derived by Definitions 2.1 and 2.5, (4), $\Delta_\ell k^n = \sum_{r=0}^{n} nC_r \ell^r k^{n-r}$, $\Delta_{\ell_i} = (E^{\ell_i} - 1)$.

Lemma 2.6. *If $P_{np-1}(k) = c_{np-1}k^{np-1} + c_{np-2}k^{np-2} + \cdots + c_1 k + c_0$ is any polynomial in k of degree $(np-1)$, then,*

$$\Delta^p_{\ell_1,\ell_2,\cdots\ell_n} P_{np-1}(k) = 0. \tag{6}$$

Lemma 2.7. *If p and q are positive integers and c is a constant, then $\Delta^p_{\ell_1,\ell_2,\cdots\ell_n}\Delta^q_{\ell_1,\ell_2,\cdots\ell_n} = \Delta^q_{\ell_1,\ell_2,\cdots\ell_n}\Delta^p_{\ell_1,\ell_2,\cdots\ell_n}$ and*

$$\Delta^p_{\ell_1,\ell_2,\cdots\ell_n}(cu(k)) = c\Delta^p_{\ell_1,\ell_2,\cdots\ell_n}(u(k)).$$

Lemma 2.8. *If p and q are positive integers, then*

$$\Delta^{p}_{\ell,\ell,\cdots,\ell} k^{q} = \begin{cases} 0, & \text{if } q < np, \\ q!\ell^{q}, & \text{if } q = np. \end{cases} \quad (7)$$

Lemma 2.9. *If $p_k = a_0 k^{np} + a_1 k^{np-1} + a_2 k^{np-2} + \cdots + a_n$ is any polynomial in k of degree np, then $\Delta^{p}_{\ell_1,\ell_2,\cdots\ell_n} P_k = a_0 (np)! \ell^{np}$.*

Lemma 2.10. *If ℓ_j's are positive reals and r is a positive integer, then*

$$\Delta^{r}_{\ell_1,\ell_2,\cdots,\ell_n} = \Big(\sum_{i_1=0}^{r} (-1)^{i_1} rC_{i_1} E^{\ell_1}(r-i_1) \Big) \Big(\sum_{i_2=0}^{r} (-1)^{i_2} rC_{i_2} E^{\ell_2}(r-i_2) \Big) \cdots \times \Big(\sum_{i_n=0}^{r} (-1)^{i_n} rC_{i_n} E^{\ell_n}(r-i_n) \Big).$$

Lemma 2.11. *Let $\ell_{1i}, \ell_{2i}, \cdots, \ell_{ni}$, $i = 1, 2, \ldots, m$ are reals and $\ell_j = \sum\limits_{i=1}^{m} \ell_{ji}$, where $j = 1, 2, \cdots, n$. Then,*

$$\Delta_{\ell_1,\ell_2,\cdots,\ell_n} = \prod_{j=1}^{n} \Big[\prod_{i=1}^{m} (\Delta_{ji+1} - 1) \Big].$$

Lemma 2.12. *If m, ℓ_j's, $j = 1, 2, \cdots, n$ are positive integers, then*

$$(i) \quad \Delta_{m\ell_1,m\ell_2,\cdots,m\ell_n} = E^{m(\ell_1+\ell_2+\cdots+\ell_n)} - \{E^{m(\ell_1+\ell_2+\cdots+\ell_{n-1})} + E^{m(\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n)} + \cdots + E^{m(\ell_2+\ell_3+\cdots+\ell_n)}\} + \{E^{m(\ell_1+\ell_2+\cdots+\ell_{n-2})} + E^{m(\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1})} + \cdots + E^{m(\ell_2+\ell_3+\cdots+\ell_{n-1})} + E^{m(\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n)} + E^{m(\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n)} + \cdots + E^{m(\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n)} + E^{m(\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n)} + E^{m(\ell_1+\ell_2+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n)} + \cdots + E^{m(\ell_2+\ell_3+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n)} + \cdots + E^{m(\ell_1+\ell_4+\cdots+\ell_n)} + E^{m(\ell_2+\ell_4+\cdots+\ell_n)} + E^{m(\ell_3+\ell_4+\cdots+\ell_n)}\} - \cdots + (-1)^{n-1}\{E^{m\ell_1} + E^{m\ell_2} + \cdots + E^{m\ell_n}\} + (-1)^{n}.$$

$$(ii) \quad \Delta_{m\ell_1,m\ell_2,\cdots,m\ell_n} = (1 + \Delta_{\ell_1+\ell_2+\cdots+\ell_n})^m - \{(1 + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-1}})^m + (1 + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n})^m + \cdots + (1 + \Delta_{\ell_2+\ell_3+\cdots+\ell_n})^m\} + \{(1 + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}})^m + (1 + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}})^m + \cdots + (1 + \Delta_{\ell_2+\ell_3+\cdots+\ell_{n-1}})^m + (1 + \Delta_{\ell_1+\cdots+\ell_{n-3}+\ell_n})^m + (1 + \Delta_{\ell_1+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n})^m + \cdots + (1 + \Delta_{\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n})^m + (1 + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n})^m + (1 + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n})^m + \cdots + (1 + \Delta_{\ell_2+\ell_3+\cdots+\ell_{n-3}\ell_{n-1}+\ell_n})^m$$

$$+\cdots+(1+\Delta_{\ell_1+\ell_4+\cdots+\ell_n})^m+(1+\Delta_{\ell_2+\ell_4+\cdots+\ell_n})^m+(1+\Delta_{\ell_3+\ell_4+\cdots+\ell_n})^m\}$$
$$-\cdots+(-1)^{n-1}\{(1+\Delta_{\ell_1})^m+(1+\Delta_{\ell_2})^m+\cdots+(1+\Delta_{\ell_n})^m\}+(-1)^n.$$

$$(iii)\quad \Delta_{m\ell_1,m\ell_2,\cdots,m\ell_n}=\sum_{r=0}^{m} mC_r\{\Delta^r_{\ell_1+\ell_2+\cdots+\ell_n}-\big(\Delta^r_{\ell_1+\ell_2+\cdots+\ell_{n-1}}+$$
$$\Delta^r_{\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n}+\cdots+\Delta^r_{\ell_2+\ell_3+\cdots+\ell_n}\big)+\big(\Delta^r_{\ell_1+\ell_2+\cdots+\ell_{n-2}}+$$
$$\Delta^r_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}}+\cdots+\Delta^r_{\ell_2+\ell_3+\cdots+\ell_{n-1}}+\Delta^r_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n}$$
$$+\Delta^r_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n}+\cdots+\Delta^r_{\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n}+\Delta^r_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n}$$
$$+\Delta^r_{\ell_1+\ell_2+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n}+\cdots+\Delta^r_{\ell_2+\ell_3+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n}$$
$$+\Delta^r_{\ell_1+\ell_4+\cdots+\ell_n}+\Delta^r_{\ell_2+\ell_4+\cdots+\ell_n}+\Delta^r_{\ell_3+\ell_4+\cdots+\ell_n}\big)-\cdots+(-1)^{n-1}$$
$$\big(\Delta^r_{\ell_1}+\Delta^r_{\ell_2}+\cdots+\Delta^r_{\ell_n}\big)\}+(-1)^n.$$

$$(iv)\quad \Delta^m_{\ell_1,\ell_2,\cdots,\ell_n}=\sum_{r=0}^{m}(-1)^r mC_r\Delta^{m-r}_{\ell_1+\ell_2+\cdots+\ell_n}-\{\sum_{i=0}^{r}(-1)^i rC_i$$
$$\big(\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-1}}+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n}+\cdots$$
$$+\Delta_{\ell_2+\ell_3+\cdots+\ell_n}\big)^{r-i}\big(\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}}+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}}$$
$$+\cdots+\Delta_{\ell_2+\ell_3+\cdots+\ell_{n-1}}+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n}+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n}+$$
$$\cdots+\Delta_{\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n}+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n}$$
$$+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n}+\cdots+\Delta_{\ell_2+\ell_3+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n}+$$
$$\cdots+\Delta_{\ell_1+\ell_4+\cdots+\ell_n}+\Delta_{\ell_2+\ell_4+\cdots+\ell_n}+\Delta_{\ell_3+\ell_4+\cdots+\ell_n}-\cdots+(-1)^n$$
$$(\Delta_{\ell_1}+\Delta_{\ell_2}+\cdots+\Delta_{\ell_n})\big)^i\}.$$

$$(v)\quad \Delta^m_{\ell_1,\ell_2,\cdots,\ell_n}=\prod_{j=1}^{n}\Big(\sum_{i_j=0}^{m-1}(-1)^{i_j}mC_{i_j}\Delta_{(m-i_j)\ell_j}\Big).$$

Theorem 2.13. (see [4]) *If $\{x_k\}$ and $\{y_k\}$ are two sequences and $\ell,n\in N(1)$, then*

$$\Delta^n_\ell(x_ky_k) = (x_k)\Delta^n_\ell(y_k) + nC_1\Delta_\ell(x_k)\Delta^{n-1}_\ell(y_{k+\ell})$$
$$+nC_2\Delta^2_\ell(x_k)\Delta^{n-2}_\ell(y_{k+2\ell})+...+nC_n\Delta^n_\ell(x_k)(y_{k+n\ell}).$$

The following is the discrete version of Leibnitz Theorem according to $\Delta_{\ell_1,\ell_2,\cdots,\ell_n}$.

Theorem 2.14. *If $u(k)$, $v(k), k \in [0, \infty)$ are two functions, then*

$$\Delta^m_{\ell_1,\ell_2,\cdots,\ell_n}(u(k)v(k)) = \Delta^m_{\ell_1}(\Delta^m_{\ell_2}(\cdots(\Delta^m_{\ell_{n-1}}(u(k)\Delta^m_{\ell_n}v(k)))\cdots))$$
$$+ mC_1\Delta^m_{\ell_1}(\Delta^m_{\ell_2}(\cdots(\Delta^m_{\ell_{n-1}}(\Delta_{\ell_n}u(k)\Delta^{m-1}_{\ell_n}u(k-\ell_n)))\cdots))$$
$$+ mC_2\Delta^m_{\ell_1}(\Delta^m_{\ell_2}(\cdots(\Delta^m_{\ell_{n-1}}(\Delta^2_{\ell_n}u(k)\Delta^{m-2}_{\ell_n}v(k+2\ell_n)))\cdots))$$
$$+ \cdots + mC_m\Delta^m_{\ell_1}(\Delta^m_{\ell_2}(\cdots(\Delta^m_{\ell_{n-1}}(\Delta^m_{\ell_n}u(k)v(k+m\ell_n))))). \quad (8)$$

Proof. The proof follows by Theorem 2.13 and (4). □

The following is consequence of Definition 2.1.

Lemma 2.15. *For $m \in \mathbb{N}(1)$ and the reals $\ell_i > 0, i = 1, 2, \ldots, n$,*

$$\Delta_{m\ell_1,m\ell_2,\cdots,m\ell_n} = \{\sum_{r=0}^{m} mC_r\{\sum_{i=0}^{r-1}(-1)^i rC_i\big(\Delta_{(r-i)\ell_1+\ell_2+\cdots+\ell_n}$$
$$- \big(\Delta_{(r-i)\ell_1+\ell_2+\cdots+\ell_{n-1}} + \Delta_{(r-i)\ell_1+\cdots+\ell_{n-2}+\ell_n} + \cdots + \Delta_{(r-i)\ell_2+\ell_3+\cdots+\ell_n}\big)$$
$$+ \big(\Delta_{(r-i)\ell_1+\ldots+\ell_{n-2}} + \Delta_{(r-i)\ell_1+\cdots+\ell_{n-3}+\ell_{n-1}} + \cdots + \Delta_{(r-i)\ell_2+\ell_3+\cdots+\ell_{n-1}}$$
$$+ \Delta_{(r-i)\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n} + \Delta_{(r-i)\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n} + \cdots$$
$$+ \Delta_{(r-i)\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n} + \Delta_{(r-i)\ell_1+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n}$$
$$+ \Delta_{(r-i)\ell_1+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n} + \cdots + \Delta_{(r-i)\ell_2+\ell_3+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n}$$
$$+ \Delta_{(r-i)\ell_1+\ell_4+\cdots+\ell_n} + \Delta_{(r-i)\ell_2+\ell_4+\cdots+\ell_n} + \Delta_{(r-i)\ell_3+\ell_4+\cdots+\ell_n}\big)$$
$$- \cdots + (-1)^{n-1}\big(\Delta_{(r-i)\ell_1} + \Delta_{(r-i)\ell_2} + \cdots + \Delta_{(r-i)\ell_n}\big)\big)\}\} + 1. \quad (9)$$

Proof. The proof follows from (1) and Lemma 2.12. □

Theorem 2.16. *If p is a positive integer, then*

$$(k + (m\ell_i+)_{1-n})^p - \big\{(k + (m\ell_i+)_{1-(n-1)})^p + (k + (m\ell_i+)_{1-(n-2),n})^p + \cdots$$
$$+ (k + (m\ell_i+)_{2-n})^p\big\} + \big\{(k + (m\ell_i+)_{1-(n-2)})^p + (k + (m\ell_i+)$$
$$_{1-(n-3),(n-1)})^p + \cdots + (k + (m\ell_i+)_{2-(n-1)})^p + (k + (m\ell_i+)_{1-(n-3),n})^p$$
$$+ (k + (m\ell_i+)_{1-(n-4),(n-2),n})^p + \cdots + (k + (m\ell_i+)_{2-(n-2),n})^p$$
$$+ (k + (m\ell_i+)_{1-(n-4),(n-1),n})^p + (k + (m\ell_i+)_{1-(n-5),(n-3),(n-1),n})^p + \cdots$$
$$+ (k + (m\ell_i+)_{2-(n-3),(n-1),n})^p + \cdots + (k + (m\ell_i+)_{1,4-n})^p + (k + (m\ell_i+)_{2,4-n})^p$$
$$+ (k + (m\ell_i+)_{3-n})^p\big\} - \cdots + (-1)^n\big\{(k + m\ell_1)^p + (k + m\ell_2)^p + \cdots + (k + \ell_n)^p\big\}$$
$$= \sum_{r=0}^{m} mC_r\{\big((k + r(\ell_i+)_{1-n})^p - \big((k + r(\ell_i+)_{1-(n-1)})^p$$
$$+ (k + r(\ell_i+)_{1-(n-2),n})^p + \cdots + (k + r(\ell_i+)_{2-n})^p\big) + \big((k + r(\ell_i+)_{1-(n-2)})^p$$

$$+(k+r(\ell_i+)_{1-(n-3),(n-1)})^p+\cdots+(k+r(\ell_i+)_{2-(n-1)})^p+(k+r(\ell_i+)_{1-(n-3),n})^p$$
$$+(k+r(\ell_i+)_{1-(n-4),(n-2),n})^p+\cdots+(k+r(\ell_i+)_{2-(n-2),n})^p$$
$$+(k+r(\ell_i+)_{1-(n-4),(n-1),n})^p+(k+r(\ell_i+)_{1-(n-5),(n-3),(n-1),n})^p+\cdots$$
$$+(k+r(\ell_i+)_{2-(n-3),(n-1),n})^p+\cdots+(k+r(\ell_i+)_{1,4-n})^p+(k+r(\ell_i+)_{2,4-n})^p$$
$$+(k+r(\ell_i+)_{3-n})^p\big)-\cdots+(-1)^{n-1}\big((k+r\ell_1)^p+(k+r\ell_2)^p+\cdots$$
$$+(k+r\ell_n)^p)\big)+(-1)^n k^p\big)-rC_1\big((k+(r-1)(\ell_i+)_{1-n})^p-\big((k+(r-1)(\ell_i+)_{1-(n-1)})^p$$
$$+(k+(r-1)(\ell_i+)_{1-(n-2),n})^p+\cdots+(k+(r-1)(\ell_i+)_{2-n})^p\big)+\big((k+(r-1)(\ell_i+)_{1-(n-2)})^p$$
$$+(k+(r-1)(\ell_i+)_{1-(n-3),(n-1)})^p+\cdots+(k+(r-1)(\ell_i+)_{2-(n-1)})^p$$
$$+(k+(r-1)(\ell_i+)_{1-(n-3),n})^p+(k+(r-1)(\ell_i+)_{1-(n-4),(n-2),n})^p+\cdots$$
$$+(k+(r-1)(\ell_i+)_{2-(n-2),n})^p+(k+(r-1)(\ell_i+)_{1-(n-4),(n-1),n})^p$$
$$+(k+(r-1)(\ell_i+)_{1-(n-5),(n-3),(n-1),n})^p+\cdots+(k+(r-1)(\ell_i+)_{2-(n-3),(n-1),n})^p+\cdots$$
$$+(k+(r-1)(\ell_i+)_{1,4-n})^p+(k+(r-1)(\ell_i+)_{2,4-n})^p+(k+(r-1)(\ell_i+)_{3-n})^p\big)-\cdots$$
$$+(-1)^{n-1}\big((k+(r-1)(\ell_1)^p+(k+(r-1)\ell_2)^p+\cdots+(k+(r-1)\ell_n)^p)\big)$$
$$+(-1)^n k^p\big)+\cdots+(-1)^{r-1}rC_{r-1}\big((k+(\ell_i+)_{1-n})^p-\big((k+(\ell_i+)_{1-(n-1)})^p$$
$$+(k+(\ell_i+)_{1-(n-2),n})^p+\cdots+(k+(\ell_i+)_{2-n})^p\big)+\big((k+(\ell_i+)_{1-(n-2)})^p$$
$$+(k+(\ell_i+)_{1-(n-3),(n-1)})^p+\cdots+(k+(\ell_i+)_{2-(n-1)})^p+(k+(\ell_i+)_{1-(n-3),n})^p$$
$$+(k+(\ell_i+)_{1-(n-4),(n-2),n})^p+\cdots+(k+(\ell_i+)_{2-(n-2),n})^p$$
$$+(k+(\ell_i+)_{1-(n-4),(n-1),n})^p+(k+(\ell_i+)_{1-(n-5),(n-3),(n-1),n})^p+\cdots$$
$$+(k+(\ell_i+)_{2-(n-3),(n-1),n})^p+\cdots+(k+(\ell_i+)_{1,4-n})^p+(k+(\ell_i+)_{2,4-n})^p$$
$$+(k+(\ell_i+)_{3-n})^p\big)-\cdots+(-1)^{n-1}\big((k+\ell_1)^p+(k+\ell_2)^p$$
$$+\cdots+(k+\ell_n)^p)\big)+(-1)^n k^p\big)\}, \quad (10)$$

where $(\ell_i+)_{a-b}=\ell_a+\ell_{a+1}+...+\ell_b$ *and* $(\ell_i+)_{a-b,c,d}=\ell_a+...+\ell_b+\ell_c+\ell_d$.

Proof. The proof follows from the relation $\Delta_\ell=E^\ell-1$ and (9). □

Example 2.17. If $\theta_1,\theta_2,\cdots,\theta_n$ are in degrees assuming only integer values in the anticlockwise direction, then

$$\sin(k+(m\theta_i+)_{1-n})-\big(\sin(k+(m\theta_i+)_{1-(n-1)})+\sin(k+(m\theta_1+)_{1-(n-2),n})$$
$$+\cdots+\sin(k+(m\theta_i+)_{2-n})\big)+\big(\sin(k+(m\theta_i+)_{1-(n-2)})$$
$$+\sin(k+(m\theta_i+)_{1-(n-3),(n-1)})+\cdots+\sin(k+(m\theta_i+)_{2-(n-1)})$$
$$+\sin(k+(m\theta_i+)_{1-(n-3),n})+\sin(k+(m\theta_i+)_{1-(n-4),(n-2),n})+\cdots$$
$$+\sin(k+(m\theta_i+)_{2-(n-2),n})+\sin(k+(m\theta_i+)_{1-(n-4),(n-1),n})$$

$$+\sin(k+(m\theta_i+)_{1-(n-5),(n-3),(n-1),n})+\cdots+\sin(k+(m\theta_i+)_{2-(n-3),(n-1),n})+\cdots$$
$$+\sin(k+(m\theta_i+)_{1,4-n})+\sin(k+(m\theta_i+)_{2,4-n})+\sin(k+(m\theta_i+)_{3-n})\big)-\cdots$$
$$+(-1)^n\big(\sin(k+m\theta_1)+\sin(k+m\theta_2)+\cdots+\sin(k+\theta_n)\big)$$
$$=\sum_{r=0}^{m} mC_r\{\big(\sin(k+r(\theta_i+)_{1-n})-\big(\sin(k+r(\theta_i+)_{1-(n-1)})$$
$$+\sin(k+r(\theta_i+)_{1-(n-2),n})+\cdots+\sin(k+r(\theta_i+)_{2-n})\big)+\big(\sin(k+r(\theta_i+)_{1-(n-2)})$$
$$+\sin(k+r(\theta_1+)_{1-(n-3),(n-1)})+\cdots+\sin(k+r(\theta_i+)_{2-(n-1)})$$
$$+\sin(k+r(\theta_i+)_{1-(n-3),n})+\sin(k+r(\theta_i+)_{1-(n-4),(n-2),n})+\cdots$$
$$+\sin(k+r(\theta_i+)_{2-(n-2),n})+\sin(k+r(\theta_i+)_{1-(n-4),(n-1),n})$$
$$+\sin(k+r(\theta_i+)_{1-(n-5),(n-3),(n-1),n})+\cdots+\sin(k+r(\theta_i+)_{2-(n-3),(n-1),n})+\cdots$$
$$+\sin(k+r(\theta_i+)_{1,4-n})+\sin(k+r(\theta_i+)_{2,4-n})+\sin(k+r(\theta_i+)_{3-n})\big)-\cdots$$
$$+(-1)^{n-1}\big(\sin(k+r(\theta_1)+\sin(k+r\theta_2)+\cdots+\sin(k+r\theta_n))\big)$$
$$+(-1)^n\sin k\big)-rC_1\big(\sin(k+(r-1)(\theta_i+)_{1-n})-\big(\sin(k+(r-1)(\theta_i+)_{1-(n-1)})$$
$$+\sin(k+(r-1)(\theta_i+)_{1-(n-2),n})+\cdots+\sin(k+(r-1)(\theta_i+)_{2-n})\big)$$
$$+\big(\sin(k+(r-1)(\theta_i+)_{1-(n-2)})+\sin(k+(r-1)(\theta_i+)_{1-(n-3),(n-1)})+\cdots$$
$$+\sin(k+(r-1)(\theta_i+)_{2-(n-1)})+\sin(k+(r-1)(\theta_i+)_{1-(n-3),n})$$
$$+\sin(k+(r-1)(\theta_i+)_{1-(n-4),(n-2),n})+\cdots+\sin(k+(r-1)(\theta_i+)_{2-(n-2),n})$$
$$+\sin(k+(r-1)(\theta_i+)_{1-(n-4),(n-1),n})+\sin(k+(r-1)(\theta_i+)_{1-(n-5),(n-3),n})+\cdots$$
$$+\sin(k+(r-1)(\theta_i+)_{2-(n-3),(n-1),n})+\cdots+\sin(k+(r-1)(\theta_i+)_{1,4-n})$$
$$+\sin(k+(r-1)(\theta_i+)_{2,4-n})+\sin(k+(r-1)(\theta_i+)_{3,4-n})\big)-\cdots$$
$$+(-1)^{n-1}\big(\sin(k+(r-1)(\theta_1)+\sin(k+(r-1)\theta_2)+\cdots+\sin(k+(r-1)\theta_n))\big)$$
$$+(-1)^n\sin k\big)+\cdots+(-1)^{r-1}rC_{r-1}\big(\sin(k+(\theta_i+)_{1-n})$$
$$-\big(\sin(k+(\theta_i+)_{1-(n-1)})+\sin(k+(\theta_i+)_{1-(n-2),n})+\cdots$$
$$+\sin(k+(\theta_i+)_{2-n})\big)+\big(\sin(k+(\theta_i+)_{1-(n-2)})+\sin(k+(\theta_i+)_{1-(n-3),(n-1)})+\cdots$$
$$+\sin(k+(\theta_i+)_{2-(n-1)})+\sin(k+(\theta_i+)_{1-(n-3),n})+\sin(k+(\theta_i+)_{1-(n-4),(n-2),n})+\cdots$$
$$+\sin(k+(\theta_i+)_{2-(n-2),n})+\sin(k+(\theta_i+)_{1-(n-4),(n-1),n})$$
$$+\sin(k+(\theta_i+)_{1-(n-5),(n-3),(n-1),n})+\cdots+\sin(k+(\theta_i+)_{2-(n-3),(n-1),n})+\cdots$$
$$+\sin(k+(\theta_i+)_{1,4-n})+\sin(k+(\theta_i+)_{2,4-n})+\sin(k+(\theta_i+)_{3-n})\big)-\cdots$$
$$+(-1)^{n-1}\big(\sin(k+\theta_1)+\sin(k+\theta_2)+\cdots+\sin(k+\theta_n))\big)+(-1)^n\sin k\big)\},$$

where $(\theta_i+)_{a-b}=\theta_a+\theta_{a+1}+...+\theta_b$ and $(\theta_i+)_{a-b,c,d}=\theta_a+...+\theta_b+\theta_c+\theta_d$.

Proof. The proof follows from the relation $\Delta_\ell = E^\ell - 1$ and by operating (9) on $u(k) = \sin k$. □

3. Generalized Polynomial Factorial of the n-th Kind

Definition 3.1. For the positive integer r and the reals $\ell_1, \ell_2, \cdots, \ell_n$, the generalized polynomial factorial of the n-th kind is defined as
$k^{(r)}_{\ell_1,\ell_2,\cdots,\ell_n} = (k+\ell_2+\ell_3+\cdots+\ell_n)^{(r)}_{\ell_1} + (k+\ell_1+\ell_3+\cdots+\ell_n)^{(r)}_{\ell_2} + \cdots + (k+\ell_1+\ell_2+\cdots+\ell_{n-1})^{(r)}_{\ell_n} - \{(k+\ell_2+\ell_3+\cdots+\ell_{n-1})^{(r)}_{\ell_1} + (k+\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n)^{(r)}_{\ell_1} + \cdots + (k+\ell_3+\ell_4+\cdots+\ell_n)^{(r)}_{\ell_1} + (k+\ell_1+\ell_3+\cdots+\ell_{n-1})^{(r)}_{\ell_2} + (k+\ell_1+\ell_3+\cdots+\ell_{n-2}+\ell_n)^{(r)}_{\ell_2} + \cdots + (k+\ell_3+\ell_4+\cdots+\ell_n)^{(r)}_{\ell_2} + \cdots + (k+\ell_1+\ell_2+\cdots+\ell_{n-2})^{(r)}_{\ell_n} + (k+\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1})^{(r)}_{\ell_n} + \cdots + (k+\ell_2+\ell_3+\cdots+\ell_{n-1})^{(r)}_{\ell_n} + \cdots\} + (-1)^{n-2}\{(k+\ell_2)^{(r)}_{\ell_1} + (k+\ell_3)^{(r)}_{\ell_1} + \cdots + (k+\ell_n)^{(r)}_{\ell_1} + (k+\ell_1)^{(r)}_{\ell_2} + (k+\ell_3)^{(r)}_{\ell_2} + \cdots + + (k+\ell_n)^{(r)}_{\ell_2} + \cdots + (k+\ell_1)^{(r)}_{\ell_n}$

$$+ (k+\ell_2)^{(r)}_{\ell_n} + \cdots + (k+\ell_{n-1})^{(r)}_{\ell_n}\} + (-1)^{n-1}\{k^{(r)}_{\ell_1} + k^{(r)}_{\ell_2} + \cdots + k^{(r)}_{\ell_n}\}, \quad (11)$$

and first kind

$$k^{(r)}_\ell = k(k-\ell)(k-2\ell)\cdots(k-(r-1)\ell). \quad (12)$$

Lemma 3.2. *If $\ell_1, \ell_2, \cdots, \ell_n$ are positive reals, then*

$$k^{(r)}_{\ell_1,\ell_2,\cdots,\ell_n} = \Delta_{\ell_2,\ell_3,\cdots,\ell_n} k^{(r)}_{\ell_1} + \Delta_{\ell_1,\ell_3,\cdots,\ell_n} k^{(r)}_{\ell_2} + \cdots + \Delta_{\ell_1,\ell_2,\cdots,\ell_{n-1}} k^{(r)}_{\ell_n}. \quad (13)$$

Using the Stirling numbers of the first kind s^n_r, the following can be easily obtained.

Lemma 3.3. (see [4]) *Let r be positive integer and ℓ is positive real. Then*

$$\Delta_\ell k^{(r)}_\ell = r\ell k^{(r-1)}_\ell. \quad (14)$$

Lemma 3.4. *Let s^n_r be the Stirling numbers of the first kind. Then,*
(i) $k^{(r)}_\ell = \sum\limits_{r=1}^{n} (s^n_r t^{n-r} k^r)$, *(ii)* $k^n = \sum\limits_{r=1}^{n} S^n_r \ell^{n-r} k^{(r)}_\ell$,
(iii) $\sum\limits_{r=1}^{m} \Delta_{\ell_1,\ell_2,\cdots,\ell_n} s^m_r t^{m-r} k^r = \Delta_{\ell_1,\ell_2,\cdots,\ell_n} k^{(m)}_t$.
In particular then $t = \ell_n$,

$$\sum_{r=1}^{m} s^m_r \ell_n^{m-r} \Delta_{\ell_1,\ell_2,\cdots,\ell_n} k^r = m\ell_n \Delta_{\ell_1,\ell_2,\cdots,\ell_{n-1}} k^{(m-1)}_{\ell_n}.$$

Lemma 3.5. *If $\ell_1, \ell_2, \cdots, \ell_n$ are positive reals then,*

$$\Delta_{\ell_1,\ell_2,\cdots,\ell_n} k^{(r)}_{\ell_i} = (r\ell_i)\Delta_{\ell_1,\ell_2,\cdots,\ell_{i-1},\ell_{i+1},\cdots,\ell_n} k^{(r-1)}_{\ell_i}. \quad (15)$$

Proof. The proof follows from (14). □

Lemma 3.6. *Let m is a positive integer and ℓ_i's, $i = 1, 2, \ldots, n$ are positive reals. Then,*

$$\Delta_{\ell_1,\ell_2,\cdots,\ell_n} k^{(m)}_{\ell_1,\ell_2,\cdots,\ell_n} = m\ell_1 \Delta^2_{\ell_2,\ell_3,\cdots,\ell_n} k^{(m-1)}_{\ell_1} + m\ell_2 \Delta^2_{\ell_1,\ell_3,\cdots,\ell_n} k^{(m-1)}_{\ell_2} + \cdots + m\ell_n \Delta^2_{\ell_1,\ell_2,\cdots,\ell_{n-1}} k^{(m-1)}_{\ell_n}. \quad (16)$$

Proof. The proof follows from (4), (14) and using (11) for $k^{(m)}_{\ell_1,\ell_2,\cdots,\ell_{i-1},\ell_{i+1},\cdots,\ell_n}$. □

Corollary 3.7. *Let m and n are positive integers and $m \geq n$. Then,*

$$\Delta_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}} k^{(m)}_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}} = (m\ell)^{(n)}_{\ell} k^{(m-n)}_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}}. \quad (17)$$

The following theorem is the generalized version of Newton's formula with reference to $\Delta_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}}$.

Theorem 3.8. *Let $f(k)$ be a polynomial of degree $q(= mn)$ in k. Then $f(k)$ can be expressed as*

$$f(k) = f(0) + \frac{\Delta_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}} f(0)}{n!\ell^n} k^{(n)}_{\ell} + \frac{\Delta^2_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}} f(0)}{(2n)!\ell^{2n}} k^{(2n)}_{\ell} + \cdots + \frac{\Delta^m_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}} f(0)}{q!\ell^q} k^{(q)}_{\ell}. \quad (18)$$

Proof. Assume that

$$f(k) = a_0 + a_1 k^{(n)}_{\ell} + a_2 k^{(2n)}_{\ell} + \cdots + a_m k^{(q)}_{\ell}, \quad \text{where} \quad q = mn. \quad (19)$$

The coefficients are determined from the relation

$$\Delta^r_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}} f(0) = a_r (nr)!\ell^{nr}. \quad (20)$$

Now, the proof follows from (19) and (20). □

Corollary 3.9. *Let $f(k)$ be a polynomial of degree $q(= mn)$ in k. Then $f(k - t)$ can be expressed as*

$$f(k-t) = f(k) + \frac{\Delta_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}} f(t)}{n!\ell^n}(k-t)_\ell^{(n)} + \frac{\Delta^2_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}} f(t)}{(2n)!\ell^{2n}}(k-t)_\ell^{(2n)}$$
$$+ \cdots + \frac{\Delta^m_{\underbrace{\ell,\ell,\cdots,\ell}_{n,times}} f(t)}{q!\ell^q}(k-t)_\ell^{(q)}. \quad (21)$$

Proof. The proof follows by replacing k by $k-t$, 0 by t in (18). □

4. Inverse of the Generalized Difference Operator of the n-th Kind and its Applications

Definition 4.1. The inverse of the generalized difference operator of the n-th kind denoted by $\Delta^{-1}_{\ell_1,\ell_2,\cdots,\ell_n}$ is defined as follows. If $\Delta_{\ell_1,\ell_2,\cdots,\ell_n} z(k) = y(k)$, then

$$z(k) = \Delta^{-1}_{\ell_1,\ell_2,\cdots,\ell_n} y(k) + c_{(n-1)j}\frac{k_{\ell_{n-1}}^{(n-1)}}{(n-1)!\ell_{n-1}^{n-1}} + c_{(n-2)j}\frac{k_{\ell_{n-2}}^{(n-2)}}{(n-2)!\ell_{n-2}^{n-2}}$$
$$+ \cdots + c_{2j}\frac{k_{\ell_2}^{(2)}}{(2)!\ell_2^2} + c_{1j}\frac{k}{\ell_1} + c_{0j}, j \in \{0,1,2,\cdots,\ell-1\}. \quad (22)$$

In general, $\Delta^{-m}_{\ell_1,\ell_2,\cdots,\ell_n} = \Delta^{-1}_{\ell_1,\ell_2,\cdots,\ell_n}(\Delta^{-(m-1)}_{\ell_1,\ell_2,\cdots,\ell_n})$.

Lemma 4.2. *Let $\ell_i \in (0,\infty), i = 1,2,...,n$, $m \in \mathbb{N}(1)$ and $k \in [n\ell,\infty)$. Then,*

$$\Delta^{-1}_{\ell_1,\ell_2,\cdots,\ell_n} k^{(m)}_{\ell_1,\ell_2,\cdots,\ell_n} = \frac{k_{\ell_1}^{(m+1)}}{\ell_{1(m+1)}} + \frac{k_{\ell_2}^{(m+1)}}{\ell_{2(m+1)}} + \cdots + \frac{k_{\ell_n}^{(m+1)}}{\ell_{n(m+1)}} + c_{(n-1)j}\left(\frac{k_{\ell_{n-1}}^{(n-1)}}{(n-1)!\ell_{n-1}^{n-1}}\right)$$
$$+ c_{(n-2)j}\left(\frac{k_{\ell_{n-2}}^{(n-2)}}{(n-2)!\ell_{n-2}^{n-2}}\right) + \cdots + c_{2j}\left(\frac{k_{\ell_2}^{(2)}}{2!\ell_2^2}\right) + c_{1j}\left(\frac{k_{\ell_1}^{(1)}}{\ell_1}\right) + c_{0j}. \quad (23)$$

Proof. The proof follows from (22), (4) and operating $\Delta^{-1}_{\ell_1,\ell_2,\cdots,\ell_n}$ on both sides of (13). □

Corollary 4.3. *For $\ell \in (0,\infty)$ and $m \in \mathbb{N}(1)$, then*

$$\Delta^{-1}_{\ell,\ell,\cdots,\ell} k^{(m)}_{\ell,\ell,\cdots,\ell} = n\frac{k_\ell^{(m+1)}}{\ell(m+1)} + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right)$$
$$+ c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right) + \cdots + c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right) + c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right) + c_{0j}. \quad (24)$$

Theorem 4.4. *There exist constants $c_{0j}, c_{1j}, \cdots, c_{(n-1)j}$'s such that*

$$\Delta^{-1}_{\underbrace{\ell,\ell,\cdots,\ell}_{n-times}} u(k) = \sum_{r_n=2}^{r_n^*} \sum_{r_{n-1}=1}^{r_{n-1}^*} \sum_{r_{n-2}=1}^{r_{n-2}^*} \cdots \sum_{r_2=1}^{r_2^*} \sum_{r_1=0}^{r_1^*} u(k - r_n\ell - r_{n-1}\ell - \cdots - r_2\ell - r_1\ell) + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right) + \cdots + c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right) + c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right) + c_{0j}, \quad (25)$$

where $r_n^* = \left[\frac{k}{\ell}\right], r_{n-1}^* = r_n^* - r_n, r_{n-2}^* = r_{n-1}^* - r_{n-1}, \ldots, r_1^* = r_2^* - r_2.$

Proof. The proof follows by (22) and the relation

$$\Delta_{\underbrace{\ell,\ell,\cdots,\ell}_{n-times}} \left\{ \sum_{r_n=2}^{r_n^*} \sum_{r_{n-1}=1}^{r_{n-1}^*} \sum_{r_{n-2}=1}^{r_{n-2}^*} \cdots \sum_{r_2=1}^{r_2^*} \sum_{r_1=0}^{r_1^*} u(k - r_n\ell - r_{n-1}\ell - \cdots - r_2\ell - r_1\ell) + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right) + \cdots + c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right) + c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right) + c_{0j} \right\} = u(k). \quad \square$$

Lemma 4.5. *Let $\lambda \neq 1, k \geq n\ell$ and P_k is any function of k. Then,*

$$\sum_{r_n=2}^{r_n^*} \sum_{r_{n-1}=1}^{r_{n-1}^*} \sum_{r_{n-2}=1}^{r_{n-2}^*} \cdots \sum_{r_2=1}^{r_2^*} \sum_{r_1=0}^{r_1^*} \lambda^{(k-r_n\ell-\cdots-r_2\ell-r_1\ell)} P_{(k-r_n\ell-\cdots-r_2\ell-r_1\ell)} = \frac{\lambda^k}{(\lambda^\ell-1)^n}\left\{1 - \frac{\lambda^\ell\Delta_\ell}{(\lambda^\ell-1)} + \frac{\lambda^\ell\Delta_\ell^2}{(\lambda^\ell-1)^2} - \cdots\right\}^n P_k + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + \cdots + c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right) + c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right) + c_{0j}. \quad (26)$$

Proof. Let F_k be any function such that

$$\Delta^{-1}_{\underbrace{\ell,\ell,\cdots,\ell}_{n-times}} (\lambda^k F_k) = \lambda^k P_k. \quad (27)$$

By (4) and E, we find $P_k = (\lambda^\ell E^\ell - 1)^n$. From (27) and (22),

$$\Delta^{-1}_{\underbrace{\ell,\ell,\cdots,\ell}_{n-times}}(\lambda^k P_k) = \lambda^k(\lambda^\ell E^\ell - 1)^{-n}P_k + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right) + \cdots + c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right) + c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right) + c_{0j}.$$

Now, the proof follows from (25) and the binomial theorem. □

Lemma 4.6. *Let $m, n \in \mathbb{N}(1)$ and $\ell \in (0,\infty)$. Then*

(i) $$\Delta^{-1}_{\underbrace{\ell,\ell,\cdots,\ell}_{(n-1)-times}} k^m = (k-(n-1)\ell)^m - (n-1)(k-(n-2)\ell)^m + \cdots + (-1)^{n-1}k^m = \frac{1}{n}\sum_{r=1}^{m} S_r^m \ell^{m-r} k^{(r)}_{\underbrace{\ell,\ell,\cdots,\ell}_{n-times}}. \quad (28)$$

(ii) $$\Delta^{-1}_{\underbrace{\ell,\ell,\cdots,\ell}_{n-times}} k_\ell^{(m)} = \frac{k^{(m+2n-1)}_{\underbrace{\ell,\ell,\cdots,\ell}_{n-times}}}{n(m+1)\cdots(m+2n-1)\ell^{2n-1}} + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right) + \cdots + c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right) + c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right) + c_{0j}. \quad (29)$$

The following theorem is the methodology to find $P^{n-1}S^m$, the $(n-1)$-th partial sum of m-th powers of arithmetic progression.

Theorem 4.7. *If n is a positive integer and $k \in [n\ell, \infty)$ then,*

$$\sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\cdots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(k - r_n\ell - r_{n-1}\ell - \cdots - r_2\ell - r_1\ell)^m = \sum_{t=1}^{m}\frac{S_t^m \ell^{m-(t+n)} k_\ell^{(t+n)}}{(t+1)(t+2)\cdots(t+n)} - \left\{c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right) + \cdots + c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right) + c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right) + c_{0j}\right\}, \quad (30)$$

where c_{ij}'s for $i = 0, 1, 2, \ldots, (n-1)$ are obtained by solving the system of equations

$$\sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\cdots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(k-r_n\ell-r_{n-1}\ell-\cdots-r_2\ell-r_1\ell)^m$$
$$=\sum_{t=1}^{m}\frac{S_t^m\ell^{m-(t+n)}((m+a)\ell+j)_\ell^{(t+n)}}{(t+1)(t+2)\cdots(t+n)}-\Bigg\{c_{(n-1)j}\left(\frac{((m+a)\ell+j)_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right)$$
$$+c_{(n-2)j}\left(\frac{((m+a)\ell+j)_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right)+\cdots+c_{2j}\left(\frac{((m+a)\ell+j)_\ell^{(2)}}{2!\ell^2}\right)$$
$$+c_{1j}\left(\frac{((m+a)\ell+j)_\ell^{(1)}}{\ell}\right)+c_{0j}\Bigg\}, \tag{31}$$

for $a=n-1,n,n+1,\dots,2n-2$.

Proof. From (4) and (25), we have

$$\sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\cdots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(k-r_n\ell-r_{n-1}\ell-\cdots-r_2\ell-r_1\ell)^m$$
$$+c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right)+c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right)+\cdots$$
$$+c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right)+c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right)+c_{0j}=\sum_{t=1}^{m}\frac{S_t^m\ell^{m-(t+n)}k_\ell^{(t+n)}}{(t+1)(t+2)\cdots(t+n)}. \tag{32}$$

Replace k by $(m+a)\ell+j$ for $a=n-1,n,n+1,\dots,2n-2$ in (32) and solve the system of n equations with n unknowns. Now, the proof follows by (32). □

Example 4.8. Fourth partial sum of 6-th powers of arithmetic progression $(0.3)^6+(0.7)^6+(1.1)^6+\cdots+k^6$ is

$$P^4S^6=\sum_{r_5=2}^{r_5^*}\sum_{r_4=1}^{r_4^*}\sum_{r_3=1}^{r_3^*}\cdots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}$$
$$=\sum_{p=1}^{6}\frac{S_p^6(0.4)^{6-(p+5)}k_{0.4}^{p+5}}{(p+1)(p+2)(p+3)(p+4)(p+5)}-\sum_{i=1}^{4}c_{ij}\frac{k_{0.4}^{(i)}}{i!(0.4)^i}. \tag{33}$$

Solution. In (30), taking $n=5$, $j=0.3,\ell=0.4$ and $m=6$, we get P^4S^6. To find the values c_{ij}'s, $i=0,1,2,3,4$, we put $k=4.3,4.7,5.1,5.5,$ and 5.9 simultaneously which yields

$$\sum_{r_5=2}^{r_5^*}\sum_{r_4=1}^{r_4^*}\sum_{r_3=1}^{r_3^*}\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(4.3-(0.4)r_5-(0.4)r_4-(0.4)r_3-(0.4)r_2$$

$$-(0.4)r_1)^6+c_{4j}\left(\frac{(4.3)^{(4)}_{0.4}}{(4!)(0.4)^4}\right)+c_{3j}\left(\frac{(4.3)^{(3)}_{0.4}}{(6)(0.4)^3}\right)+c_{2j}\left(\frac{(4.3)^{(2)}_{0.4}}{(2)(0.4)^2}\right)$$

$$+c_{1j}\left(\frac{4.3}{0.4}\right)+c_{0j}=\sum_{p=1}^{6}\frac{S^6_p(0.4)^{6-(p+5)}(4.3)^{(p+5)}_{0.4}}{(p+1)(p+2)(p+3)(p+4)(p+5)}, \tag{34}$$

$$\sum_{r_5=2}^{r_5^*}\sum_{r_4=1}^{r_4^*}\sum_{r_3=1}^{r_3^*}\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(4.7-(0.4)r_5-(0.4)r_4-(0.4)r_3-(0.4)r_2$$

$$-(0.4)r_1)^6+c_{4j}\left(\frac{(4.7)^{(4)}_{0.4}}{(4!)(0.4)^4}\right)+c_{3j}\left(\frac{(4.7)^{(3)}_{0.4}}{(6)(0.4)^3}\right)+c_{2j}\left(\frac{(4.7)^{(2)}_{0.4}}{(2)(0.4)^2}\right)$$

$$+c_{1j}\left(\frac{4.7}{0.4}\right)+c_{0j}=\sum_{p=1}^{6}\frac{S^6_p(0.4)^{6-(p+5)}(4.7)^{(p+5)}_{0.4}}{(p+1)(p+2)(p+3)(p+4)(p+5)}, \tag{35}$$

$$\sum_{r_5=2}^{r_5^*}\sum_{r_4=1}^{r_4^*}\sum_{r_3=1}^{r_3^*}\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(5.1-(0.4)r_5-(0.4)r_4-(0.4)r_3-(0.4)r_2$$

$$-(0.4)r_1)^6+c_{4j}\left(\frac{(5.1)^{(4)}_{0.4}}{(4!)(0.4)^4}\right)+c_{3j}\left(\frac{(5.1)^{(3)}_{0.4}}{(6)(0.4)^3}\right)+c_{2j}\left(\frac{(5.1)^{(2)}_{0.4}}{(2)(0.4)^2}\right)$$

$$+c_{1j}\left(\frac{5.1}{0.4}\right)+c_{0j}=\sum_{p=1}^{6}\frac{S^6_p(0.4)^{6-(p+5)}(5.1)^{(p+5)}_{0.4}}{(p+1)(p+2)(p+3)(p+4)(p+5)}, \tag{36}$$

$$\sum_{r_5=2}^{r_5^*}\sum_{r_4=1}^{r_4^*}\sum_{r_3=1}^{r_3^*}\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(5.5-(0.4)r_5-(0.4)r_4-(0.4)r_3-(0.4)r_2$$

$$-(0.4)r_1)^6+c_{4j}\left(\frac{(5.5)^{(4)}_{0.4}}{(4!)(0.4)^4}\right)+c_{3j}\left(\frac{(5.5)^{(3)}_{0.4}}{(6)(0.4)^3}\right)+c_{2j}\left(\frac{(5.5)^{(2)}_{0.4}}{(2)(0.4)^2}\right)$$

$$+c_{1j}\left(\frac{5.5}{0.4}\right)+c_{0j}=\sum_{p=1}^{6}\frac{S^6_p(0.4)^{6-(p+5)}(5.5)^{(p+5)}_{0.4}}{(p+1)(p+2)(p+3)(p+4)(p+5)}, \tag{37}$$

$$\sum_{r_5=2}^{r_5^*}\sum_{r_4=1}^{r_4^*}\sum_{r_3=1}^{r_3^*}\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(5.9-(0.4)r_5-(0.4)r_4-(0.4)r_3-(0.4)r_2$$

$$-(0.4)r_1)^6+c_{4j}\left(\frac{(5.9)^{(4)}_{0.4}}{(4!)(0.4)^4}\right)+c_{3j}\left(\frac{(5.9)^{(3)}_{0.4}}{(6)(0.4)^3}\right)+c_{2j}\left(\frac{(5.9)^{(2)}_{0.4}}{(2)(0.4)^2}\right)$$

$$+c_{1j}\left(\frac{5.9}{0.4}\right)+c_{0j}=\sum_{p=1}^{6}\frac{S^6_p(0.4)^{6-(p+5)}(5.9)^{(p+5)}_{0.4}}{(p+1)(p+2)(p+3)(p+4)(p+5)}. \tag{38}$$

By solving equations (34), (35), (36), (37) and (38) we get constants c_{ij}'s, for $i = 0, 1, 2, 3, 4$ and the equation (30) yields P^4S^6.

In particular, when $k = 12.3$, the 4-th partial sum of 6-th powers of A.P. $(0.3)^6 + (0.7)^6 + (1.1)^6 + \cdots + (12.3)^6$ as $P^4S^6 = 656471620.9$.

Similarly one can find P^2S^6, PS^6 and S^6 using $\Delta_{\ell,\ell,\ell}$, $\Delta_{\ell,\ell}$, Δ_ℓ (see [4], [9], [10]).

Acknowledgments

This research is supported by University Grants Commission, New Delhi.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York (2000).

[2] Saber N. Elaydi, *An Introduction to Difference Equations*, Second Edition, Springer (1999).

[3] Walter G. Kelley, Allan C. Peterson, *Difference Equations, An Introduction with Applications*, Academic Press (1991).

[4] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Theory of generalized difference operator and its applications, *Far East Journal of Mathematical Sciences*, **20**, No. 2 (2006), 163-171.

[5] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Qualitative properties of solutions of certain classes of difference equations, *Far East Journal of Mathematical Sciences*, **23**, No. 3 (2006), 295-304.

[6] M. Maria Susai Manuel, A. George Maria Selvam, G. Britto Antony Xavier, Rotatory and boundedness of solutions of certain class of difference equations, *International Journal of Pure and Applied Mathematics*, **33**, No. 3 (2006), 333-343.

[7] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Generalized Bernoulli polynomials through weighted Pochhammer symbols, *Far East Journal of Applied Mathematics*, **26**, No. 3 (2007), 321-333.

[8] M. Maria Susai Manuel, G. Britto Antony Xavier, Recessive, dominant and spiral behaviours of solutions of certain classes of generalized difference equations, *International Journal of Differential Equations and Applications*, **10**, No. 4 (2007), 423-433.

[9] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, Generalized difference operator of the second kind and its application to number theory, *International Journal of Pure and Applied Mathematics*, **47**, No. 1 (2008), 127-140.

[10] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, R. Pugalarasu, S. Elizabeth, On generalized difference operator of third kind and its applications in number theory, *International Journal of Pure and Applied Mathematics*, **53**, No. 1 (2009), 69-81.

[11] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, R. Pugalarasu, S. Elizabeth, Generalized difference operator of the fourth kind and its applications in number theory, Part-I, *International Journal of Pure and Applied Mathematics*, **56**, No. 1 (2009), 97-109.

[12] Ronald E. Mickens, *Difference Equations*, Van Nostrand Reinhold Company, New York (1990).