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ON SHARKOVSKII'S THEOREM

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Abstract: Let f be a continuous real valued function defined over the real line. If f has a periodic point of least period n, the existence of any other periodic point of least period m different from n is completely discovered by Russian mathematician Alexander Nikolaevich Sharkovskii. In this paper we have proved Sharkovskii's Theorems in a very simple way. In the last section we have discussed some important consequences and generalization of these theorems.

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1. Introduction

In 1964, the Russian mathematician Alexander Nikolaevich Sharkovskii [11] discovered a spectacular and beautiful result on the continuous maps of the intervals. He gave a complete answer to the existence of which particular period implies the existence of other periods. The original paper of Sharkovskii [11] was published in Russian and translated in English in [12]. There are three parts of the full Sharkovskii's Theorem, which we have stated as Theorem 2.1, Theorem 2.2 and Theorem 2.3. But in many papers and books (as for example [2, 6, 10]) dealing with Sharkovskii's Theorem, only Theorem 3.1 has been applied to his name. It is also noticeable that one of the most striking features of the Sharkovskii's Theorem is that the existence of a period three point implies

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the existence of periodic points of every period. However, in 1975, T.Y. Li and J. Yorke [9] published their surprising result that if $f : R \rightarrow R$ has a point of period three, it has points of all other periods. But the proof of Li and Yorke's paper is different from the proof of the original paper of Sharkovskii. In the original Sharkovskii's paper the proof is very complicated. In the past 40 years there emerged various types of proofs of this beautiful theorem [3, 4, 5, 6, 8]. In 1996, S. Elaydi [7] proved Theorem 3.2 and he named this theorem as the converse of Sharkovskii's Theorem.

In our present paper, we have given a simpler and altogether different proof of Sharkovskii's Theorem, ending our work by discussing some important consequences and generalization of those theorems.

2. Mathematical Preliminaries

In this section we give some definitions, lemmas and notations, which are essential for this paper.

Definition 2.1. (Fixed Point, see [2]) Let $f : I \rightarrow I$ be a continuous map. If a point a is such that $f(a) = a$, then a is called a fixed point for f.

Definition 2.2. (Periodic Point, see [2]) Let $f : I \rightarrow I$ be a continuous map. The point x is called a periodic point of least period n if $f^n(x) = x$ and $f^m(x) \neq x$, for all $m < n$ where m and n are positive integers.

Definition 2.3. (Sharkovskii Ordering, see [11]) Sharkovskii ordering of natural numbers is defined as follows: $1 \triangleleft 2 \triangleleft 2^2 \triangleleft 2^3 \triangleleft \cdots \triangleleft 2^2.7 \triangleleft 2^2.5 \triangleleft 2^2.3 \triangleleft \cdots \triangleleft 2.7 \triangleleft 2.5 \triangleleft 2.3 \triangleleft \cdots \triangleleft 7 \triangleleft 5 \triangleleft 3$.

Definition 2.4. (Double Operator, see [10]) Let $f : I' \rightarrow I'$ be a map where I'=[1,0] is the unit interval. Then the double operator of f is defined by

$$D_f(x) = \begin{cases} \frac{2}{3} + \frac{1}{3} f(3x), & 0 \leq x \leq \frac{1}{3}, \\ (2 + f(1))(\frac{2}{3} - x), & \frac{1}{3} \leq x \leq \frac{2}{3}, \\ x - \frac{2}{3}, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

We now give the statement of Intermediate Value Theorem which is very useful for this paper.

Theorem 2.1. (Intermediate Value Theorem) *Let f be continuous in each point of a closed interval $[a, b]$. Also let $x_1,\ x_2$ be any two points of $[a, b]$, such that $f(x_1) \neq f(x_2)$. Then f takes on every values between $f(x_1)$ and $f(x_2)$ somewhere in the interval (x_1, x_2).*

The following notations are used in this paper.

Throughout this paper f is always a continuous map.

Let I be a closed interval and $f : I \to I$ be a continuous map. Then a set $P = \{x_1, x_2, \ldots, x_n\}$ be such that $f(x_1) = x_2$, $f(x_2) = x_3, \ldots, f(x_n) = x_1$ is called a cycle of n period or a n-periodic orbit. If K and L are closed intervals such that, $f(K) \supset L$ then, we call $f(K)$ covers L and denote this by $K \to L$. Also $f^n(x)$ is defined by $f^1(x) = f(x)$ and $f^n(x) = f(x) \circ f^{n-1}(x)$, when $n \geq 2$. If A is a set, interior of A is denoted by $\text{Int}(A)$. Lastly, if m and n are two positive integers, the greatest common divisor of m and n is denoted by $\gcd(m, n)$.

We also need the following well known lemmas. We give the following alternative proofs of these lemmas.

Lemma 2.1. *Let I be a closed interval and $f(I)$ covers I, that is, $I \to I$, then f has a fixed point in I.*

Proof. Let $I = [a, b]$, then $f([a, b]) \supset [a, b]$. Since f is continuous, by Intermediate Value Theorem, there exist points a_1, b_1 in I such that $f(a_1) = a$ and $f(b_1) = b$.

Also, $a \leq a_1, a \leq b_1$ and $b \geq b_1, b \geq a_1$.

We now consider the continuous function $f(x) - x$. Then we get $f(a_1) - a_1 \leq 0$ and $f(b_1) - b_1 \geq 0$.

Since $f(x) - x$ is a continuous function, we get from the above arguments, that the graph of $f(x) - x$ must cut the line $f(x) - x = 0$ between a_1 and b_1 (or between b_1 and a_1) at least once. Hence f has a fixed point in I. □

Lemma 2.2. *If I and J are closed intervals with $I \subset J$ and $f(I)$ covers J, that is, $I \to J$, then f has a fixed point in I.*

Proof. Let $I = [a_1, b_1]$ and $J = [a_2, b_2]$. Since $I \to J$, by Intermediate Value Theorem there exist points a_3 and b_3 in I such that $f(a_3) = a_2$ and $f(b_3) = b_2$. So, $a_2 \leq a_3$ and $b_2 \geq b_3$. We now consider the continuous function $f(x) - x$. Now $f(a_3) - a_3 = a_2 - a_3 \leq 0$ and $f(b_3) - b_3 = b_2 - b_3 \geq 0$. Hence $f(x) - x$ must cut the line $f(x) - x = 0$ at least once between a_3 and b_3 (or between b_3 and a_3). Therefore $f(x)$ has a fixed point in I. □

Lemma 2.3. *If K and L are closed intervals and $f(K)$ covers L, that is, $K \to L$, then there exists a closed sub-interval $K_1 \subset K$ such that $f(K_1) = L$.*

Proof. Let $K = [a, b]$ and $L = [c, d]$. So, by Intermediate Value Theorem there exist at least one point a_1 in K such that $f(a_1) = c$ and at least one point b_1 in K such that $f(b_1) = d$. We take $S_1 = \{x : x \in K \text{ and } f(x) = c\}$ and $S_2 = \{x : x \in K \text{ and } f(x) = d\}$. Then $S_1 \cap S_2 = \phi$. There are two possibilities

$\max S_1 < \min S_2$ or $\max S_1 > \min S_2$.

Case I. Let $\max S_1 < \min S_2$.

We now consider the closed interval $K_1 = [\max S_1, \min S_2]$.

Then obviously $f(K_1) = L$.

Case II. Let $\max S_1 > \min S_2$.

Here we consider the closed interval $[\min S_2, \max S_1]$. Let p and q be two points in $[\min S_2, \max S_1]$ such that $f(p) = c$ and $f(q) = d$ and there exists no point in [p,q] of S_1 and S_2. Then obviously $K_1 = [p, q]$ such that $f(K_1) = L$. □

Lemma 2.4. *Let $J_0, J_1, \ldots, J_{n-1}$ be closed intervals such that $J_i \subset f(J_{i-1})$, for $i = 1, 2, \ldots, n-1$ and $f(J_{n-1}) \supset J_0$. Then there exists a fixed point p of f^n such that $f^i(p) \in J_i$, for $i = 0, 1, 2, \ldots, n-1$.*

Proof. Let $J_0 = A_n$. Given, $f(J_{n-1}) \supset J_0 = A_n$. Then by Lemma 2.3 we get a closed sub-interval A_{n-1} of J_{n-1} such that $f(A_{n-1}) = A_n$. Similarly, $f(J_{n-2}) \supset J_{n-1}$. By applying Lemma 2.3 again we get a closed sub-interval A_{n-2} of J_{n-2} such that $f(A_{n-2}) = A_{n-1}$. Proceeding similarly, we obtain a closed sub-interval A_i of J_i such that $f(A_i) = A_{i+1}$, for $i = 0, 1, 2, \ldots, n-1$. Also note that $f(A_0) = A_1, f^2(A_0) = A_2, \ldots, f^n(A_0) = A_n$, that is, $f^n(A_0) = A_n = J_0 \supset A_0$. So by Lemma 2.1 there exists a point p in A_0 such that $f^n(p) = p$.

Since $p \in A_0 \Rightarrow f^i(p) \in f^i(A_0), i = 0, 1, 2, \ldots, n-1$

We conclude that $f^i(p) \in A_i \subset J_i, i = 0, 1, 2, \ldots, n-1$. □

Lemma 2.5. *If y is a periodic point of f with least period m, it is a periodic point of f^n with least period $\frac{m}{\gcd(m,n)}$.*

Proof. Let the least period of f^n be k. Then $(f^n)^k(y) = y \Rightarrow f^{nk}(y) = y$ $\Rightarrow m$ divides $nk \Rightarrow nk = mr$, for some $r \in Z$.

Let $\gcd(m, n) = d$. Then there exist integers u and v such that $m = du, n = dv$ and $\gcd(u, v) = 1$. Now $nk = mr \Rightarrow dvk = dur \Rightarrow vk = ur \Rightarrow u$ divides vk, but $\gcd(u, v) = 1 \Rightarrow u$ divides $k \Rightarrow \frac{m}{d}$ divides k.

Now, $(f^n)^{\frac{m}{d}}(y) = f^{\frac{mn}{d}}(y) = f^{mv}(y) = (f^m)^v(y) = y$. Since least period of f^n is k, k divides $\frac{m}{d}$. Hence we get that $k = \frac{m}{d}$, that is, y is a periodic point of f^n with least period $\frac{m}{\gcd(m,n)}$.

Lemma 2.6. *If y is a periodic point of f^n with least period k, it is also a periodic point of f with least period $\frac{kn}{s}$, where s divides n and is relatively prime to k.*

Proof. By the given condition $(f^n)^k(y) = y \Rightarrow f^{nk}(y) = y$. Let the least period of f for y be d. So $f^d(y) = y$ and d divides nk. Then $nk = ds$, for some $s \in Z$. By applying Lemma 2.5 we get that $k = \frac{d}{\gcd(d,n)} = \frac{\frac{kn}{s}}{\gcd(\frac{kn}{s},n)}$. So $\frac{n}{s} = \gcd(\frac{kn}{s}, n) = \gcd(\frac{n}{s}k, \frac{n}{s}s)$. That is, $\frac{n}{s} = \frac{n}{s}\gcd(k, s)$. Hence $\gcd(k, s) = 1$. Since, $\frac{nk}{s} = d$ is a positive integer and $\gcd(k, s) = 1$, obviously s divides n. Lastly, $\gcd(k, s) = 1$ implies that k, s are relatively prime.

Lemma 2.7. *If f has a periodic point of least period $m \geq 3$, f has also a periodic point of least period 2.*

Proof. First we prove that if f has a periodic point of least period $m \geq 3$, it has a periodic point less than m which is also not fixed. Let $x_1 < x_2 < \cdots < x_m$ be the m periodic orbit of f. We now apply the directed graph proof. We consider the $p-1$ number of vertices, named $1, 2, \ldots, p-1$ in such a way that i-th vertex is joined to the j-th vertex if $[x_i, x_{i+1}]$ covers $[x_j, x_{j+1}]$. Since the set $\{x_1, x_2, \ldots, x_m\}$ forms a m periodic orbit of f with $m \geq 3$, each vertex i must be joined to at least one vertex j such that $i \neq j$. We now discuss how to choose an edge. Starting from the first vertex choose an edge that joins the first vertex to a different vertex in the way defined above. Then we join this vertex to another and get another edge. Continuing this process one by one, we note that after at most $m-1$ vertices it must return to a previously taken vertex. So this process gives us a cycle which visits at least 2 vertices and at most $m-1$ vertices. That is, we get q-edges namely, $i_1, i_2, \ldots, i_q$ such that $2 \leq q \leq m-1$. So, if we now take $A_k = [x_{i_k}, x_{i_{k+1}}]$, A_k covers A_{k+1} for $k = 1, 2, \ldots, q$ and A_q covers A_1. We now prove this result in two different ways.

Case I. Since $f(A_1) \supset A_2$, by Lemma 5.2.3, there exists an interval $I_1 \subset A_1$ such that $f(I_1) = A_2$. Again $f^q(I_1) = f^{q-1}(f(I_1)) = f^{q-1}(A_2)$.

Also note that, $f^{q-1}(A_2) \supset f^{q-2}(A_3) \supset \cdots \supset f(A_q) \supset A_1$, that is, $f^q(I_1) \supset A_1 \supset I_1$. So $f^q(I_1) \supset I_1$. Hence from Lemma 2.1, f^q has a fixed point y in I_1. Clearly, y is a periodic point for f whose least period is a factor of q and therefore less than m.

Case II. Since $A_1 \to A_2 \to \cdots \to A_q \to A_1$, by Lemma 2.4, there exists a fixed point y in A_1 of f^q. For same reason y is a periodic point of f whose least period is a factor of q and therefore less than m.

We are ending this proof now by showing that y is not a fixed point of f. In both the cases it will be noticeable that $y \in A_1 \Rightarrow f(y) \in A_2$. Because $f(y)$ cannot belong to A_1 since, $q \geq 2$. If $f(y) = y$, y must be an element of $A_1 \cap A_2$. But this is impossible because $\text{Int}(A_1) \cap \text{Int}(A_2) = \phi$ and their end points form a period m cycle with least period $m > 1$. This shows that y is not a fixed point of f.

It is to be noted that if we did not get q such that $q = 2$ then obviously $2 < q \leq m - 1$. Continuing this process again and again at some finite step we get $q = 2$, that is, we get a period two point of f.

Lemma 2.8. If *f has a periodic point of least period m with $m \geq 3$ and odd, f has a period $(m + 2)$ point.*

Lemma 2.9. *If f has a periodic point of least period m with $m \geq 3$ and odd, f has a period 6 point and a period $2m$ point.*

Here we do not prove Lemma 2.8 and Lemma 2.9, as there are many beautiful proofs of those results, see [4, 6].

Lemma 2.10. *Let $I' = [0, 1]$ be the unit interval and $f : I' \to I'$ be a function. Also assume D_f as the double operator of f. Then $\mathrm{Per}(D_f) = 2\mathrm{Per}(f) \cup \{1\}$.*

Proof of Lemma 2.10 is found in [10].

3. Main Theorems

Theorem 3.1. *Let $f : I \to I$ be a continuous map. If f has a cycle of period m, it also has cycle of any period n such that $n \lhd m$ in Sharkovskii ordering.*

Proof. By Lemma 2.8, we get that if f has a cycle of period m, it also has a cycle of period n with $n \lhd m$ in the following ordering $\cdots \lhd 7 \lhd 5 \lhd 3$.

Applying Lemma 2.9, we get that f has also a period 2.3 point, that is, $2.3 \lhd \cdots \lhd 7 \lhd 5 \lhd 3$.

Again if f has a period $2m$ point with $m \geq 3$ and odd, by Lemma 2.5, f^2 has a period m point. Applying Lemma 2.8 again we can say that f^2 has a period $(m + 2)$ point. This gives by Lemma 2.6, that f has either period $2(m + 2)$ point or period $(m + 2)$ point. If f has a period $(m + 2)$ point, by Lemma 2.9, it has also a period $2(m + 2)$ point. So in either case f has period $2(m + 2)$ point. Hence we get $\ldots 2.5 \lhd 2.3 \lhd \cdots \lhd 7 \lhd 5 \lhd\ 3$.

We now prove the theorem for $2^m, m > 0$. So f has a period of 2^m point, for $m > 0$. Let $k = 2^l$ such that $l < m$. We now consider $g = f^{\frac{k}{2}}$.

Now, $f^{2^m}(y_1) = y_1$, for some $y_1 \Rightarrow f^{2^{l-1}2^{m-l+1}}(y_1) = y_1 \Rightarrow (f^{\frac{k}{2}})^{2^{m-l+1}}(y_1) = y_1 \Rightarrow g^{2^{m-l+1}}(y_1) = y_1$.

Since $2^{m-l+1} > 2$, by Lemma 2.7 we get that g has a periodic point of period 2, that is, $g^2(y_2) = y_2$, for some $y_2 \Rightarrow f^{2^l}(y_2) = y_2$. Hence existence of period of 2^m point of $f \Rightarrow$ existence of period of 2^l point of f, where $l < m$ and $m > 0$.

Next we prove the theorem for $p.2^m$ with $p \geq 3$ as an odd number and $m > 1$. The following cases are to be considered.

Case I. Let $k = q.2^m$ with $q > p$ and q is odd.

So in this case f has a period of $p.2^m$ point with p as an odd number and $m > 1$. We now consider $g = f^{2^m}$. Then $g^p = f^{p.2^m}$. So g has a p periodic point. Since p is odd and $q > p$, by Lemma 2.7 there exists a periodic point of period q of g. Now $g^q = f^{q.2^m}$.

Hence existence of period of $p.2^m$ point of $f \Rightarrow$ existence of period of $q.2^m$ point of f with $q > p$ and p, q are odd numbers such that $p > 1$, $m \geq 2$.

Case II. Let $g = f^{2^{m-1}}$.

In this case $g^{2.p}(y_3) = y_3$, for some y_3. So g has a period $2.p$ point. By this and by our earlier proof, g has a period $2.(p+2)$ point.

On the other hand, by Lemma 2.5, g^2 has a period p point. From which we get g^2 has a period 2.3 point, by applying Lemma 2.9. Then $(g^2)^{2.3}(y_4) = y_4$, for some $y_4 \Rightarrow g^{2^2.3}(y_4) = y_4 \Rightarrow f^{2^{m+1}.3}(y_4) = y_4$.

So f has a period $2^{m+1}.3$ point. Now if $p = 3$, this proves the existence of period of $p.2^m$ point of f, that is, we get existence of period of $p.2^l$ point of f, where $l > m$ and $m > 1$.

If $p > 3$, by applying Case I we get a periodic point of period $p.2^{m+1}$ of f. So in either case existence of period of $p.2^m$ point of $f \Rightarrow$ existence of period of $p.2^l$ point of f, where $l > m$ and $m > 1$.

Lastly, we prove that if f has a periodic point of period $m.2^p, m \geq 3$ and odd, it has also a periodic point of period $2^r, r > p$.

We now take $g = f^{2^p}$. Again $f^{m.2^p}(y_5) = (y_5)$, for some y_5, that is, $g^m(y_5) = y_5$. Hence we get that, g has a periodic point of period m with $m \geq 3$. So by Lemma 2.7, we get that, g has a period 2 point, that is, $g^2(y_6) = y_6$, for some y_6. This gives $f^{2^{p+1}}(y_6) = y_6$. Hence existence of period of $m.2^p$, $m \geq 3$, point of $f \Rightarrow$ existence of period of 2^r point of f with $r > p$.

By virtue of above those arguments the theorem is proved. □

Theorem 3.2. *For each positive integer n there exists a continuous map $f : I \to I$, that has a cycle of period n, but no cycle of period m for any $n \lhd m$ in Sharkovskii ordering.*

Proof. Let $I = [-1, 1]$ be a closed interval and we consider the continuous map $F(x) = 1 - 2x^2$ on I. Then $F^n(x) = x$ has exactly 2^n distinct solution in I. Hence there are finitely many period n-orbits of $F(x)$. Let O_n be one of period n-orbits such that $(\max O_n - \min O_n)$ is smallest among all the period n-orbits.

We now define the function,

$$F_n(x) = \begin{cases} \min O_n, & F(x) \le \min O_n, \\ \max O_n, & F(x) \ge \max O_n, \\ F(x), & \min O_n \le F(x) \le \max O_n. \end{cases}$$

Then obviously $F_n(x)$ is a continuous function for all n. Since $(\max O_n - \min O_n)$ is smallest, $F_n(x)$ has exactly one period n orbit. It has also no period m orbit for any $n \lhd m$ in Sharkovskii ordering.

This proves the theorem. $\square$

Theorem 3.3. (see [10]) *There exists a continuous map $f : I \to I$ that has cycle of period 2^i for $i \ge 0$ and has no cycle of any other period.*

Proof. Let $I' = [0, 1]$ be the unit interval and we consider the continuous function $f : I' \to I'$ by $f(x) = \frac{1}{3}$. Then we get, $\mathrm{Per}(f) = \{1\}$. We now consider the double operator of f,

$$f_1(x) = D_f(x) = \begin{cases} \frac{7}{9}, & 0 \le x \le \frac{1}{3}, \\ \frac{7}{3}(\frac{2}{3} - x), & \frac{1}{3} < x \le \frac{2}{3}, \\ x - \frac{2}{3}, & \frac{2}{3} < x \le 1. \end{cases}$$

We first prove that $f_1(x)$ is a continuous function. From our construction it is clear that $f_1(x)$ is continuous on $[0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3}) \cap (\frac{2}{3}, 1]$. So we are ready to prove that $f_1(x)$ is continuous at $\frac{1}{3}$ and $\frac{2}{3}$.

Now

$$\lim_{x \to \frac{1}{3}^-} f_1(x) = \lim_{x \to \frac{1}{3}^+} f_1(x) = \frac{7}{9} = f_1(\frac{1}{3})$$

and

$$\lim_{x \to \frac{2}{3}^-} f_1(x) = \lim_{x \to \frac{2}{3}^+} f_1(x) = 0 = f_1(\frac{2}{3}).$$

Hence $f_1(x)$ is a continuous function on I'.

Now from Lemma 2.10 we get that, the periods of $f_1(x)$ are $\mathrm{Per}(f_1) = \{1, 2\}$. Let $f_2(x) = D_f^2$, then the periods of $f_2(x)$ are $\mathrm{Per}(f_2) = \{1, 2, 2^2\}$. Hence by mathematical induction periods of $f_n(x)$ are $\mathrm{Per}(f_n) = \{1, 2, 2^2, \ldots, 2^n\}$, where $f_n(x) = D_f^n(x)$.

Lastly, we define $f_\infty(x) = \lim_{n\to\infty} f_n(x)$ and claim that $f_\infty(x)$ is a continuous function.

We have proved before that $f_1(x)$ is a continuous function on I' and noted that $f_1(x) = x_1$, for some $x_1 \in I'$. Then $f_2(x) = f_1(x_1)$ is again continuous on I', since $x_1 \in I'$. Let $f_m(x)$ be continuous on I', that is, $f_m(x) = x_{m+1}$, for some $x_{m+1} \in I'$. Then $f_{m+1}(x) = f_1(x_{m+1})$ is also continuous on I', since $x_{m+1} \in I'$.

Hence by mathematical induction we get $f_n(x) = D_f^n(x)$ is continuous for all n. Hence $\{f_n(x)\}$ is a sequence of continuous functions defined on I' such that $\lim_{x\to c} f_\infty(x) = f_\infty(c)$, for all $c \in I'$. So $f_\infty(x)$ is a continuous function on I'.

Lastly, we get from the definition of $f_\infty(x)$, that $\mathrm{Per}(f_\infty) = \{1, 2, 2^2, \dots\}$ and it contains no other period.

Hence the theorem is proved. □

4. Consequences of Sharkovskii's Theorem

The Sharkovskii's Theorem has several consequences. We note those one by one.

1. If a continuous map has a cycle of least period $\neq 2^i, i \geq 0$, it has also a cycle of least period $2^l, l \geq 0$.

2. If a continuous map has only finitely many periodic points, they all necessarily have periods which are powers of two.

3. In Sharkovskii ordering the greatest period is three. So, period three implies all other periods of a continuous map.

4. Sharkovskii's Theorem does not tell about the stability of cycles, just tell that there are cycles of those periods.

5. From Sharkovskii ordering we can easily prove that if a continuous map has a period point of least period $p.2^m$ with p as odd, it has also a period point of least period $q.2^m$ with q as even.

We will now discuss the generalization of this theorem. Sharkovskii's Theorem does not immediately apply to dynamical systems on other topological spaces. It is generally an one-dimensional result. In fact, this theorem does not even hold on the circle. We consider the unit circle S^1 on the plane. Let $f : S^1 \to S^1$ be defined by $f(\theta) = 72^\circ + \theta$. Then it is a continuous map and this map makes all points of S^1 periodic with least period five. There is no other periodic point. Hence Sharkovskii's Theorem does not hold on S^1. Although this theorem can be extended to wider classes of maps such as discontinuous, multi-valued, and random maps and so on, to different types of phase spaces. Recently, P. Szuca [13] has proved that Sharkovskii ordering holds even for discontinuous map whose graphs are connected G_δ sets of the plane. C. Bernhardt [1] suggested a multi-valued version of Sharkovskii's Theorem. On the other hand, even in R^n there is a class of maps for which the ordering holds, these are so called 'Triangular' maps.

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