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THEORY OF GENERALIZED DIFFERENCE OPERATOR OF n-TH KIND AND ITS APPLICATIONS IN NUMBER THEORY (PART - II)

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Abstract: In this paper, we extend the theory of generalized difference operator of the n-th kind, $\Delta_{\ell_1,\ell_2,\ldots,\ell_n}$, where ℓ_i's are real and present the discrete version of Leibnitz Theorem and Newton's Formula with reference to $\Delta_{\ell_1,\ell_2,\ldots,\ell_n}$. Using the Stirling numbers of the second kind S_i^n's we establish a formula for the sum of the general partial sums of consecutive terms of an arithmetic progression and sum of the general partial sums of an arithmetico-geometric progression in number theory. Suitable examples are provided to illustrate the main results.

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1. Introduction

The thoery of difference equations is based on the definition of the difference operator

$$\Delta y(n) = y(n+1) - y(n). \tag{1}$$

Many authors [1], [12], [2], [3] also defined Δ as

$$\Delta y(n) = y(n+\ell) - y(n), \quad \ell \in \mathbb{N} = \{1, 2, 3, \ldots\}, \tag{2}$$

but no significant work took place based on [4]. So, we took up the study of the difference operator defined by (2) and continued the study. We have for conenience defined Δ as given in (2) and denote it as Δ_ℓ and named it as the generalized difference operator. By defining Δ_ℓ^{-1} properly, many results in number theory were obtained. One can refer [4, 7] to have a glimpse of the theory. With the definition of Δ given (1) one can study the qualitative properties like monotonicity, oscillation and weakly oscillation of solutions of any type of difference equation. But, for difference equations involving Δ_ℓ, new types of properties like rotatory, spiral, weblike, recessive and dominance were noticed. For a comprehensive study on this line, one can refer [5, 6, 8]. The theory is further extended to $\ell \in \mathbb{R}$, the set of reals. Later we extended the theory for Δ_ℓ by introducing higher kinds namely, $\Delta_{\ell,m}, \Delta_{\ell_1,\ell_2,\ell_3}, \Delta_{\ell_1,\ell_2,\ell_3,\ell_4}$, generalized difference operators of the second, third and fourth kinds respectively. For a detailed study on this area one can refer [9, 10]. We further generalize the theory to $\Delta_{\ell_1,\ell_2,\ldots,\ell_n}$, the generalized difference operator of the n-th kind and many interesting results were obtained in [11]. In this paper, we use the definition of $\Delta_{\ell_1,\ell_2,\ldots,\ell_n}$ (see [11]) and obtain sums $P^{n-1}C_m$ and $P^{n-1}mS^m$, where $C_m = (i+(m-1)\ell)_\ell^{(m)} + (i+m\ell)_\ell^{(m)} + (i+(m+1)\ell)_\ell^{(m)} + \cdots + (i+(k+m-1)\ell)_\ell^{(m)}$, $PC_m = (i+(m-1)\ell)_\ell^{(m)} + [(i+(m-1)\ell)_\ell^{(m)} + (i+m\ell)_\ell^{(m)}] + \cdots + [(i+(m-1)\ell)_\ell^{(m)} + (i+m\ell)_\ell^{(m)} + (i+(m+1)\ell)_\ell^{(m)}] + \cdots + [(i+(m-1)\ell)_\ell^{(m)} + (i+m\ell)_\ell^{(m)} + \cdots + (i+(k+m-1)\ell)_\ell^{(m)}]$, $mS^m = ja^j + (j+\ell)a^{j+\ell} + \cdots + (j+m\ell)a^{j+m\ell}$, and $PmS^m = ja^j + [ja^j + (j+\ell)a^{j+\ell}] + \cdots + [ja^j + (j+\ell)a^{j+\ell} + \ldots + (j+m\ell)a^{j+m\ell}]$.

Similarly, $P^{n-1}C_m$ is the sum of all partial sums of $P^{n-2}C_m$ and $P^{n-1}mS^m$ is the sum of all partial sums of $P^{n-2}mS^m$, etc. Recurresively taking $n = 1, 2, 3, \ldots$, we get values of $C_m, PC_m, \ldots, P^{n-1}C_m$ and $mS^m, PmS^m, \ldots, P^{n-1}mS^m$, respectively.

Throughout this paper, we make use of the following assumptions:

(i) $\left[\frac{k}{\ell}\right]$ is the integer part of $\frac{k}{\ell}$,

(ii) $c, c_{0_j}, c_{1_j}, c_{2_j}, \ldots$ are constants and

(iii) $rC_i = \frac{r!}{(r-i)!\,i!}$, where $0! = 1,\ r! = 1\cdot 2\cdot \cdots \cdot r$.

2. Perliminaries

In this section, we present some preliminary definitions and results which will be useful for deriving our main results.

Definition 2.1. (see [11]) For the real or complex valued function $u(k), k \in [0,\infty)$, the generalized difference operator of n-th kind for $u(k)$ is defined as

$$\begin{aligned}\Delta_{\ell_1,\ell_2,\cdots\ell_n}u(k) &= u(k+\ell_1+\ell_2+\cdots+\ell_n) - \big\{u(k+\ell_1+\ell_2+\cdots+\ell_{n-1})+\\ &u(k+\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n)+u(k+\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n)+\cdots+\\ &u(k+\ell_1+\ell_3+\cdots+\ell_n)+u(k+\ell_2+\ell_3+\cdots+\ell_n)\big\}+\big\{u(k+\ell_1+\cdots+\ell_{n-2})\\ &+u(k+\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1})+\cdots+u(k+\ell_1+\ell_3+\cdots+\ell_{n-1})+\\ &u(k+\ell_2+\ell_3+\cdots+\ell_{n-1})+u(k+\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n)+u(k+\ell_1+\ell_2+\\ &\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n)+\cdots+u(k+\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n)+\cdots+u(k+\ell_1+\\ &\ell_4+\cdots+\ell_n)+u(k+\ell_2+\ell_4+\cdots+\ell_n)+u(k+\ell_3+\ell_4+\cdots+\ell_n)\big\}\\ &+\cdots+(-1)^{n-1}\big\{u(k+\ell_1)+\cdots+u(k+\ell_n)\big\}+(-1)^n u(k). \qquad (3)\end{aligned}$$

The following are easily deductions.

Lemma 2.2. (see [11]) *Let E be the usual shift operator. Then,*

$$\begin{aligned}(i)\quad \Delta_{\ell_1,\ell_2,\cdots\ell_n} &= E^{\ell_1+\ell_2+\cdots+\ell_n} - \big\{E^{\ell_1+\ell_2+\cdots+\ell_{n-1}} + E^{\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n}+\cdots\\ &+E^{\ell_2+\ell_3+\cdots+\ell_n}\big\}+\big\{E^{\ell_1+\ell_2+\cdots+\ell_{n-2}}+E^{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}}+\cdots\cdots+\\ &E^{\ell_2+\ell_3+\cdots+\ell_{n-1}}+E^{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n}+E^{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n}+\cdots\cdots+\\ &E^{\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n}+E^{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n}+E^{\ell_1+\ell_2+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n}\\ &+\cdots+E^{\ell_2+\ell_3+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n}+\cdots\cdots+E^{\ell_1+\ell_4+\cdots+\ell_n}+E^{\ell_2+\ell_4+\cdots+\ell_n}+\\ &E^{\ell_3+\ell_4+\cdots+\ell_n}\big\}+\cdots+(-1)^{n-1}\big\{E^{\ell_1}+E^{\ell_2}+\cdots+E^{\ell_n}\big\}+(-1)^n. \qquad (4)\end{aligned}$$

$$\begin{aligned}(ii)\quad \Delta_{\ell_1,\ell_2,\cdots\ell_n} &= \Delta_{\ell_1+\ell_2+\cdots+\ell_n} - \big\{\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-1}}+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}+\ell_n}+\\ &\cdots+\Delta_{\ell_2+\ell_3+\cdots+\ell_n}\big\}+\big\{\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-2}}+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1}}+\cdots+\\ &\Delta_{\ell_2+\ell_3+\cdots+\ell_{n-1}}+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_n}+\Delta_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-2}+\ell_n}+\cdots+\end{aligned}$$

$$\Delta_{\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n} + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-4}+\ell_{n-1}+\ell_n} + \Delta_{\ell_1+\ell_2+\cdots+\ell_{n-5}+\ell_{n-3}+\ell_{n-1}+\ell_n} + \cdots + \Delta_{\ell_2+\ell_3+\cdots+\ell_{n-3}+\ell_{n-1}+\ell_n}\big\} + \cdots + (-1)^{n-1}\big\{\Delta_{\ell_1} + \Delta_{\ell_2} + \cdots + \Delta_{\ell_n}\big\}. \quad (5)$$

$$(iii) \qquad \Delta_{\ell_1,\ell_2,\cdots\ell_n} = \Delta_{\ell_1}\Delta_{\ell_2}\cdots\Delta_{\ell_n}. \quad (6)$$

Lemma 2.3. (see [11]) *If ℓ_j's, $j = 1, 2, \ldots, n$ are positive integers, then*

$$\Delta_{\ell_1,\ell_2,\cdots\ell_n} = \prod_{j=1}^{n}\Big(\sum_{i_j=1}^{\ell_j} \ell_j C_{i_j}\Delta^{i_j}\Big). \quad (7)$$

Definition 2.4. (see [11]) If $\ell_1, \ell_2, ..., \ell_n$ are positive reals, then the generalized polynomial factorial of the n-th kind denoted by $k^{(t)}_{\ell_1,\ell_2,\ldots,\ell_n}$ is defined as $k^{(r)}_{\ell_1,\ell_2,\cdots,\ell_n} = (k+\ell_2+\ell_3+\cdots+\ell_n)^{(r)}_{\ell_1} + (k+\ell_1+\ell_3+\cdots+\ell_n)^{(r)}_{\ell_2} + \cdots + (k+\ell_1+\ell_2+\cdots+\ell_{n-1})^{(r)}_{\ell_n} - \{(k+\ell_2+\ell_3+\cdots+\ell_{n-1})^{(r)}_{\ell_1} + (k+\ell_2+\ell_3+\cdots+\ell_{n-2}+\ell_n)^{(r)}_{\ell_1} + \cdots + (k+\ell_3+\ell_4+\cdots+\ell_n)^{(r)}_{\ell_1} + (k+\ell_1+\ell_3+\cdots+\ell_{n-1})^{(r)}_{\ell_2} + (k+\ell_1+\ell_3+\cdots+\ell_{n-2}+\ell_n)^{(r)}_{\ell_2} + \cdots + (k+\ell_3+\ell_4+\cdots+\ell_n)^{(r)}_{\ell_2} + \cdots + (k+\ell_1+\ell_2+\cdots+\ell_{n-2})^{(r)}_{\ell_n} + (k+\ell_1+\ell_2+\cdots+\ell_{n-3}+\ell_{n-1})^{(r)}_{\ell_n} + \cdots + (k+\ell_2+\ell_3+\cdots+\ell_{n-1})^{(r)}_{\ell_n} + \cdots\} + (-1)^{n-2}\{(k+\ell_2)^{(r)}_{\ell_1} + (k+\ell_3)^{(r)}_{\ell_1} + \cdots + (k+\ell_n)^{(r)}_{\ell_1} + (k+\ell_1)^{(r)}_{\ell_2} + (k+\ell_3)^{(r)}_{\ell_2} + \cdots + + (k+\ell_n)^{(r)}_{\ell_2} + \cdots + (k+\ell_1)^{(r)}_{\ell_n}$

$$+ (k+\ell_2)^{(r)}_{\ell_n} + \cdots + (k+\ell_{n-1})^{(r)}_{\ell_n}\} + (-1)^{n-1}\{k^{(r)}_{\ell_1} + k^{(r)}_{\ell_2} + \cdots + k^{(r)}_{\ell_n}\}, \quad (8)$$

and first kind

$$k^{(r)}_{\ell} = k(k-\ell)(k-2\ell)\cdots(k-(r-1)\ell). \quad (9)$$

Definition 2.5. (see [11]) The inverse of the generalized difference operator of the n-th kind, denoted by $\Delta^{-1}_{\ell_1,\ell_2,\cdots,\ell_n}$ is defined as if $\Delta_{\ell_1,\ell_2,\cdots,\ell_n}\, z(k) = u(k)$, then

$$z(k) = \Delta^{-1}_{\ell_1,\ell_2,\cdots,\ell_n}u(k) + c_{(n-1)j}\frac{k^{(n-1)}_{\ell_{n-1}}}{(n-1)!\ell^{n-1}_{n-1}} + c_{(n-2)j}\frac{k^{(n-2)}_{\ell_{n-2}}}{(n-2)!\ell^{n-2}_{n-2}} + \cdots + c_{2j}\frac{k^{(2)}_{\ell_2}}{(2)!\ell_2^2} + c_{1j}\frac{k}{\ell_1} + c_{0j}, j \in \{0, 1, 2, \cdots, \ell-1\}. \quad (10)$$

In general, $\Delta^{-m}_{\ell_1,\ell_2,\cdots,\ell_n} = \Delta^{-1}_{\ell_1,\ell_2,\cdots,\ell_n}(\Delta^{-(m-1)}_{\ell_1,\ell_2,\cdots,\ell_n})$.

Theorem 2.6. (see [11]) *There exist constants $c_{oj}, c_{1j}, \cdots c'_{(n-1)j}s$ such that*

$$\Delta^{-1}_{\underbrace{\ell,\ell,\cdots,\ell}_{n-times}}u(k) = \sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\cdots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*} u(k - r_n\ell - r_{n-1}\ell - \cdots$$

$$-r_2\ell - r_1\ell) + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right) + \cdots + c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right) + c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right) + c_{0j}, \quad (11)$$

where $r_n^* = \left[\frac{k}{\ell}\right], r_{n-1}^* = r_n^* - r_n, r_{n-2}^* = r_{n-1}^* - r_{n-1}, \ldots, r_1^* = r_2^* - r_2$.

Proof. The proof follows by (6) and the relation

$$\Delta_{\underbrace{\ell,\ell,\cdots,\ell}_{n-times}}\left\{\sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\cdots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*} u(k - r_n\ell - r_{n-1}\ell - \cdots - r_2\ell - r_1\ell) + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right) \cdots + c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right) + c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right) + c_{0j}\right\} = u(k). \quad \square$$

3. Main Results

In this section, we present the generalized discrete version of the Leibnitz Theorem and Newton's formula with respect to the generalized difference operator of the n-th kind.

The following is the discrete version of the Leibnitz Theorem according to $\Delta_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}}$.

Theorem 3.1. *For two real or complex valued functions* $u(k)$ *and* $v(k), k \in [0,\infty)$,

$$\Delta^m_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}}(u(k)\,v(k)) = \Delta^m_{\underbrace{\ell,\ell,\ldots,\ell}_{(n-1)-times}}(u(k)\Delta_\ell^m v(k)) + mC_1\Delta^m_{\underbrace{\ell,\ell,\ldots,\ell}_{(n-1)-times}}(\Delta_\ell u(k)\Delta_\ell^{m-1}v(k+\ell)) + mC_2\Delta^m_{\underbrace{\ell,\ell,\ldots,\ell}_{(n-1)-times}}(\Delta_\ell^2 u(k)\Delta_\ell^{m-2}v(k+2\ell)) + \cdots + \Delta^m_{\underbrace{\ell,\ell,\ldots,\ell}_{(n-1)-times}}(\Delta_\ell^m u(k)v(k+m\ell)). \quad (12)$$

Proof. The proof follows by the generalized Leibnitz Theorem (see [4], Theorem 2.5) and (6). ☐

Lemma 3.2. (see [11]) *Let m and n are positive integers and $\ell \in (0,\infty)$. Then,*

$$\Delta_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}} k^{(m)}_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}} = (m\ell)^{(n)}_{\ell} \quad k^{(m-n)}_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}}. \tag{13}$$

The following is the generalized version of Newton's formula with reference to $\Delta_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}}$ and $k^{(m)}_{\underbrace{\ell,\ell,\ldots\ell}_{n-times}}$.

Theorem 3.3. *Let $f(k)$ be a polynomial of degree q (where $q = (m(r+1)-1)$) in k. Then, $f(k)$ can be expressed as*

$$f(k) = f(0) + \frac{\Delta_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}} f(0)}{(2r-1)!\, r\ell^{(2r-1)}}\ k^{(2r-1)}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}} + \frac{\Delta^2_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}} f(0)}{(3r-1)!\, r\ell^{(3r-1)}} k^{(3r-1)}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}}$$
$$+\cdots+ \frac{\Delta^r_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}} f(0)}{q!\, r\ell^{q}}\ k^{(q)}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}}. \tag{14}$$

Proof. Assume that

$$f(k) = a_0 + a_1 k^{(2r-1)}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}} + a_2 k^{(3r-1)}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}} + \cdots + a_r k^{(q)}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}}, \tag{15}$$

where $q = (m(r+1)-1)$. Clearly $f(0) = a_0$. The coefficients a_i, $i = 1,2,3,\ldots,r$ are determined from the relation

$$\Delta^r_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}} f(0) = a_r\,(m(r+1)-1)!r\ \ell^{(m(r+1)-1)},\ r \geq 1. \tag{16}$$

The proof now follows from (15) and (16). ☐

Corollary 3.4. *Let $f(k)$ be a polynomial of degree q (where $q = (m(r+1)-1)$) in k. Then $f(k-t)$ can be expressed as*

$$f(k-t) = f(t) + \frac{\Delta_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}} f(t)}{(2r-1)!r\,\ell^{(2r-1)}}\ (k-t)^{(2r-1)}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}} + \frac{\Delta^2_{\underbrace{\ell,\ell,\,\ldots\,,\ell}_{n-times}} f(t)}{(3r-1)!r\,\ell^{(3r-1)}}$$

$$(k-t)^{(3r-1)}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}} + \cdots + \frac{\Delta^{r}_{\underbrace{\ell,\ell,\ \ldots\ ,\ell}_{n-times}} f(t)}{q!\, r\ell^{q}} \quad (k-t)^{(q)}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-times}}. \quad (17)$$

Proof. The proof follows by replacing k by $(k-t)$ and 0 by t in (14). □

Lemma 3.5. (see [11]) *Let m be a positive integer and $k \in [0,\infty)$. Then,*

$$\Delta^{-1}_{\ell_1,\ell_2,\cdots,\ell_n} k^{(m)}_{\ell_1,\ell_2,\cdots,\ell_n} = \frac{k^{(m+1)}_{\ell_1}}{\ell_{1(m+1)}} + \frac{k^{(m+1)}_{\ell_2}}{\ell_{2(m+1)}} + \cdots + \frac{k^{(m+1)}_{\ell_n}}{\ell_{n(m+1)}} + c_{(n-1)j}\left(\frac{k^{(n-1)}_{\ell_{n-1}}}{(n-1)!\ell^{n-1}_{n-1}}\right)$$
$$+ c_{(n-2)j}\left(\frac{k^{(n-2)}_{\ell_{n-2}}}{(n-2)!\ell^{n-2}_{n-2}}\right) + \cdots + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2!\ell_2^2}\right) + c_{1j}\left(\frac{k^{(1)}_{\ell_1}}{\ell_1}\right) + c_{0j}. \quad (18)$$

Proof. The proof follows from (10) and $\Delta^{-1}_{\ell} k^{(n)}_{\ell} = \frac{k^{(n+1)}_{\ell}}{\ell(n+1)} + c$. □

Corollary 3.6. (see [11]) *Let m and n are positive integer and $\ell \in (0,\infty)$. Then,*

$$\Delta^{-1}_{\ell,\ell,\cdots,\ell} k^{(m)}_{\ell,\ell,\cdots,\ell} = n\frac{k^{(m+1)}_{\ell}}{\ell(m+1)} + c_{(n-1)j}\left(\frac{k^{(n-1)}_{\ell}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}$$
$$\left(\frac{k^{(n-2)}_{\ell}}{(n-2)!\ell^{n-2}}\right) + \cdots + c_{2j}\left(\frac{k^{(2)}_{\ell}}{2!\ell^2}\right) + c_{1j}\left(\frac{k^{(1)}_{\ell}}{\ell}\right) + c_{0j}. \quad (19)$$

Lemma 3.7. (see [11]) *If p and q are positive integers and ℓ is positive real then,*

$$\Delta^{p}_{\underbrace{\ell,\ell,\ldots,\ell}_{n-\ times}} k^{q} = \begin{cases} q!\ell^{q}, & \text{if} \quad q = np, \\ 0, & \text{if} \quad q < np. \end{cases} \quad (20)$$

Proof. The proof follows from (6) and induction on p. □

4. Applications

Here, we derive the formulae for the sum of general partial sums of the products of m consecutive terms of an arithmetic progression and an arithmetico-geometric progression using the generalized difference operator of the n-th kind and its inverse operator. Also, we present suitable examples to illustrate the formulae.

The following theorem gives a general rule to find the sum of general partial sums of the products of m consecutive terms of an arithmetic progression.

Theorem 4.1. *If $j \geq (n-1)\ell$, then*

$$\sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\cdots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(k-r_n\ell-r_{n-1}\ell-\cdots-r_2\ell-r_1\ell)_\ell^{(m)}$$

$$=\frac{k_\ell^{(m+n)}}{\ell^n(m+1)(m+2)\dots(m+n)}-c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right)$$

$$-c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right)-\cdots-c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right)-c_{1j}\left(\frac{k_\ell^{(1)}}{\ell}\right)-c_{0j}, \tag{21}$$

where, the constants are obtained by solving the system of n equations which are obtained by substituting $k=(m+a)\ell+j$, for $a=n-1,n,n+1,\dots,2n-2$.

Proof. We have

$$\Delta^{-1}_{\underbrace{\ell,\ell,\dots,\ell}_{n-times}}k_\ell^{(m)}=\sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\cdots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}$$

$$(k-r_n\ell-r_{n-1}\ell-\cdots-r_2\ell-r_1\ell)_\ell^{(m)}+c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right)$$

$$+c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right)+\cdots+c_{2j}\left(\frac{k_\ell^{(2)}}{2!\ell^2}\right)+c_{1j}\left(\frac{k_\ell}{\ell}\right). \tag{22}$$

Now, the theorem follows by substituting $k=(m+a)\ell+j$, for $a=n-1,n,n+1,\dots,2n-2$ in (21) and solving the system of n equations. □

Example 4.2. Sum of fourth partial sums of product of 4 consecutive terms of the A.P. $2+5+8+11+\cdots+k$ is

$$P^4C_4=\sum_{r_5=2}^{r_5^*}\sum_{r_4=1}^{r_4^*}\sum_{r_3=1}^{r_3^*}\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(k-3r_5-3r_4-3r_3-3r_2-3r_1)_3^{(4)}$$

$$=\frac{k_3^{(9)}}{3^5(5)(6)(7)(8)(9)}-\sum_{i=1}^{4}c_{ij}\frac{k_3^{(i)}}{i!3^i}.$$

Solution. In (21), by taking $n=5$, $m=4$, $\ell=3$, $j=11$, we get P^4C_4.

To find the values c_{ij}'s, $i = 0, 1, 2, 3, 4$, we put $k = 35, 38, 41, 44$ and 47 in (21) simultaneously. By solving the above equations we get constants c_{ij}'s, for $i = 0, 1, 2, 3, 4$ and the equation (21) yields P^4C_4.

In particular, when $k = 77$, $P^4C_4 = 5284285776$.

Lemma 4.3. *If $a \neq 0$, then there exists constants c_{ij}'s such that*

$$\Delta^{-1}_{\underbrace{\ell,\ell,\dots,\ell}_{n-times}}(ka^k) = \frac{a^k}{(a^\ell - 1)^n}\left[k - \frac{n\ell a^\ell}{(a^\ell - 1)}\right] + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{(n-2)}}\right) + \cdots + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}. \quad (23)$$

Proof. The proof follows from

$$\Delta_{\underbrace{\ell,\ell,\dots,\ell}_{n-times}}\left\{\frac{a^k}{(a^\ell - 1)^n}\left[k - \frac{n\ell a^\ell}{(a^\ell - 1)}\right] + c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{(n-2)}}\right) + \cdots + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}\right\} = ka^k. \quad \square$$

The following theorem is to find the formula for the sum of the general partial sums of an arithmetico-geometric progression.

Theorem 4.4. *If $k > n\ell$, then*

$$\sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\cdots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(k - r_n\ell - r_{n-1}\ell - \cdots - r_2\ell - r_1\ell) a^{(k - r_n\ell - r_{n-1}\ell - \cdots - r_2\ell - r_1\ell)} = \frac{a^k}{(a^\ell - 1)^n}\left(k - \frac{n\ell a^\ell}{(a^\ell - 1)}\right) - \left\{c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right) + \cdots + c_{0j}\right\}, \quad (24)$$

where the constants are obtained by solving the system of n equations arrived by substituting $k = (m + a)\ell + j$, for $a = n - 1, n, n + 1, \dots, 2n - 2$.

Proof. From (6) and (23), we get

$$\sum_{r_n=2}^{\left[\frac{k-j}{\ell}\right]}\sum_{r_{n-1}=1}^{\left[\frac{k-j}{\ell}\right]-2}\cdots\sum_{r_2=1}^{\left[\frac{k-j}{\ell}\right]-n+1}\sum_{r_1=0}^{\left[\frac{k-j}{\ell}\right]-n}(k - r_n\ell - r_{n-1}\ell - \cdots - r_2\ell - r_1\ell)$$

$$a^{(k-r_n\ell-r_{n-1}\ell-\cdots-r_2\ell-r_1\ell)}+c_{(n-1)j}\left(\frac{k_\ell^{(n-1)}}{(n-1)!\ell^{n-1}}\right)+c_{(n-2)j}$$

$$\left(\frac{k_\ell^{(n-2)}}{(n-2)!\ell^{n-2}}\right)+\cdots+c_{1j}\left(\frac{k}{\ell}\right)+c_{0j}=\frac{a^k}{(a^\ell-1)^n}\left(k-\frac{n\ell a^\ell}{(a^\ell-1)}\right). \quad (25)$$

The theorem follows by substituting $k=(p\ell+j)$, for $p=n-2,n-1,n,\ldots,2n-3$ in (25) and solving the system of n equations. □

Example 4.5. Sum of fourth partial sums of arithmetico-geometric progression $(1)2^1+(5)2^5+(9)2^9+\cdots+(k)2^k$ is

$$P^4\frac{k-1}{4}S^{\frac{k-1}{4}}=\sum_{r_5=2}^{r_5^*}\sum_{r_4=1}^{r_4^*}\sum_{r_3=1}^{r_3^*}\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}(k-3r_5-3r_4-3r_3-3r_2-3r_1)$$
$$=2^{(k-3r_5-3r_4-3r_3-3r_2-3r_1)}\frac{2^k}{(2^4-1)^n}\left(k-\frac{(5)(4)2^4}{(2^4-1)}\right).$$

Solution. In (24), by taking $n=5$, $a=2,\ell=4$, and $j=1$, we get $P^4\frac{k-1}{4}S^{\frac{k-1}{4}}$.

To find the values c_{ij}'s, $i=0,1,2,3,4$, we put $k=13,17,21,25$ and 29 in (24) simultaneously.

By solving the above equations we get constants c_{ij}'s, for $i=0,1,2,3,4$ and the equation (24) yields $P^4\frac{k-1}{4}S^{\frac{k-1}{4}}$.

In particular, when $k=73$, the fourth partial sums of arithmetico-geometric progression $(1)2^1+(5)2^5+(9)2^9+\cdots+(k)2^k$ as

$$P^4 13S^{13}=6.426046022\times10^{27}.$$

Similarly one can find the values $C_m, PC_m,\ldots,P^{n-1}C_m$ and

$$mS^m, PmS^m,\ldots,P^{n-1}mS^m$$

using $\Delta_\ell,\Delta_{\ell,\ell},\ldots$, respectively (see [4, 9, 10]).

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